Engineering Hydrology
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Module 1
Lecture 6: Conservations Laws

Hello all, welcome back. Till now we were discussing about Reynolds transport theorem. We have seen the derivation of Reynolds transport theorem and we have seen the relationship between the Lagrangian and Eulerian concept of fluid flow.

Now, what we will do we will make use of the theorem, which we have derived in the previous lecture that is Reynolds transport theorem for the derivation of conservation laws. The conservation laws are mass conservation, energy conservation and momentum conservation.

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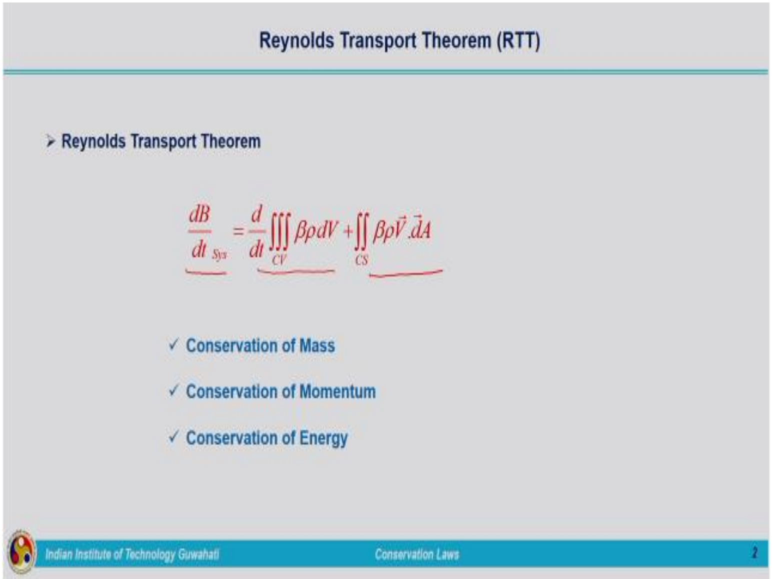

So, let us move on to that.

$$\frac{dB}{dt}_{Syt} = \frac{d}{dt}\iiint_{CV} \beta\rho d\cancel{V} + \iint_{CS} \beta\rho\vec{V}\cdot\vec{dA}$$

This is the Reynolds theorem which we have derived in the previous lecture. So, Reynolds theorem is relating the time rate of change of extensive property $\frac{dB}{dt}$ of a system to the time rate of change of extensive property within the control volume and the net outflow of extensive property across the control surface.

So, this we have explained in detail in the previous lecture. So, now, we will make use of this theorem for the derivation of conservation of mass, conservation of momentum and conservation of energy.

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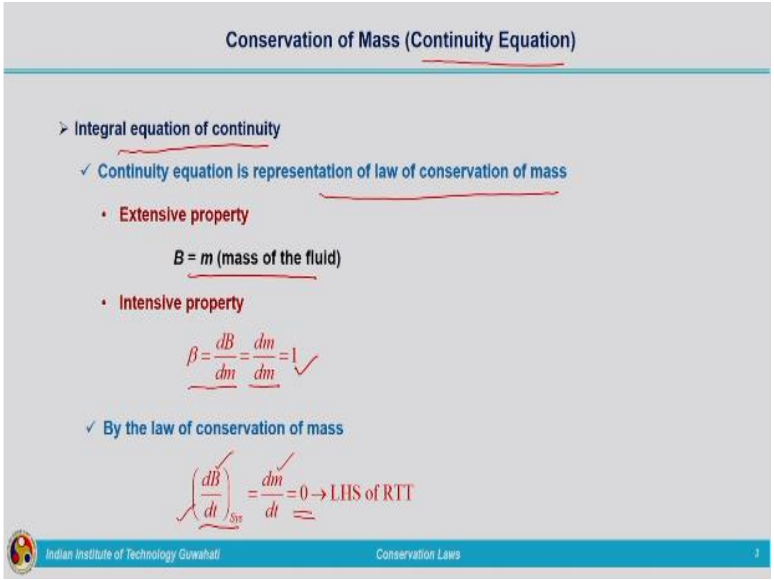

Let us start with the derivation of conservation of mass by making use of Reynolds transport theorem, that is, we are going to derive the integral equation of continuity. Conservation of mass equation is also known as continuity equation. We commonly make use of the term continuity equation in hydrology, it is nothing but the mass conservation equation.

So, here we are going to make use of Reynolds transport theorem. What continuity equation is expressing? It is the representation of law of conservation of mass. Here in this case, when we are dealing with we need to consider the extensive property. Extensive property is nothing but mass of the fluid, because we have derived the Reynolds transport theorem for extensive property.

