Engineering Hydrology

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Lecture 39

Infiltration Green-Ampt Equation

Hello, all. Welcome back. In the couple of previous lectures, we were discussing about subsurface water and infiltration process. Now, we know what is meant by actual infiltration, potential infiltration, also we have seen the well-known equation that is termed as Richard's equation, which describes the flow through the unsaturated porous medium. Today, we are going to see another equation which can be utilized for finding out infiltration rate and cumulative infiltration, that is, Green-Ampt equation.

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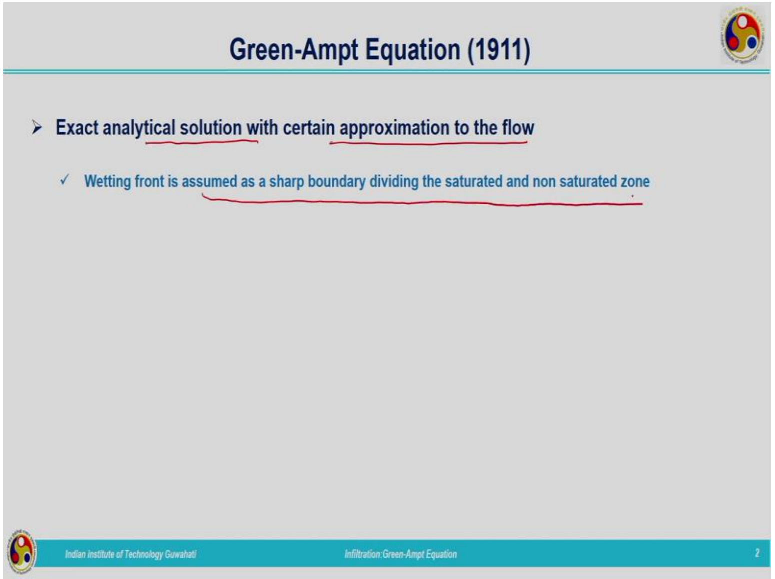

Green-Ampt equation is derived in the year of 1911. So, this provides an exact analytical solution with certain approximation to the flow. Some of the approximations are made related to flow through unsaturated zone, and then this equation has been derived. This is derived based on the fundamental principles, that is, based on the continuity and momentum equations.

First, we will derive the continuity equation, and then momentum equation. After that, combining these two fundamental equations, Green-Ampt equation is derived. So, in this, wetting front is assumed as a sharp boundary dividing the saturated and unsaturated zone. While we were discussing about the soil moisture profile, we have seen different zones beneath the ground surface. It has been divided into four zones.

So, in this case, we are not considering different zones. We are considering two zones only, that is, the unsaturated zone and the saturated zone. Saturated zone is above the wetting front and unsaturated zone is with an initial moisture content, which is coming below the wetting front. So other zones, which are transmission zone, transition zone, these zones are not taken into account.

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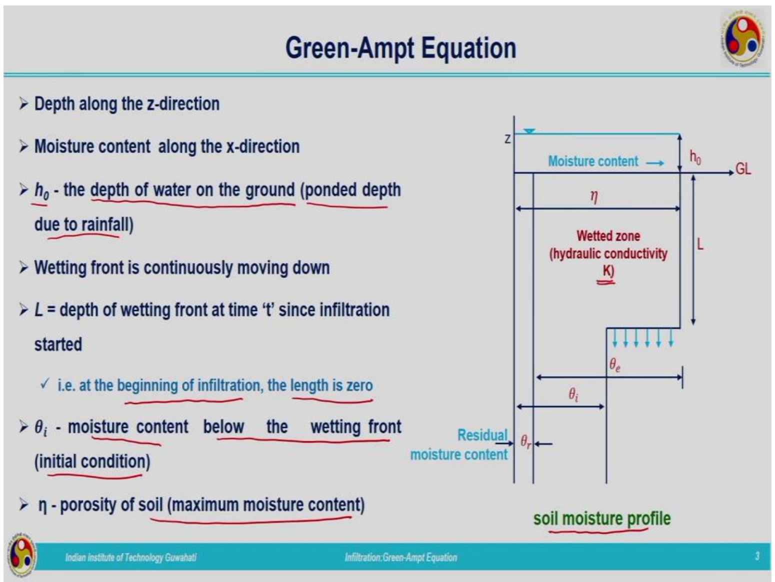

So, before starting the derivation, we need to have the idea about different parameters considered. So first let us have a look into the soil moisture profile. This is the ground level, and along the vertical axis, that is, the z-axis we are marking the depth, depth beneath the ground surface. Then along the x-direction we are marking the moisture content.

