Engineering Hydrology
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Module 1
Lecture 5: Reynolds Transport Theorem Part - II

Hello all, welcome back. Today's lecture is the continuation of the previous lecture. In the previous lecture, we have started with the derivation of Reynolds Transport Theorem. Reynolds Transport Theorem, we have seen the introduction part of it and the Reynolds Transport Theorem relates the time rate of change of extensive property within the control mass or system to the reasons behind these changes, i.e, the reasons which are causing the changes. So, yesterday in the previous lecture, we have started with the derivation.

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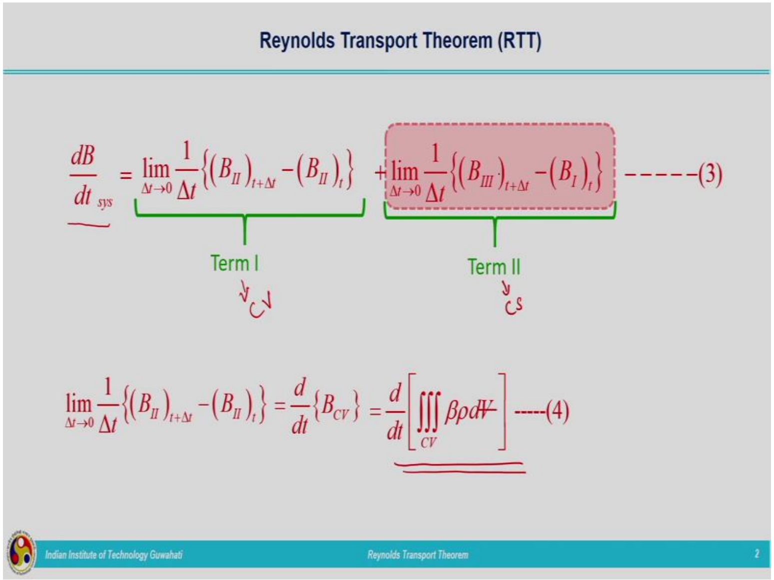

And the expression came out to be like this.

$$\frac{dB}{dt}_{Sys} = \underbrace{\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{(B_{II})_{t+\Delta t} - (B_{II})_t\right\}}_{\text{Term1}} + \underbrace{\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{(B_{III})_{t+\Delta t} - (B_{I})_t\right\}}_{\text{Term2}} ------(3)$$

We have considered a fluid flow field within that we have considered a control mass and for the analysis point of view under the Eulerian perspective, we have considered a control volume. At time *t*, these control volume and the system were coinciding each other. After a small interval of time Δt, the system has moved to the right-hand side that is in the flow direction. And we have, we were trying to find out the relationship with the time rate of change of extensive property, whatever be that extensive property, time rate of change of extensive property within the system to that of a control volume. So, let us continue from the point where we stopped yesterday.

Time rate of change of extensive property of the system is related to two terms, first term was something related to the control volume and the second term was something related to the control surface i.e., you can see here i.e., B_{II} at time $\Delta t \to 0$, region II is coinciding with the control volume. So, $\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{(B_{II})_{t+\Delta t} - (B_{II})_t\right\}$ was found out to be time rate of change of extensive property stored within the control volume. So, this term we have derived yesterday, now we have to move on with the second term.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{(B_{II})_{t+\Delta t} - (B_{II})_t\right\} = \frac{d}{dt}\{B_{CV}\} = \frac{d}{dt}\left[\iiint_{CV} \beta \rho d\forall\right] \text{-----------------------------(4)}$$

This is the expression representing time rate of change of extensive property within the control volume.

Now, let us look into term 2 in detail. Here it is in terms of extensive property B_{III}, B_I, etcetera. So, that we need to write in terms of intensive property.

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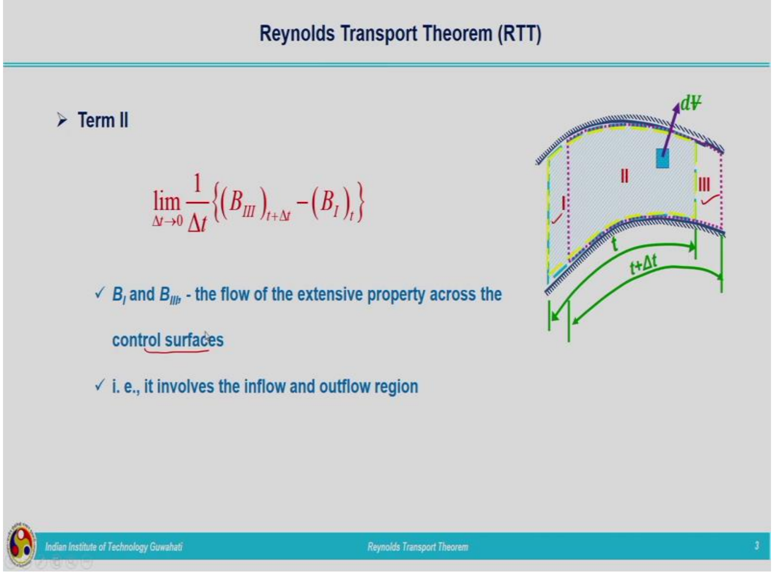

So, again the same figure is depicted here and term 2, we are going to look into in detail. So, here we are having B_{III} that is region III and also region I. Both are representing the outflow and inflow control surfaces. B_I and B_{III}, the extensive properties in regions I and III across the control surfaces. So, this involves the inflow and outflow regions.