It is applicable to extensive property. Extensive property is the property which is dependent on the mass. So, here in while deriving these conservation laws, we will consider different, different extensive properties. For example, in the case of continuity equation or mass conservation equation, our extensive property is mass of the fluid.

Now, corresponding intensive property, we know the relationship

$$\beta = \frac{dB}{dm} = \frac{dm}{dm} = 1$$

Now, we know by the law of conservation of mass, $\frac{dB}{dt}_{Syt}$ is

$$\left(\frac{dB}{dt}\right)_{Syt} = \frac{dm}{dt} = 0 \rightarrow \text{LHS of RTT}$$

because in the case of a system, we are considering a control mass. There is no mass transfer taking place from the system to the surrounding or from surrounding to the system. That is the mass within the system is a control mass or the constant mass.

So, when we take the derivative of a constant value it will be equal to 0. So, the left-hand side of Reynolds transport theorem is $\frac{dB}{dt}$ that is time rate of change of extensive property of the system. Here in our case extensive properties mass of the fluid, so $\frac{dm}{dt} = 0$. So, left hand side of RTT is 0.

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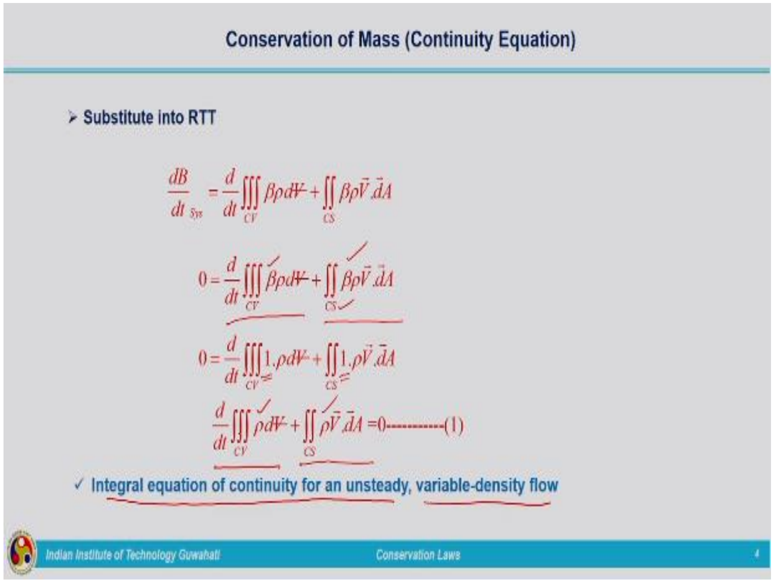

Now, we will look into the RTT expression.

$$\frac{dB}{dt}_{Syt} = \frac{d}{dt}\iiint_{CV} \beta\rho d\forall + \iint_{CS} \beta\rho\vec{V}\cdot\vec{dA}$$

$$0 = \frac{d}{dt}\iiint_{CV} \beta\rho d\forall + \iint_{CS} \beta\rho\vec{V}\cdot\vec{dA}$$

So, we are having the expression on the right-hand side, one part is related to control volume and the second part is related to control surface. Here we are going to substitute for beta. Beta is the intensive property that we have found out to be 1, if the extensive property is mass.

$$0 = \frac{d}{dt}\iiint_{CV} 1\rho d\forall + \iint_{CS} 1\rho\vec{V}\cdot\vec{dA}$$

So, that for beta we are substituting here. So, our expression will be coming out to be like this.

$$\frac{d}{dt}\iiint_{CV} \rho d\forall + \iint_{CS} \rho\vec{V}\cdot\vec{dA} = 0 \quad \text{---------------------------(1)}$$

This is the equation representing conservation of mass. Also known as the integral equation of continuity for unsteady, variable density flow. You look at the equation, we are having a term which represents the variation with respect to time, that is $\frac{d}{dt}$ term is coming it is representing that time rate of change of some quantity, so it is unsteady, definitely an unsteady flow.

Now, you look into the expression, we are having density term coming within the integrals, so that can be applied to variable density flow. Here you can see we are having the volume integral and also surface integral. In fluid mechanics, you might have seen, we used to convert this surface integral to volume integral and combine these terms together and we will proceed in that way.

But here in the hydrology perspective, I am not going to make use of Gauss divergence theorem, that is surface integral can be converted into volume integral by making use of mathematical theorems. So, here I am keeping as such. I am not changing the surface integral to volume integral, because we can make use of the theorem, Reynolds Transport Theorem as such, while we move ahead in the complex topics. So, this is our integral equation for continuity for unsteady variable density flow.