So, similar to the soil moisture profile, which we have discussed in one of the lectures, we are considering the soil moisture profile here also, but only difference is that we are not considering all the four zones, we are having the saturated zone and the unsaturated zone in this case. So, moisture content is marked along the x-axis, and here in the derivation of Green-Ampt equation, we are considering the depth of water on the ground, that is, ponded depth due to rainfall as h_0.

So, this equation is for ponded condition, that means, this equation is giving you the potential infiltration. What is potential infiltration? If sufficient quantity of water is available for infiltration, that leads to the potential infiltration, that is, maximum infiltration which can happen within a soil at a given point of time. So, when ponding depth is there, means sufficient amount of water is available for infiltration.

Now, the sharp wetting front is continuously moving in the downward direction as the rainfall is infiltrating into the ground. The wetting front is considered to have a depth of L at a given time t. Now, in the beginning of infiltration, the length is 0, and as the water infiltrates into the ground, at time t, we can consider the length of wetting front, or the depth of wetting front to be L.

Now, θ_i is the moisture content below the wetting front (that is the initial moisture condition or at time t = 0 when there was no infiltration). Once the rainfall started, and the infiltration process started, water is infiltrating into the ground and the moisture content changes to the moisture content at that particular time t. So, at time t, we are considering wetting front has reached a depth of L, and above that we are having the wetted soil, and below that we are having the dry soil which is at the initial condition.

And porosity of the soil, porosity is the maximum moisture content. Soil can hold maximum quantity of water up to porosity of the soil. That is represented by η.

Now, another term you should be familiar is θ_r, that is, the residual moisture content. Residual moisture content is the moisture content which is present in the soil, and cannot be removed by drying in ambient conditions.

Beyond θ_r, how much pore space is available for moisture to be stored, that is, represented by effective moisture content θ_e. And above the wetting front, we are having the wetted zone, which is having hydraulic conductivity K, and below that, we are having the dry soil.

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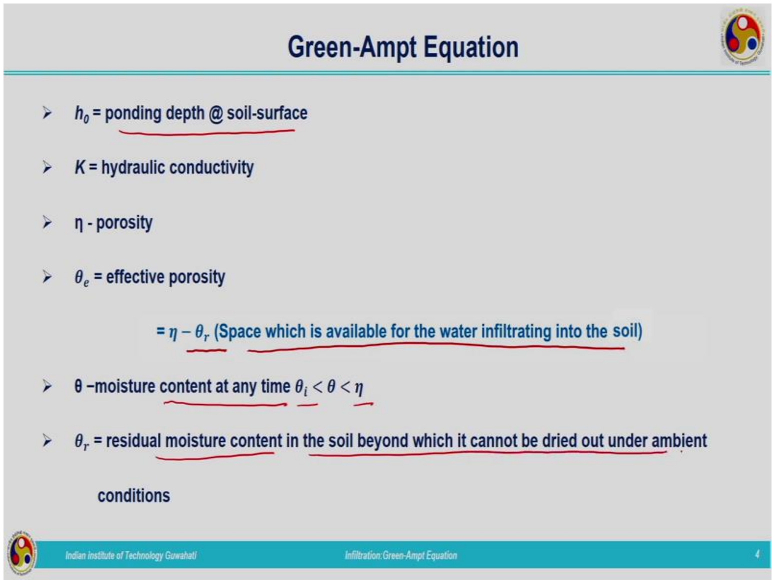

We can see different parameters.