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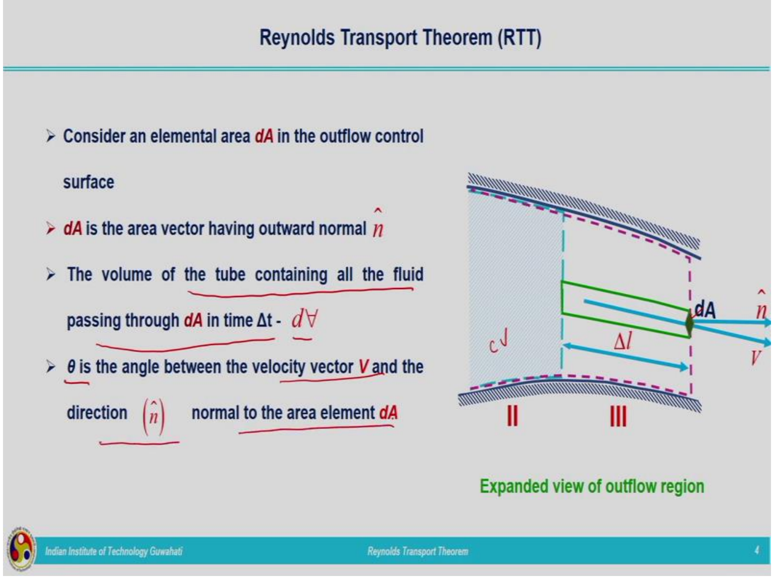

Now, what we are going to do? We are going to look at this control surfaces in a closer way i.e., let us see the expanded view of the outflow region first. So, this is our control volume that we have seen earlier at time $t=0$ and $t=t$ and all times and control mass of the system was coinciding with this control volume when time $t=t$.

And the time is increased to $t+\Delta t$, system has moved to the right-hand side represented by this pink dot. Now, what we are going to do? These are represented by region II and region III. We will consider an area *dA*, here you can see *dA* in the outflow control surface. So, area is a vector. So, area will be having a direction, the direction of an area will be taken always in the outward normal direction. So, that is marked with n.

Then we need to consider the volume of tube containing all the fluid passing through *dA* in time Δt that is represented by the notation $d\cancel{V}$. So, that is depicted here with the help of these green solid diagram, which is having a length of Δl. And here you can see I have marked the velocity vector. So, velocity is in the flow direction. This is marked as *V*.

We are considering an angle θ, which is the angle between the velocity vector and the normal to the area vector. So, we are going to consider angle θ, which is the angle between velocity vector and the direction n normal to the area element *dA*.

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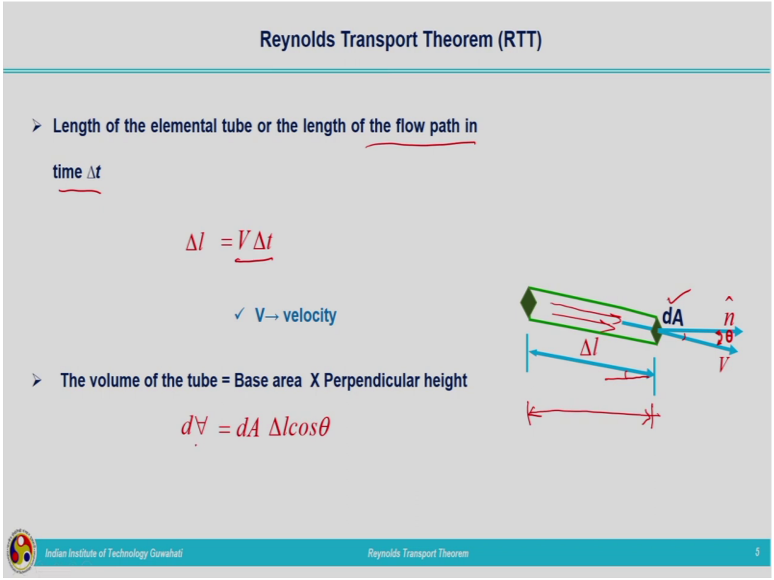

So, this is the tube which is containing the fluid in the control surface or in the outflow region which is having a length of Δl. Area vector is in the direction, marked in the direction of unit vector n and the velocity vector is in the flow direction. The angle between these two are represented by θ. What we are going to do is to get the expression for volume.

So, the length of the elemental tube or the length of the flow path in time Δt, i.e., Δl. That Δl is nothing but

$$\Delta l = V\Delta t$$

V into t will be giving you the distance travel, length of that particular tube. So, we can get Δl by taking the expression $V\Delta t$, V is nothing but our velocity.

Now, coming to the expression for volume ($d\forall$). This is a solid tube, tube in which the flow is taking place. So, the volume representing corresponding to this particular tube will be base area multiplied by the perpendicular distance between these two faces. So, what will be the value corresponding to that perpendicular distance?