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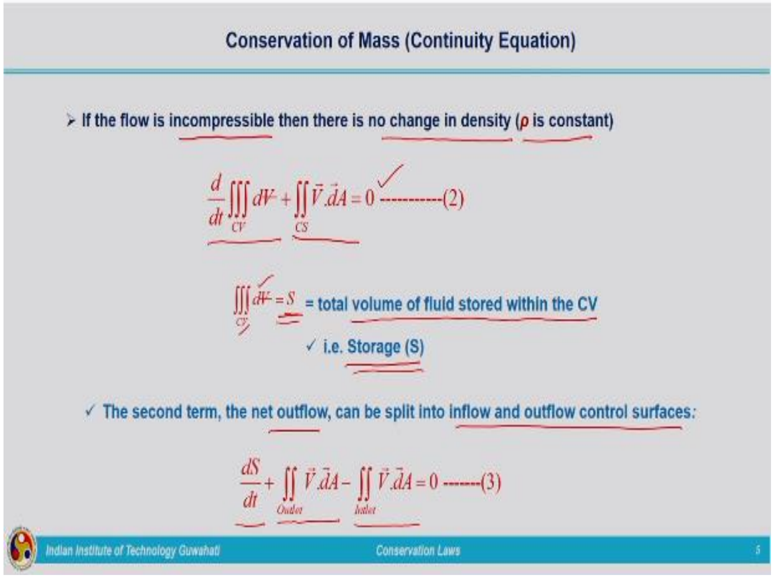

Now, if the flow is incompressible, means there will not be any change in density, that is we will be assuming ρ as constant. I am not going deep into the incompressible flow concepts. So, here if density is constant, we can take it out of the integral and our expression will be changing into this form.

$$\frac{d}{dt}\iiint_{CV} d\forall + \iint_{CS} \vec{V} \cdot \vec{dA} = 0 \quad \text{-----------------------------------(2)}$$

Now, the first term $\frac{d}{dt}\iiint_{CV} d\forall$. So that can be, that is whatever change is taking place, the volume integral is representing $\iiint_{CV} d\forall = S$, total volume of fluid within the control volume.

So, this is volume integral is across the control volume $d\forall$ that is represented by S. If you are having the notation is that is representing the storage of a control volume, because this is very important concept when hydrologic problems are concerned. Now, the second term, second term is the surface integral of $\iint_{CS} \vec{V}.\vec{d}A$ across the control surface, that is that second term itself is consisting of two terms, one is related to the outflow region and the second part is related to the inflow region.

So, net outflow considering inflow region and outflow region is combined together in the second term which is represented by surface integral of $\iint_{CS} \vec{V}.\vec{d}A$ across the control surface. So, we are having the net outflow that is split into inflow and outflow control surfaces. So, this expression is again written in this form i.e.,

$$\frac{dS}{dt} + \iint_{Outlet} \vec{V}.\vec{d}A - \iint_{Intlet} \vec{V}.\vec{d}A = 0 \text{ ----------------------------(3)}$$

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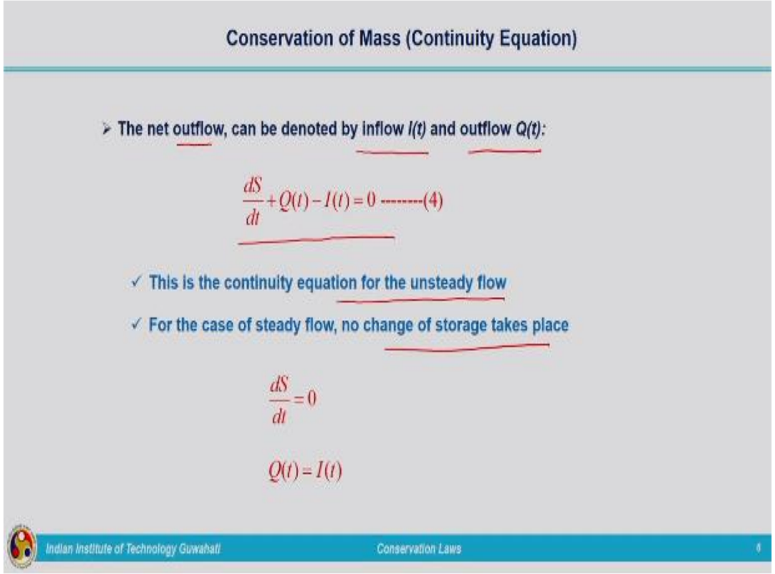

So, net outflow we are representing, that is, the terms corresponding to net outflow that is related to outflow control surface and the inflow control surface can be denoted by inflow *I(t)* and outflow *Q(t)*, and we are going to substitute in terms of *I(t)* and *Q(t)*. So, the expression will be taking this form

$$\frac{dS}{dt} + Q(t) - I(t) = 0 \text{ ----------------------(4)}$$

So, this is the continuity equation, which we use for unsteady flow.