- h_0 = ponding depth at soil surface
- K = hydraulic conductivity
- η = porosity
- θ_e= effective porosity
- θ = moisture content at any time $\theta_i < \theta < \eta$
- θ_r= residual moisture content in the soil beyond which it cannot be dried out under ambient conditions.

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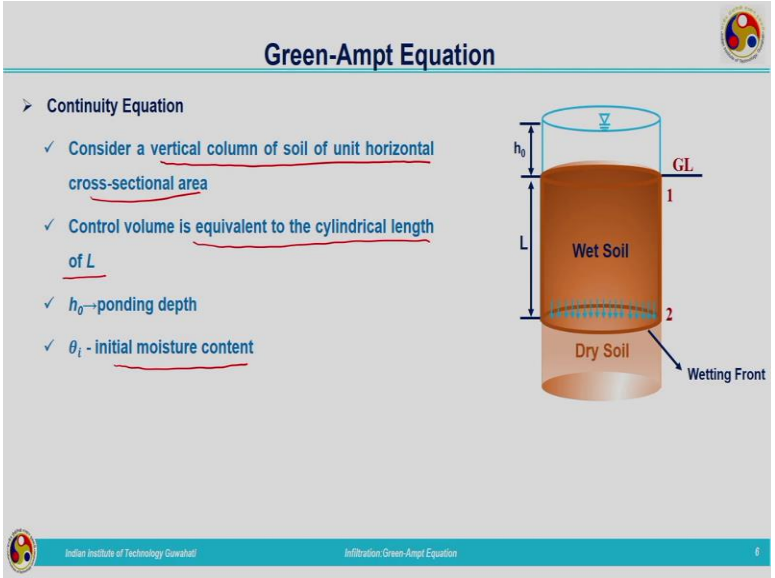

Now we are going to derive the fundamental equations, that is, the mass conservation equation and the momentum conservation equation based on control volume approach. So, for that, we need to consider the control volume. So, take a control volume, below the ground surface within the depth of infiltration water.

Water is getting infiltrated into the ground, how the soil moisture profile is approximated for the derivation of Green-Ampt equation, we have seen already. So, within that depth of infiltrated water we are considering the control volume. Green-Ampt equation will be derived as a combination of continuity equation and momentum equation.

Continuity equation

Consider a cylindrical vertical column of soil of unit horizontal cross-sectional area (A) as the control volume.

So, this is our ground level. Control volume is equivalent to the cylindrical length or depth of L which is considered between two sections, 1 and 2. And this equation is derived based on the assumption that potential infiltration is taking place that is sufficient quantity of water is available for the infiltration process. So, there is a ponding depth (h_0), or water is getting ponded above the ground surface. And, θ_i is the initial moisture content,

Now, water is infiltrating into the ground, and as it infiltrates, the soil becomes wet. And we have considered a time the wetting front has reached at the Section 2. So above this wetting front, we are having the wet soil and below that we are having the dry soil which is having the initial moisture content of θ_i. So, this will be infiltrating into the ground, again and again, deeper as time proceeds.

So, this is one assumption; sharp wetting front we are considering, that wetting front is separating dry soil and wet soil. So above the wetting front we are considering completely saturated zone. And beneath that, it is unsaturated.

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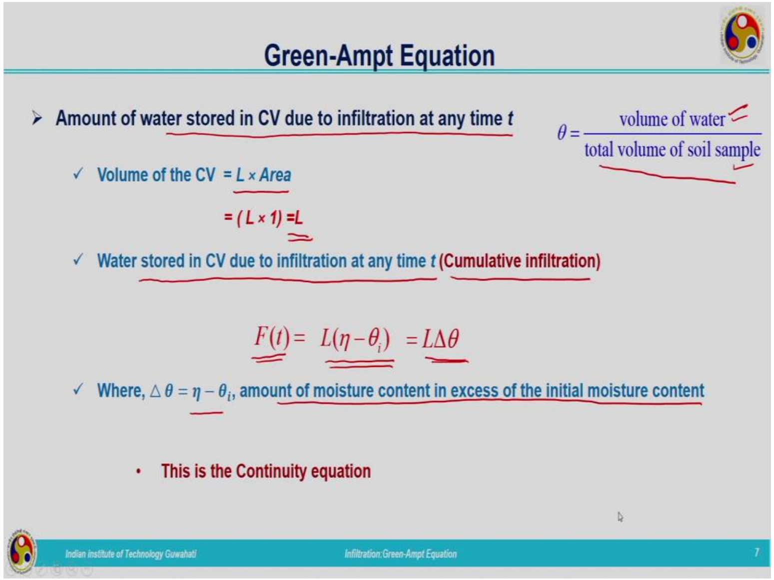

Now, we are going to derive the continuity equation, that is, the mass conservation equation.