So, $d\forall$ is given by base area into perpendicular height. So, base area is nothing but our small elemental area dA. So, $d\forall$ can be written as

$$d\forall = dA\Delta l\cos\theta$$

So, this is the slanting length, length of the fluid element Δl, Δl multiplied by, this angle is θ, so, this angle also will be θ. So, we can consider the component in the direction as $\Delta l \cos\theta$. So, the volume of the tube can be obtained by using this expression

$$d\mathcal{V} = dA\Delta l \cos\theta$$

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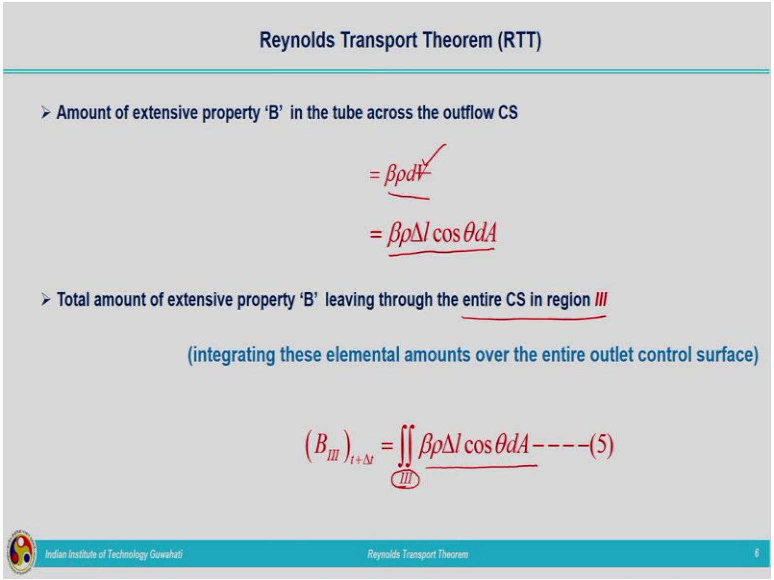

Now, we need to get the expression for extensive property in the tube across the control surface which is there in the outflow region that is region III, i.e., $\beta\rho d\mathcal{V}$. We know the expression is $\beta\rho d\mathcal{V}$, here what we are going to do? We are going to substitute for $d\mathcal{V}$. The expression for $d\mathcal{V}$ we have already found out. So, that we will substitute in the expression for amount of extensive property *B*. It will be coming out to be $\beta\rho\Delta l \cos\theta dA$

This is corresponding to an elemental volume. This expression we got after considering a small elemental area dA and the corresponding volume $d\mathcal{V}$, elemental volume we have calculated and based on that for that elemental tube, which contains the flowing fluid was having a volume of $\Delta l \cos\theta dA$ and based on that we calculated the extensive property corresponding to that small elemental area.

So, the total amount of extensive property '*B*' leaving through the control surface in the region III, outflow control surface will be, definitely we need to integrate this over that particular area. It is across the control surface. So, we will be integrating over that area. So,

$$(B_{III})_{t+\Delta t} = \iint_{III} \beta\rho\Delta l \cos\theta dA \text{---------------------(5)}$$

This is corresponding to the outflow region. So, we are having it for region III. So, we got the expression for the total amount of extensive property B leaving out the control surface as $\left(B_{III}\right)_{t+\Delta t}$. Now, in the similar way, we need to find out the expression for fluid element or the matter which is entering into the control volume. So, for that we will be considering the inflow region that is region I.

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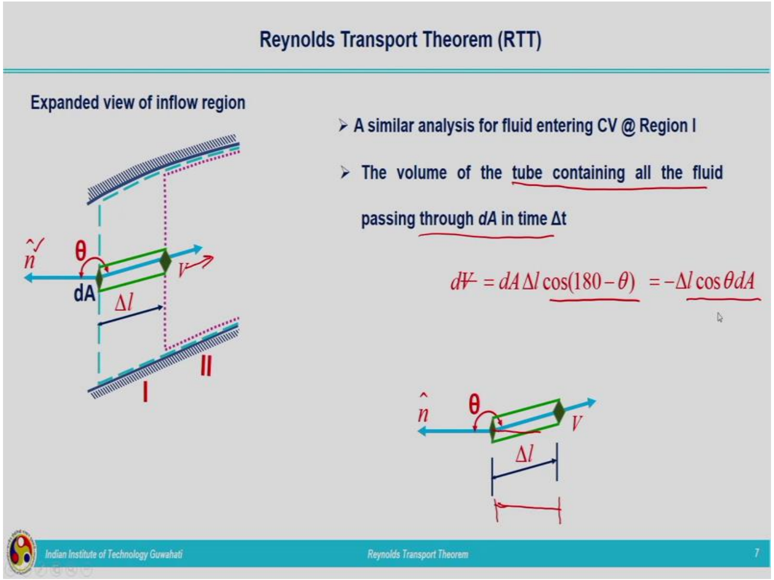

So, we will see the expanded view of inflow region. So, this is our inflow region at time $t = t$, we are having the control volume and system coinciding each other. After Δt time system has moved in the flow direction that is marked by the pink direction.

So, this is region I and we are having the region II which is contained by the control volume and the system. So, within the region I we are going to consider an elemental volume which is having an area dA and length, which is having a cross sectional area dA and length Δl.