So, if the flow is steady, sometimes we will be assuming for simplicity, we will be making the system to be steady, so in such case variation with respect to time will be 0, that term need not be considered. So, $\frac{dS}{dt}=0$. What does it mean? There won't be any change in storage takes place, that means $\frac{dS}{dt}=0$, which implies that,

$$Q(t)=I(\mathrm{t})$$

Whatever coming inside, that is going out. So, there is no change is taking place in the case of storage. Otherwise, how changes take place in the storage? That is some amount is coming inside and some amount is leaving, so both are different. So that difference will be stored within the control volume. So, that is why we are telling the mass can come inside and leave the control volume, there is changes taking place in the case of mass with respect to control volume. It is not the same with the system, in the case of system the mass is constant.

So, here in this case, in the case of steady flow, there is no change in storage taking place that means, $\frac{dS}{dt}=0$, implies that $Q(t)=I(\mathrm{t})$ that is outflow is equal to inflow. So that much about the conservation of mass or the continuity equation.

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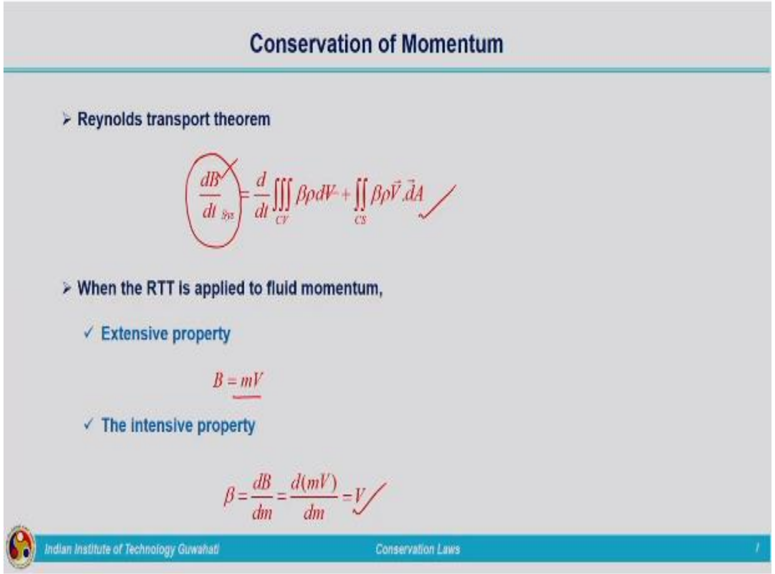

Now, we can make use of Reynolds transport theorem for deriving the equation corresponding to conservation of momentum. So, I am again repeating the equation corresponding to Reynolds transport theorem

$$\frac{dB}{dt}_{Syt} = \frac{d}{dt}\iiint_{CV} \beta\rho d\cancel{V} + \iint_{CS} \beta\rho\vec{V}\cdot\vec{dA}$$

Now we need to have an idea about the extensive and intensive properties in the case of momentum. So, when Reynolds transport theorem is applied to fluid momentum, our extensive property $B = mV$, that is momentum of fluid flow. Now, we know the relationship between B and β intensive property. Intensive property is nothing but

$$\beta = \frac{dB}{dm} = \frac{d(mV)}{dm} = V$$

Where, V is the velocity of fluid flow. Now, what we will do, we will substitute in the Reynolds transport theorem. So, from Newton's law, we know that time rate of change of momentum is nothing but the net force acting on the system.

So, that is what is coming on the left-hand side of RTT, i.e., $\left.\dfrac{dB}{dt}\right._{Syt}$, here in our case B is the momentum and when time rate of change of momentum is considered based on Lagrangian principle or control mass principle, we know it is nothing but it is the net force acting on the system. So, corresponding to this term on the left-hand side $\dfrac{dB}{dt}$ for system we can substitute net force.

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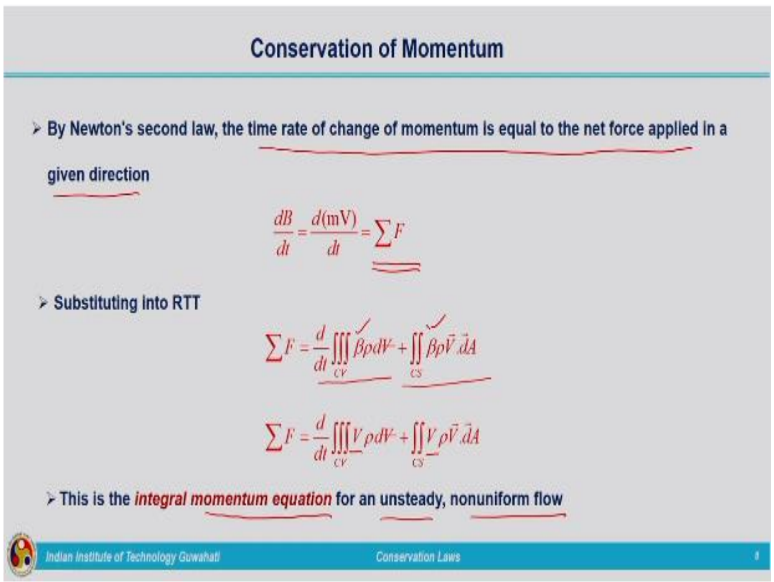