<u>Amount of water stored in CV due to infiltration at any time *t*</u>

Volume of the CV = L × Area = L × 1 = L

<u>The water stored in control volume due to infiltration at any time *t* (Cumulative infiltration)</u>

We now already, $\theta = \dfrac{\text{volume of water}}{\text{total volume of soil sample}} = \dfrac{\text{volume of water}}{L}$

$$\Rightarrow \text{volume of water} = \theta \times L$$

$$\Rightarrow F(t) = \theta \times L$$

So, $F(t) = L(\eta - \theta_i)$

Why (η - θ_i)?

So, you look at this expression, that is, for volumetric moisture content, that is, volume of water divided by total volume of soil sample. We have assumed that at time t, the wetting front has reached a depth of L. Above the wetting front we are having the saturated zone. Saturated zone means our moisture content has reached the porosity value, maximum value. But initially itself, the soil was having some initial moisture content θ_i. So how much extra is added, moisture content change has taken place from θ_i to η.

So, within this interval of time the quantity of water which is infiltrated represents a moisture content of η - θ_i. So, η - θ_i multiplied by the volume of soil sample will be giving you the water stored in the control volume within time t. So that is what is represented by equation $F(t)$, that is equal to $L \times (\eta - \theta_i)$. That can be written as $L\Delta\theta$.

$$F(t) = L(\eta - \theta_i) = L\Delta\theta$$

$\Delta\theta = \eta - \theta_i$. It is the amount of moisture content in excess of initial moisture content.

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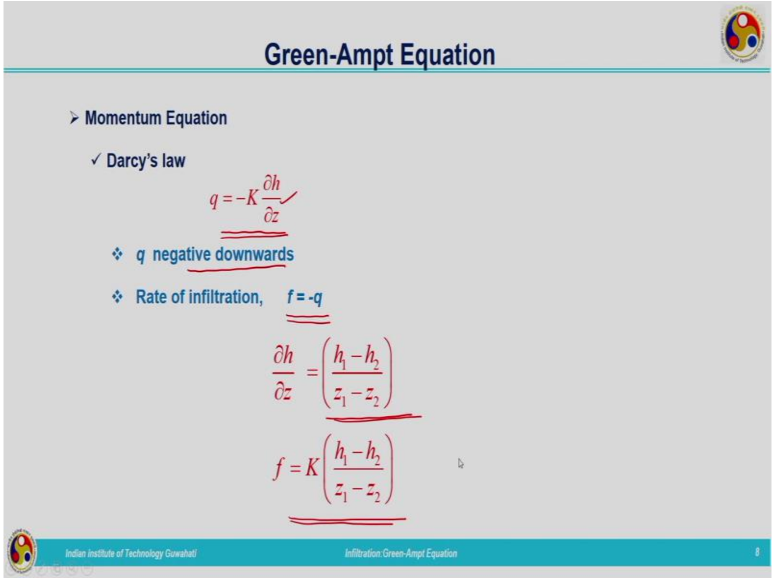

<u>Momentum equation</u>

Momentum equation is given by the Darcy's Law. That is given by

$$q = -K\frac{\partial h}{\partial z}$$

So here, we do not have to derive the momentum equation. We already have the equation. Only thing required is corresponding to ∂h and ∂z, we need to substitute here values based on the control volume which we have considered.

q (Darcy's flux) is negative downwards (upward z-direction is considered as positive and downward z-direction is negative). So, infiltration is the process which is taking place in the downward direction, so definitely it will be having negative sign.