Now, again in the similar way, what we have done for the outflow region, we are having the direction vector corresponding to area dA in the outward direction that is marked by this unit vector n and the velocity direction is along the flow direction. How the flow was taking place? Flow was taking place from left to right. So, the velocity vector will be in the flow direction. So, here you should understand that the direction vector corresponding to area and the velocity both are in the opposite direction. And the angle between these area vector and the velocity vector are represented by the θ. Now, the same similar analysis will be carried out for the region I also that is the for the fluid entering to the control volume.

So, this is a small area which is having small elemental volume, which is having a length of Δl, velocity is in the flow direction and we are having the area normal to the cross section dA and the angle between the area vector and the velocity vector is represented by θ. Now, we need to calculate the volume of the tube containing all the fluid passing through dA in time Δt.

So, the same expression will be used that is volume $d\forall$ is equal to cross sectional area multiplied by the perpendicular distance between all the faces. So, here also $d\forall$ is equal to dA multiplied by what will be the component of length that is Δl, this horizontal distance we need perpendicular distance we need.

So, this $\Delta l \cos$, what is the value corresponding to θ? θ is less than 180 degrees, so, it will be $180-\theta$, $\Delta l \cos(180-\theta)$. So, what is $\cos(180-\theta) = -\cos\theta$. So, the expression for $d\forall$, in the inflow region is having a negative sign.

$$d\forall = dA\Delta l \cos\theta = -\Delta l \cos\theta dA$$

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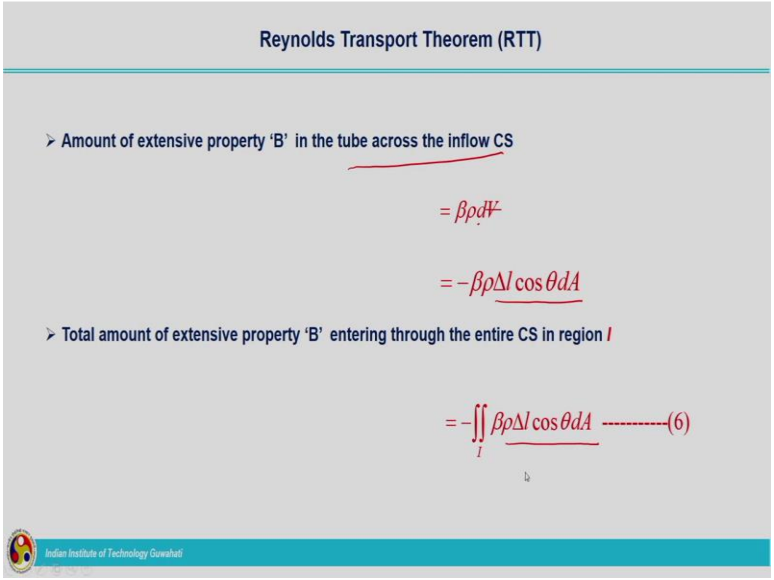

Now, the same procedure we will do for the inflow region also, amount of extensive property 'B' in the tube across the inflow control surface is given by $\beta\rho d\forall$ and then we will calculate the total amount of extensive property which is entering the control volume. So, total amount of extensive property,

$$= -\beta\rho\Delta l \cos\theta dA$$

here we have substituted for $d\forall$ as $-\Delta l \cos\theta dA$.

Now, we need to get the total amount of extensive property. Total amount of extensive property will be the integral of this particular quantity. That is given by

$$= -\iint_I \beta\rho\Delta l \cos\theta dA \text{------------------------}(6)$$

Negative has come due to the angle, angle was $180-\theta$, $\cos(180-\theta) = -\cos\theta$.

So, this is across the control surface represented by region I, inflow control surface. This is the net amount of extensive property through the entire control surface in region I. So, we got the expression corresponding to control surface representing region I and also region III i.e., the inflow region and also outflow region. Now, we can substitute corresponding to term II.

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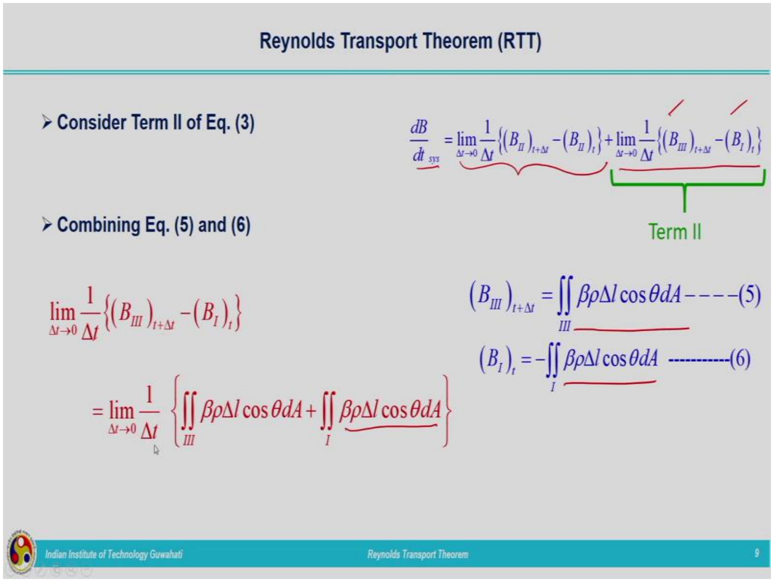