So, that is what is written over here. Newton's second law is the time rate of change of momentum is equal to net force applied in that direction. So,

$$\frac{dB}{dt} = \frac{d(mV)}{dt} = \sum F$$

Where, Net force we are representing by the notation $\sum F$. That can be substituted in RTT. So, our equation will be

$$\sum F = \frac{d}{dt}\iiint_{CV} \beta\rho d\forall + \iint_{CS} \beta\rho \vec{V}\cdot\vec{dA}$$

Now, for β we are going to substitute, we have computed the corresponding value of extensive property for momentum i.e., *V*, so beta we are substituting as *V*. So, our expression is taking this form.

$$\sum F = \frac{d}{dt}\iiint_{CV} V \rho d\cancel{V} + \iint_{CS} V \rho \vec{V} \cdot \vec{dA}$$

This is our integral form of momentum equation.

So, when we apply it to fluid flow problems, we will be finding out the corresponding expressions for $\sum F$, different types of forces will be there, body force, surface force, buoyant force that way depending on the problem, what are the forces existing, forces which are contributing to a particular type of flow that we will be computing and that will be substituting here in the left-hand side.

So, while deriving the Navier Stokes equation you might have seen in fluid mechanics, but depending on our problem we will be considering different forces in the case of hydrology that will be substituting. But as of now, I am keeping this expression like this only, whenever required we will substitute the required components in the particular equation. So, this is the integral momentum equation for unsteady non-uniform flow. You can see this is with respect to time it is changing and also it changes with respect to space. So, that is why this equation is applicable to non-uniform unsteady flow.

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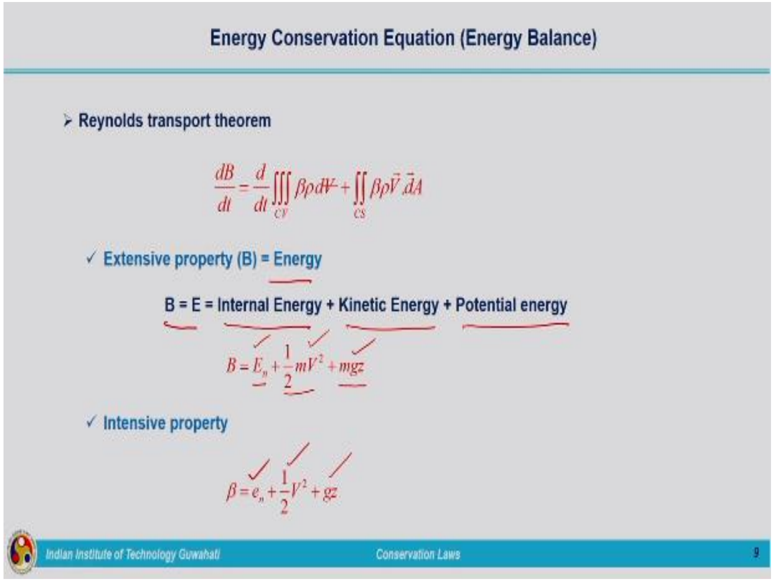

Now, next principle is the conservation of energy. So, energy conservation equation or energy balance equation. Reynolds Transport Theorem I am repeating again,

$$\left.\frac{dB}{dt}\right|_{Syt} = \frac{d}{dt}\iiint_{CV} \beta\rho d\cancel{V} + \iint_{CS} \beta\rho\vec{V}\cdot\vec{dA}$$

So here in this case we need to check what is our extensive property and the corresponding intensive property. Left hand side is time rate of change of extensive property of the system. So, extensive property B is energy because we are going to derive the energy conservation equation. So definitely our extensive property is energy.

Now, coming to intensive property will be $\frac{d}{dm}$ of this particular energy. So, we need to write this energy in the mathematical form. So, that can be written as

$$B = \text{Internal Energy} + \text{Kinetic Energy} + \text{Potential Energy}$$

Kinetic energy and potential energy are the mechanical forms of energy. The internal energy other than this kinetic energy and potential energy, any other form of energy coming is considered under internal energy.