So, rate of infiltration, $f = -q$

Now the expression for

$$\frac{\partial h}{\partial z} = \left(\frac{h_1 - h_2}{z_1 - z_2}\right)$$

Darcy's equation is then rewritten and it takes the form

$$f = K\left(\frac{h_1 - h_2}{z_1 - z_2}\right)$$

This is the equation for infiltration rate. So, we got the infiltration rate equation from the Darcy's law.

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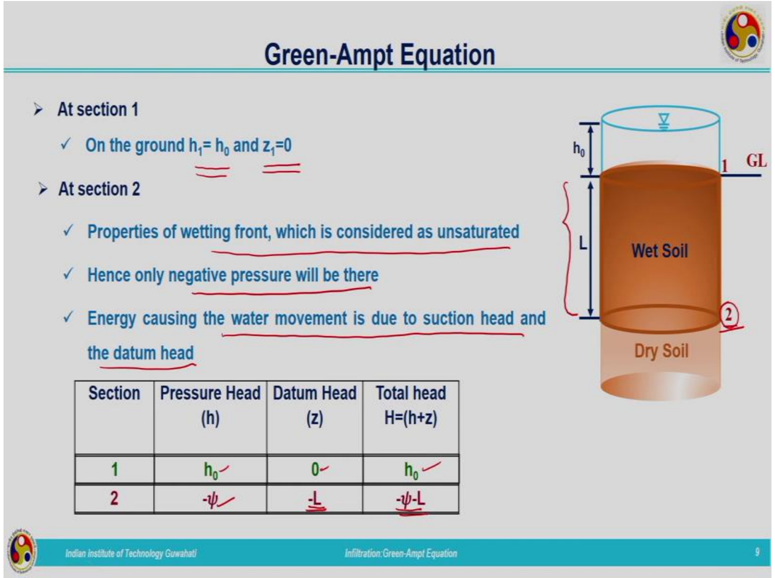

Now consider the control volume.

<u>At Section 1 (Ground surface)</u>

$h_1 = h_0$ and $z_1 = 0$ (datum head)

So that we can tabulate, at Section 1, we are having pressure head to be h_0, and datum head as $z = 0$, and total head will be h_0.

<u>At Section 2 (Wetting front)</u>

Wetting front is the sharp boundary which is separating the unsaturated and saturated region. So, at the wetting front, the properties which we are considering is that of the unsaturated region.

Wetting front is considered as unsaturated. So only negative pressure will be there. In the unsaturated zone the head causing the flow or the main component is the suction head, that is the negative pressure head. So, the suction head and the datum head, these two are the two components of head causing the flow within the unsaturated region. So now we need to write expressions for this suction head and the datum head at Section 2.

So, at Section 2, energy causing the water movement is due to suction head and datum head. Suction head can be represented by $-\Psi$, and datum head by $-L$.

So, total head = $-\Psi - L$.

So, we got the expressions corresponding to h_1, h_2, z_1, z_2, head causing the flow at Section 1, head causing the flow at Section 2, and corresponding to two sections 1 and 2, what is the datum.

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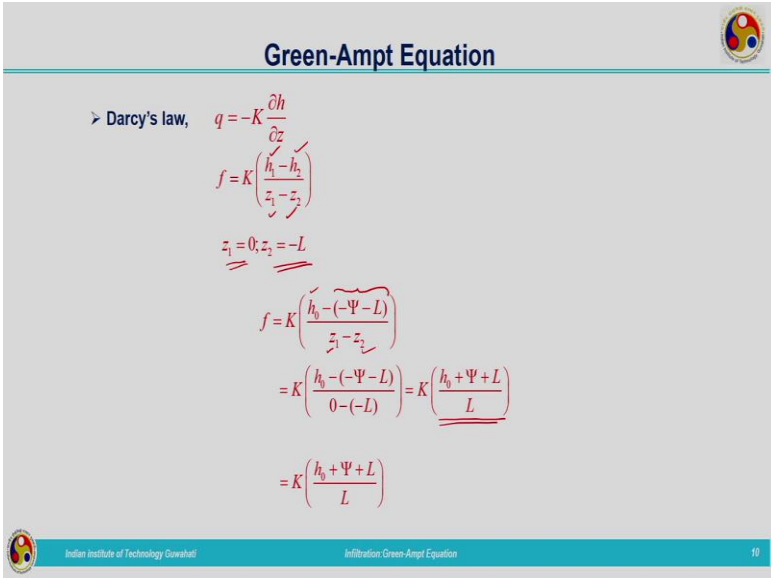