So, we are going back to our Reynolds transport theorem expression which we have derived for Reynolds transport theorem, time rate of change of extensive property within the system is related to some property related to control volume and we are having the term II here.

$$\frac{dB}{dt}_{Sys} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\{(B_{II})_{t+\Delta t} - (B_{II})_t\} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\{(B_{III})_{t+\Delta t} - (B_I)_t\}$$

So, the expressions corresponding to $(B_{III})_{t+\Delta t}$ and $(B_I)_t$, we have already found out. So, now we will substitute in term II that is the expression for

$$\left(B_{III}\right)_{t+\Delta t}=\iint_{III}\beta\rho\Delta l\cos\theta dA\text{------------------------------}(5)$$

$$\left(B_{t}\right)_{t}=-\iint_{I}\beta\rho\Delta l\cos\theta dA\text{----------------------------------}(6)$$

we will substitute in term II. So, term II is

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left(B_{III}\right)_{t+\Delta t}-\left(B_{I}\right)_{t}\right\}$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iint_{III}\beta\rho\Delta l\cos\theta dA+\iint_{I}\beta\rho\Delta l\cos\theta dA\right\}$$

We got the expression for the term II in the RTT. So, here you can see $\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iint_{III}\beta\rho\Delta l\cos\theta dA+\iint_{I}\beta\rho\Delta l\cos\theta dA\right\}$. This sign was negative here, it has become positive because we got the value corresponding to inflow region to be negative, negative and one negative sign here, together it will be positive. Now, we need to modify this expression because we are having the $\lim_{\Delta t\to 0}\frac{1}{\Delta t}$.

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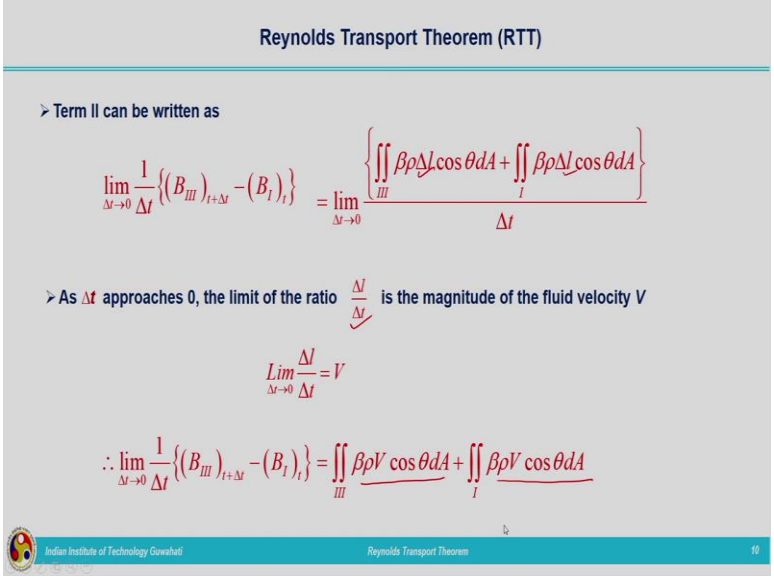

In the next step, we will do that. So, term II is this and we have written the expression like this.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\left(B_{III}\right)_{t+\Delta t} - \left(B_I\right)_t\right\} = \lim_{\Delta t \to 0} \frac{\left\{\iint_{III} \beta\rho\Delta l \cos\theta dA + \iint_{I} \beta\rho\Delta l \cos\theta dA\right\}}{\Delta t}$$

Now, you look at the numerator, we are having Δl and denominator Δt. So, limit $\Delta t \to 0$, $\frac{\Delta l}{\Delta t}$, what is it? Length divided by time and at the same time $\Delta t \to 0$, it is nothing but the magnitude of the velocity (*V*). So, we can write.

$$\operatorname*{Lim}_{\Delta t \to 0} \frac{\Delta l}{\Delta t} = V$$

So, this expression we are modifying, limit has gone. Now, we are having the term II as

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\left(B_{III}\right)_{t+\Delta t} - \left(B_I\right)_t\right\} = \iint_{III} \beta\rho V \cos\theta dA + \iint_{I} \beta\rho V \cos\theta dA$$

Both the expressions are same within the integral, only the regions are different. So, we can express this term as a net outflow that is some amount of fluid is entering the control volume and some amount is leaving the control volume. So, the summation of that, that was actually negative, minus was there, we were subtracting because of the negative sign came for the inflow value it has become positive. So, if we are summing up these two, we are getting the value corresponding to that is the extensive property which is net outflow of extensive property that is represented by this term II.