I am not going deep into the definition of internal energy, but when it comes to certain problems such as heat energy related problems, evaporation, those types of hydrological processes, we will be looking into it in detail. Here as such I am keeping as internal energy. So, total energy or the total extensive property in the case of energy conservation principle, while we are going to derive the energy conservation equation, extensive property is the energy that can be written as the sum of potential energy, kinetic energy and internal energy. So, that can be written in the mathematical form. We are representing internal energy by a notation E_n and we know the expression for kinetic energy that is $\frac{1}{2}mV^2$ and potential energies mgz.

$$B = E_n + \frac{1}{2}mV^2 + mgz$$

Now, what will be the corresponding intensive property, we have to differentiate it.

$$\beta = e_n + \frac{1}{2}V^2 + gz$$

E_n as such I have kept, capital letter is taken for internal energy when we are differentiating it with respect to mass I have put small e_n.

So, we got the expression for intensive property corresponding to an extensive property of total energy. Now, the things are very simple, what we were doing earlier the same thing we will be doing here also, that is right hand side of the RTT we will be considering this intensive property. So RHS of RTT wherever β is coming we will substitute these terms. But what is $\frac{dB}{dt}_{Syt}$? Left hand side of RTT. For that we need to have an expression.

When we were talking about momentum, we have mentioned that according to Newton's second law, time rate of change of momentum can be put as net force acting on the system. That is why we have substituted in that case as $\sum F$. But in the case of energy conservation principle, what we will substitute in the case of $\frac{dB}{dt}$? Let us see that.

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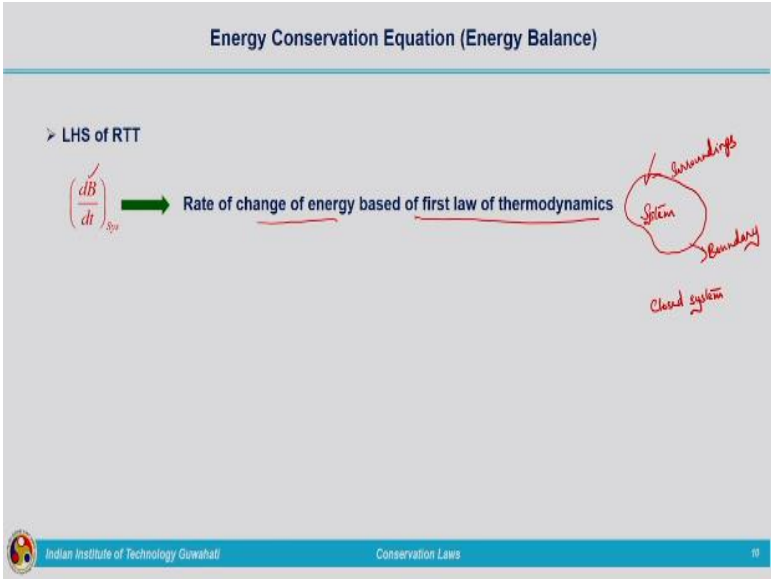

Left hand side is $\left(\frac{dB}{dt}\right)_{Syt}$. So, system we have seen very clearly while deriving the Reynolds transport theorem. Let us consider a system. I am just drawing a simple system, system is having boundary and surroundings that is near to the system boundary, what is the area present that we will be calling as surroundings.

As far as this system is concerned, it is a control mass which is having a constant mass, there is no transfer of mass taking place across the boundary in the case of a system. So, it is also known as a closed system. So certain terms synonymously used closed system, simply system or control mass.

These are not three different entities, all these three are representing a single thing that is what is the system which we are talking about. So, system is having a certain quantity of matter within that. System contains certain quantity of matter within that. That matter is represented by the mass that is a constant value as far as a system, closed system is concerned.

There is no transfer of mass taking place across the boundary in the case of a system. But that is not the case with the control volume, control volume there is a transfer of mass taking place, it is not having the constant mass. But as far as energy is concerned in the case of a system and also control volume, there is a transfer of energy taking place from the system to the surrounding and from surrounding to the system.

So, that we have studied when we were dealing with the first law of thermodynamics. We know energy cannot be created and destroyed. But from one form of energy it can be converted to another form of energy. So, here in the case of a system, there will be some amount of energy coming inside and also some work will be done by the system on the surrounding. So, that way we can write the total energy, total energy we have represented in terms of extensive property. Now we need the expression for change in energy, that can be represented by, i.e., the $\frac{dB}{dt}$, time rate of change of extensive properties, extensive property in our case is the energy.