Now, we are going to Darcy's Law, and substitute these values.

$$q = -K\frac{\partial h}{\partial z}$$

$$f = K\left(\frac{h_1 - h_2}{z_1 - z_2}\right)$$

$$z_1 = 0; z_2 = -L$$

So, once we substitute in this particular equation, equation will be

$$f = K\left(\frac{h_0 - (-\Psi - L)}{z_1 - z_2}\right)$$

$$\Rightarrow f = K\left(\frac{h_0 - (-\Psi - L)}{0 - (-L)}\right) = K\left(\frac{h_0 + \Psi + L}{L}\right)$$

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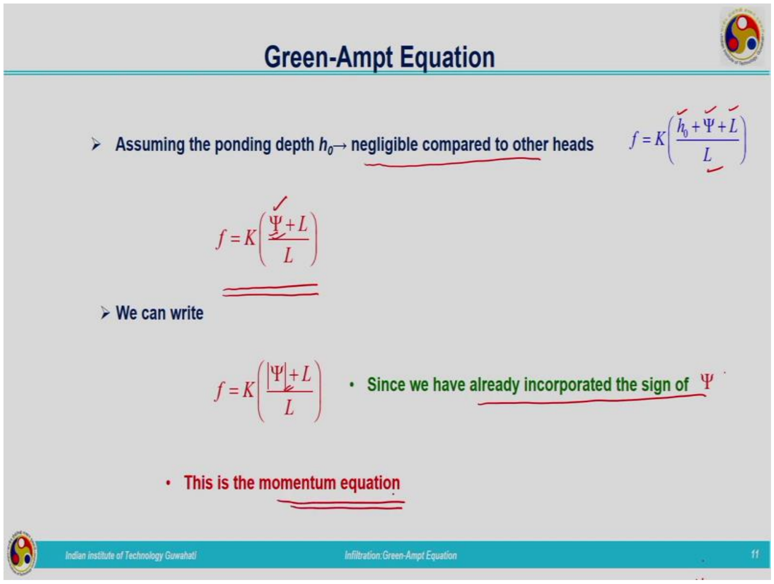

Now, we are going to assume, the ponding depth h_0 is negligible compared to Ψ and L. So, h_0 can be neglected in the equation.

So, the expression for infiltration rate or the momentum equation takes the form

$$f = K\left(\frac{\Psi + L}{L}\right)$$

Now, we can write the final expression

$$f = K\left(\frac{|\Psi| + L}{L}\right)$$

[Note- The Ψ has already incorporated the negative sign. So, if some numerical problem you are solving, suction head is given as some cm, so even though it is negative pressure head you do not have to consider the sign there because while deriving the Green-Ampt equation itself we have considered that sign].

This is the momentum equation.

So now, the fundamental equation, such as conservation of mass and conservation of momentum equations have been derived by making use of the control volume approach for the process infiltration.

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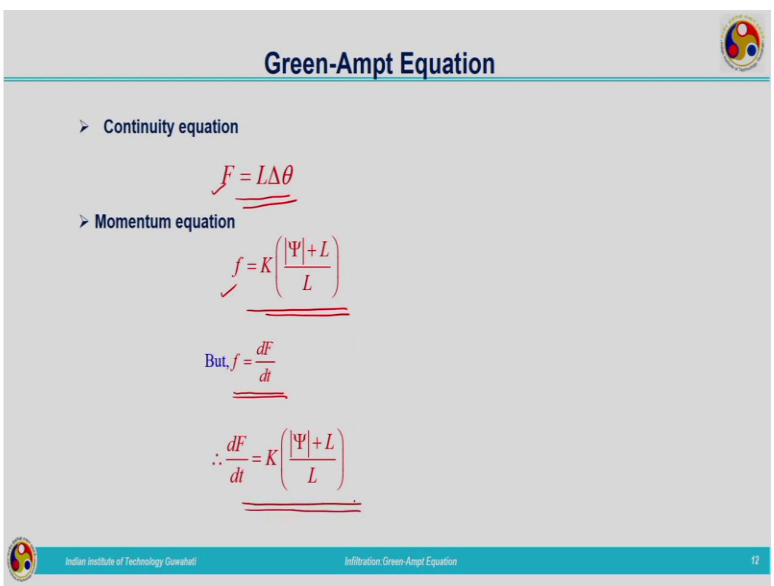