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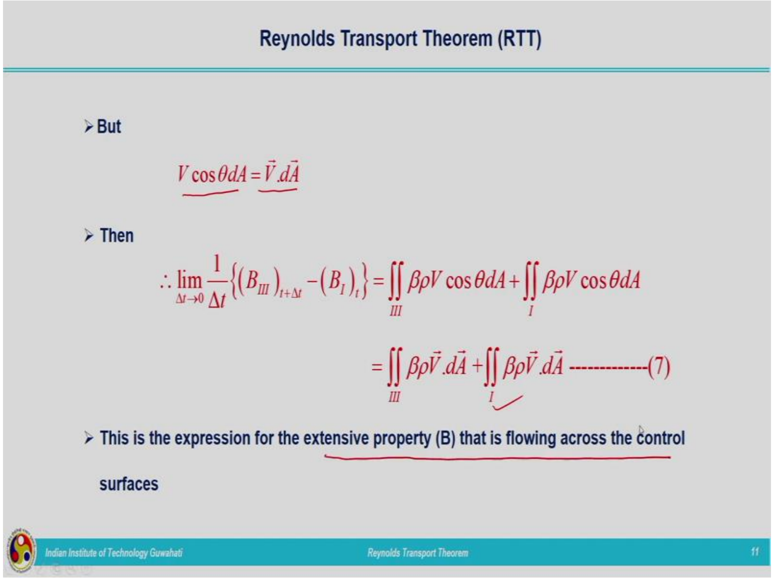

Now, we can modify that particular expression again, $V\cos\theta dA$ can be written as the dot product of these two vectors

$$V\cos\theta dA = \vec{V}.d\vec{A}$$

We know, area is a vector, velocity is a vector and $\cos\theta$ is there. So, $V\cos\theta dA$ can be written as $\vec{V}.d\vec{A}$. So, this we will substitute in the previous expression, it will be modified like this, i.e.,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{(B_{III})_{t+\Delta t}-(B_I)_t\} = \iint_{III}\beta\rho V\cos\theta dA + \iint_{I}\beta\rho V\cos\theta dA$$

$$= \iint_{III}\beta\rho\vec{V}.d\vec{A} + \iint_{I}\beta\rho\vec{V}.d\vec{A} \text{-----------------------------------}(7)$$

This is the expression for the extensive property that is flowing across the control surfaces.

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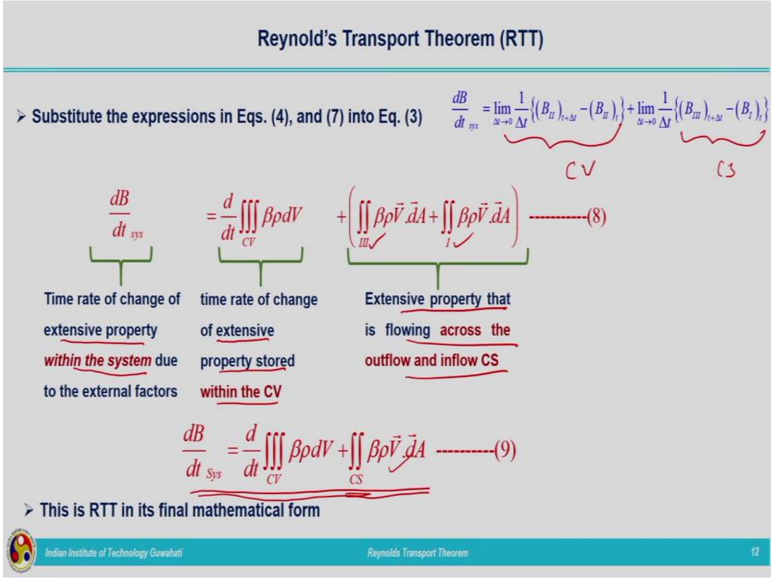

Now, what we have to do? We are having the expression for term I and also term II, that we will combine together, we will put it in equation 3, i.e., $\left.\frac{dB}{dt}\right|_{Sys}$ is equal to, this is corresponding to control volume and this is corresponding to control surface. We are having separate expressions for these two terms that we are going to substitute in equation 3 i.e.,

$$\left.\frac{dB}{dt}\right|_{Sys} = \frac{d}{dt}\iiint_{CV} \beta\rho dV + \left(\iint_{III} \beta\rho \vec{V}.d\vec{A} + \iint_{I} \beta\rho \vec{V}.d\vec{A} \right) \text{---(8)}$$

So, this is the $\left.\frac{dB}{dt}\right|_{Sys}$. It is the time rate of change of extensive property within the system due to some external factor that we do not know what is creating this change. And first term on the right-hand side is the time rate of change of extensive properties stored within the control volume. And last term, these two terms together which are representing the flow of extensive property across the control surface i.e., from the outflow and inflow region. These two can be combined together, I am removing the region III, region I notation and together I am putting across the control surface.

$$\left.\frac{dB}{dt}\right|_{Sys} = \frac{d}{dt}\iiint_{CV} \beta\rho dV + \iint_{CS} \beta\rho \vec{V}.d\vec{A} \text{---(9)}$$

So, you can write the expression for Reynolds transport theorem like this. Time rate of change of extensive property of system is equal to time rate of change of extensive property stored within the control volume plus extensive property that is flowing across the outflow and inflow region. So, this is the mathematical form of Reynolds transport theorem.