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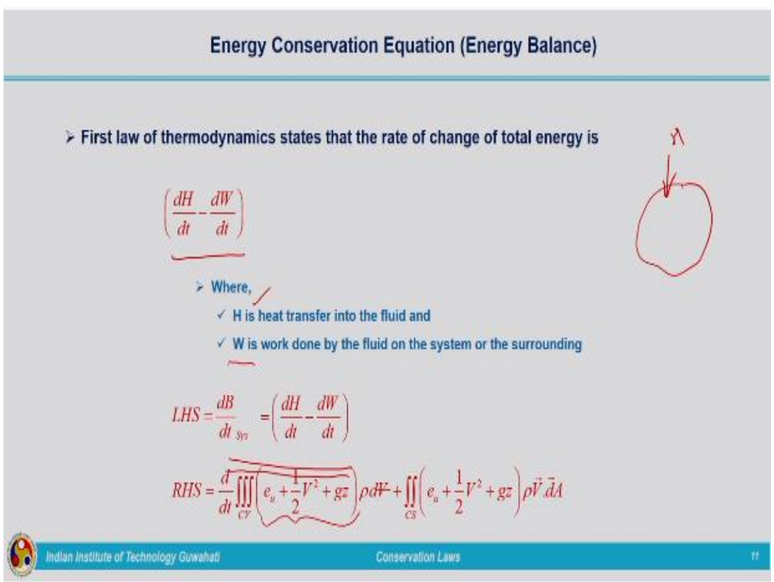

Time rate of change of energy based on first law of thermodynamics can be written as $\left(\frac{dH}{dt}-\frac{dW}{dt}\right)$. This is the expression, i.e., the time rate of change of energy can be equated to the difference between the energy which is coming inside the system and the work done by the system on the surrounding. So, H is the heat transfer into the system or into the fluid and W is the work done by the fluid on the system or the surrounding.

So, this is the expression which we will be using for the left-hand side of RTT. $\frac{dB}{dt}$, $\frac{d\left(\text{Total energy}\right)}{dt}$ we are talking about. $\frac{dB}{dt}$ is $\frac{dE}{dt}$, that is nothing but we are making use of the principle of first law of thermodynamics, based on that we know it is nothing but the expression can be rewritten by $\left(\frac{dH}{dt}-\frac{dW}{dt}\right)$.

H is the heat transfer into the system, that is if you are having the system some energy is coming inside that is the H and the work done by the system on the surrounding, by the fluid on the system or the fluid is represented by the letter W. So, we can write the time rate of change of energy of the system as $\left(\frac{dH}{dt}-\frac{dW}{dt}\right)$.

Now, we are going to substitute this into our RTT, that is left hand side of RTT can be represented by this expression.

$$LHS=\frac{dB}{dt}_{Sys}=\left(\frac{dH}{dt}-\frac{dW}{dt}\right)$$

Now, right hand side, what we have done in the case of mass conservation, momentum conservation, in the same way, we will be substituting the expression corresponding to intensive property instead of β.

So, we can write LHS based on first law of thermodynamics, RHS for β we have substituted the intensive property corresponding to extensive property total energy.

$$RHS=\frac{d}{dt}\iiint_{CV}\left(e_n+\frac{1}{2}V^2+gz\right)\rho d\cancel{V}+\iint_{CS}\left(e_n+\frac{1}{2}V^2+gz\right)\rho\vec{V}\cdot d\vec{A}$$

Total energy was consisting of internal energy, potential energy and kinetic energy. Corresponding intensive property we have written, that intensive property we are substituting on the right-hand side of the Reynolds transport theorem.

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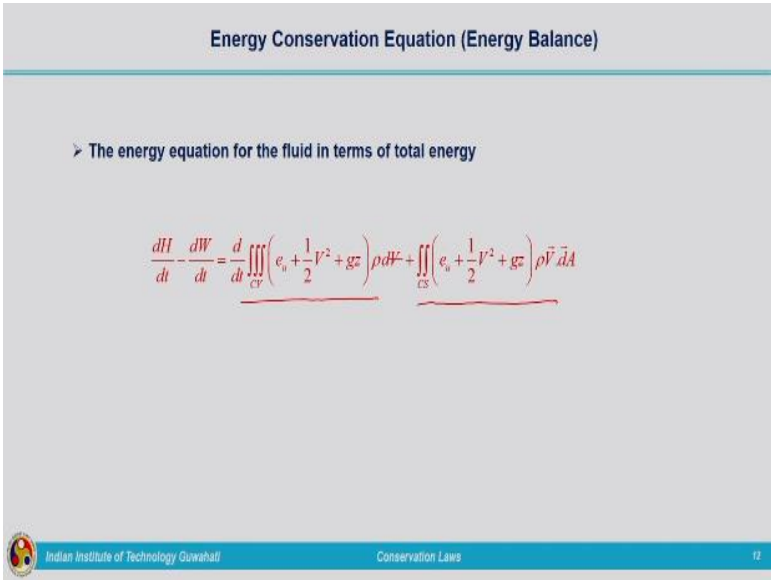

So, finally, we will get the energy equation in this form. So, the final form of energy equation can be written as

$$\frac{dH}{dt}-\frac{dW}{dt}=\frac{d}{dt}\iiint_{CV}\left(e_n+\frac{1}{2}V^2+gz\right)\rho d\cancel{V}+\iint_{CS}\left(e_n+\frac{1}{2}V^2+gz\right)\rho\vec{V}\cdot\vec{dA}$$

When you see this expression, as such you may find it as a very lengthy expression. This equation is required when we derive the energy related problems.