So now, continuity equation is

$$F = L\Delta\theta$$

and momentum equation is

$$f = K\left(\frac{|\Psi| + L}{L}\right)$$

So here we are with the expressions corresponding to cumulative infiltration and the infiltration rate.

But we are having knowledge that $f = \dfrac{dF}{dt}$

So, $\dfrac{dF}{dt} = K\left(\dfrac{|\Psi| + L}{L}\right)$

So, this equation we need to look into again.

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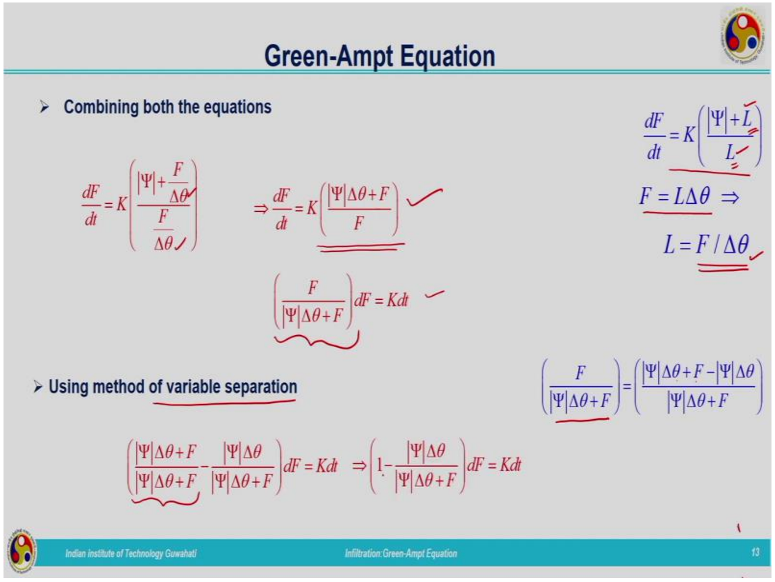

These are the two equations,

$$\frac{dF}{dt} = K\left(\frac{|\Psi| + L}{L}\right)$$

$$F = L\Delta\theta$$

Now, at a time *t*, how much is the depth of wetting front (*L*), we do not have any idea. So, what we will do, we will make use of the continuity equation and $L = F/\Delta\theta$

So, combining both the equations, we will get

$$\frac{dF}{dt} = K\left(\frac{|\Psi| + \frac{F}{\Delta\theta}}{\frac{F}{\Delta\theta}}\right)$$

$$\frac{dF}{dt} = K\left(\frac{|\Psi|\Delta\theta + F}{F}\right)$$

So, the expression changed to this form, and it's a differential equation. So, our intention is to get the expression for infiltration rate (*f*), and the cumulative infiltration (*F*).

$$\left(\frac{F}{|\Psi|\Delta\theta + F}\right)dF = Kdt$$

Now, using the method of variable-separation, we will rearrange this particular term for making the expression to be in the simple form. So,

$$\left(\frac{F}{|\Psi|\Delta\theta + F}\right) = \left(\frac{|\Psi|\Delta\theta + F - |\Psi|\Delta\theta}{|\Psi|\Delta\theta + F}\right)$$