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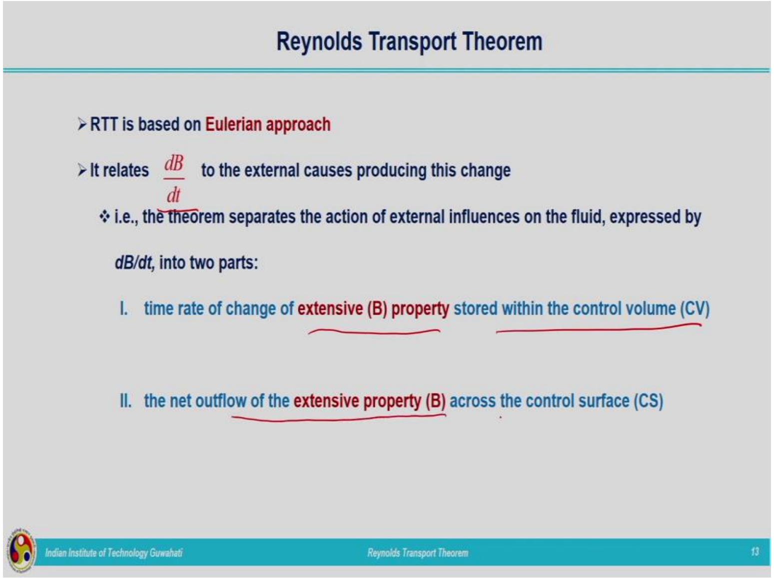

Now, we need to look at this equation for understanding the relationship. Initially itself I told you, what Reynolds transport theorem is doing? Fundamental laws are derived based on Lagrangian approach. So, we are tracing a single particle, individual particle for understanding the flow characteristics.

But majority of the engineering problems, it will not be feasible. So, that case, here what we are doing, what RTT is doing? RTT is trying to relate the quantities which we got based on the Lagrangian approach to that of the one using Eulerian approach.

So, it relates the time rate of change of $\frac{dB}{dt}$ to the external causes producing this, we got the expression for this in mathematical form i.e., it has 2 parts, one is the time rate of change of extensive properties stored within the control volume. And second part is the net outflow of the extensive property across the control surface.

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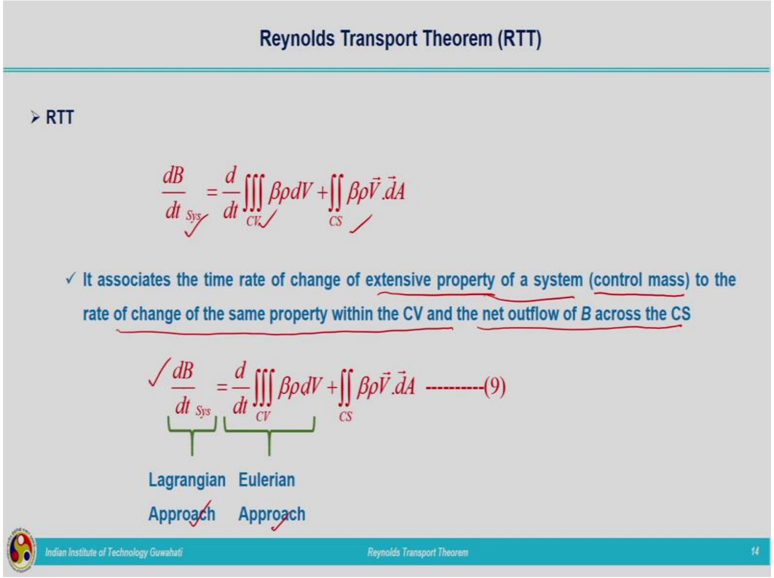

I am rewriting it again.

$$\left.\frac{dB}{dt}\right|_{Sys} = \frac{d}{dt}\iiint_{CV} \beta\rho dV + \iint_{CS} \beta\rho \vec{V}.d\vec{A}$$

So, left hand side this associate, the time rate of change of extensive property of a system, you look at the left-hand side it is something related to system. System concept means it is based on the Lagrangian approach and look at the right-hand side. Right hand side, we are having something related to control volume. Control volume approach is based on the Eulerian approach. So, it associates the time rate of change of extensive property of a system that is for a system it is the control mass to the rate of change of same property within the control volume and the net outflow of extensive property across the control surface.

So, you can see

$$\underbrace{\left.\frac{dB}{dt}\right|_{Sys}}_{\text{Lagrangian Approach}} = \underbrace{\frac{d}{dt}\iiint_{CV} \beta\rho dV}_{\text{Eulerian Approach}} + \iint_{CS} \beta\rho \vec{V}.d\vec{A}$$

So, here you can see, we have related we have considered the system. What are the changes taking place in the extensive property as time changes from t to $t+\Delta t$ i.e., $\frac{dB}{dt}$. So, based on this Lagrangian approach, we are having the fundamental laws and we have found out the relationship between the Lagrangian approach and the Eulerian approach. So, we can make use of the fundamental laws based on Lagrangian approach for engineering problems which we deal by using the Eulerian approach.