Definitely, for example, if you are talking about the evaporation, for example, you know what it is actually, but you have not seen in depth in this particular course. So, while dealing with that we will be making use of this particular equation. So, that time what has to be substituted for each and every term we will see in detail. So, this is the energy equation for the fluids in terms of total energy.

So, now, I am stopping here. We have made use of Reynolds Transport Theorem for the derivation of conservation laws. So, first equation which we have derived is the mass conservation equation. In that case, we have considered $\frac{dB}{dt}_{Sys}$ to be that is $\frac{dm}{dt}$, mass cannot be created, destroyed, that principle we have used there and $\frac{dm}{dt}$ is substituted to be zero.

So, that principally you should understand. We have assumed the system, we are not considering any changes is taking place for the mass conversion. There are certain problems in which mass will be converted to other form, i.e., in the case of water converting to vapor, so the total mass of the liquid, some part is converted to vapor. So, in that case $\frac{dm}{dt}$ cannot be considered as zero.

So, there are certain problems, when the phase change is taking place, some problems related to chemical reactions, nuclear reactions, in all such cases, if we are considering individually each case, if for example, water converting into vapor and we are considering single water phase, then we cannot substitute $\frac{dm}{dt}$ to be zero. But as such entire quantity of fluid we are considering incorporative of water and vapor, then together we can consider.

So, you have to be very careful while applying these theorems. But we have derived the mass conservation principle based on the assumption that there is no phase change or there is no chemical reaction or nuclear reaction taking place, based on the principle we have derived the expression for mass conservation.

The similar way while deriving the momentum conservation, we have made use of Newton's second law and while deriving the energy conservation principle, we have used the first law of thermodynamics into account.

So, you should understand in all these cases, the left-hand side is based on the Lagrangian concept and Lagrangian concept when we were applying we have used the concepts of fundamental laws and that is related to the component which is coming on the right-hand side which is representing the Eulerian point of view for the fluid analysis.

So, while dealing with the engineering problems, we can make use of this Reynolds transport theorem and we can use of the fundamental laws which are derived based on Lagrangian principles for solving the engineering problems which will be solved by using the control volume principles, i.e., we will be analyzing within a fixed frame of reference. That fixed frame of reference I am repeating again, here in hydrology we are considering a stationary frame of reference, this frame of reference may be moving also.

So, certain fluid mechanics problems you will be dealing with the movable control volume. So, here in this analysis, further on we will be considering the stationary control volume. So, that much about the Reynolds transport theorem and the application of that Reynolds transport theorem for the derivation of the conservation principles.

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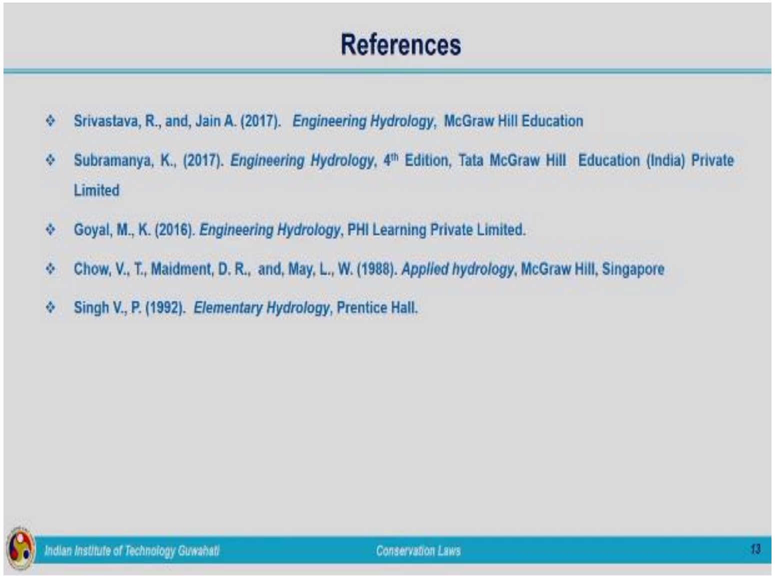

So, here I am shopping now. Some of the references which will be used for the derivation of these conservation laws are given here. Same set of references which I have given by explaining the course contents. Thank you very much. Have a nice day.