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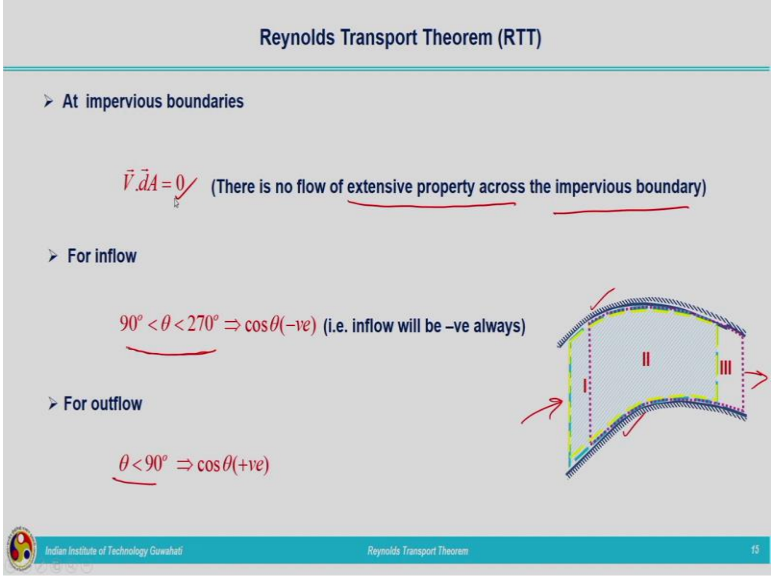

Now, some properties we need to understand related to this particular theorem i.e., we are having the control volume, control surface. Term II is representing the flow across the control surface. So, we need to understand, how the control surface is behaving. You can look into our figure that is given by this figure, the figure which we have started with.

So, here you can see this region and this region both are impervious boundaries, we were having the flow from left hand side to right hand side through this inflow and outflow regions. So, at impervious boundaries when we take $\vec{V}.d\vec{A} = 0$, i.e., there is no flow of extensive property across the impervious boundary, definitely we know. That is if it is impervious, it will not permit any flow to take place through that boundary.

Now, for inflow and outflow region, we have seen that for inflow region, it will be between 90 and 270. So, $\cos\theta$ will be negative and in the case of outflow region, θ is less than 90^0, so, it will be positive. So, depending on the flow direction, the angle will be different. For this particular case, I have written flow direction is left hand side to right hand side, if it is the case of a river from upstream to downstream. In the case of hydraulics channel, open channel cases it is from upstream to downstream. So, in that case if you are considering for the inflow region, they are always the area when we consider, area vector will be pointing in the outward normal direction, flow is in the direction from upstream to downstream. So, for the inflow region, it will be angle will be between 90 and 270. And in the case of outflow region it will be less than 90.

According to the angle, what is the angle corresponding to your particular problem based on that that particular term will be positive or negative. So, here we got the inflow region to have a negative sign and outflow to be positive. And in the case of impervious boundaries, this thing should be very clear to you, $\vec{V}.d\vec{A} = 0$.

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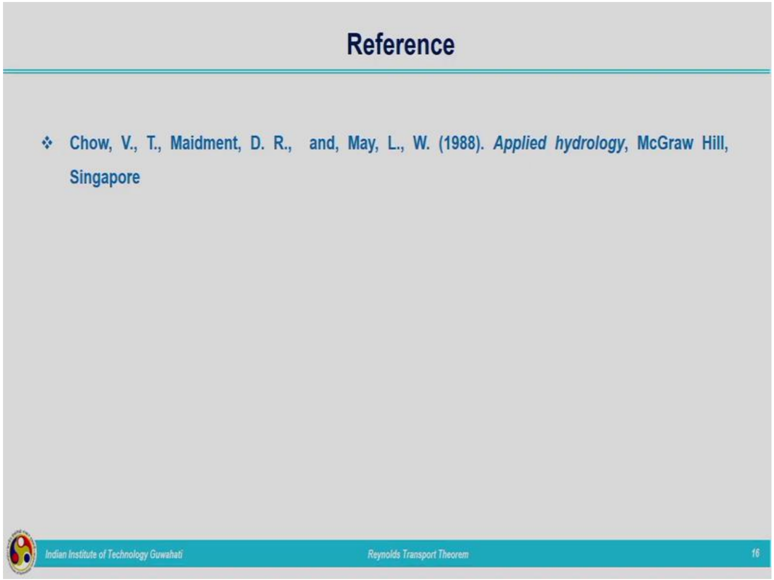

So, this particular theorem, derivation basics related to this, all this I have taken from the textbook of applied hydrology by Ven Te Chow and others. So, we have seen the detailed description about the Reynolds transport theorem in these three lectures.

We have started with the introduction of Reynolds transport theorem. In that we have seen the basic fluid properties which depends on mass and which are independent of mass that is extensive and intensive properties. Based on that, we have started the Reynolds transport theorem. And it related, in that we have seen both the approaches Lagrangian approach and also Eulerian approach. Lagrangian approach, we are considering a control mass. Eulerian approach, we are considering the control volume.

So, once we derive the expression we found the relationship between the time rate of change of extensive property within the system to that of control volume. So, this is a mathematical expression we have seen. This theorem can be utilized for deriving the basic laws based on Eulerian approach.

Actually, what we are doing? We are relating this thing to the control mass, Lagrangian concepts. So, based on this RTT derivation for the equation which we got, we will derive the basic conservation laws based on Eulerian approach. Here, I am stopping for now. Thank you. Have a nice day.