So, after substituting it can be written as

$$\left(\frac{|\Psi|\Delta\theta + F}{|\Psi|\Delta\theta + F} - \frac{|\Psi|\Delta\theta}{|\Psi|\Delta\theta + F}\right)dF = Kdt$$

$$\Rightarrow \left(1-\frac{|\Psi|\Delta\theta}{|\Psi|\Delta\theta+F}\right)dF = Kdt$$

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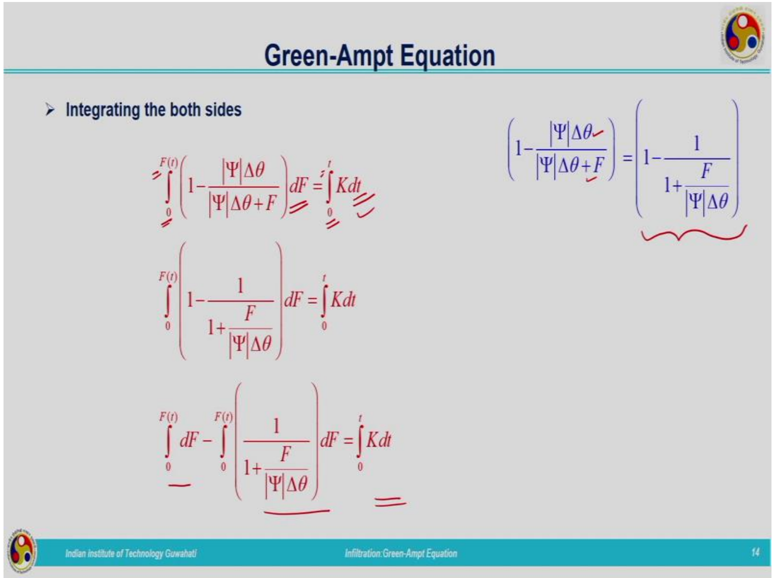

So, we are going to integrate the both sides of the equation.

$$\int_0^{F(t)}\left(1-\frac{|\Psi|\Delta\theta}{|\Psi|\Delta\theta+F}\right)dF = \int_0^t Kdt$$

$$\int_0^{F(t)}\left(1-\frac{1}{1+\frac{F}{|\Psi|\Delta\theta}}\right)dF = \int_0^t Kdt$$

$$\int_0^{F(t)} dF - \int_0^{F(t)}\left(\frac{1}{1+\frac{F}{|\Psi|\Delta\theta}}\right)dF = \int_0^t Kdt$$

And now we are having three terms, one term on the right-hand side and two terms on the left-hand side.

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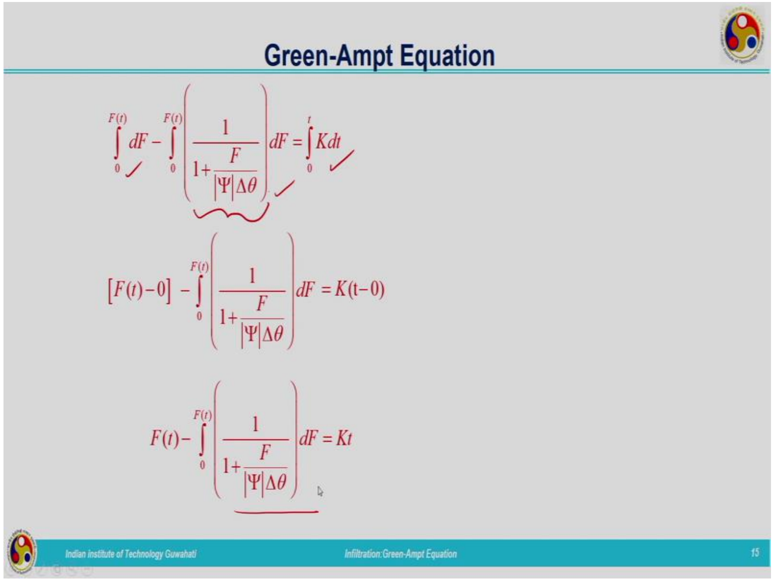

$$[F(t)-0] - \int_0^{F(t)} \left(\frac{1}{1+\frac{F}{|\Psi|\Delta\theta}} \right) dF = K(\mathrm{t}-0) ----------(1)$$

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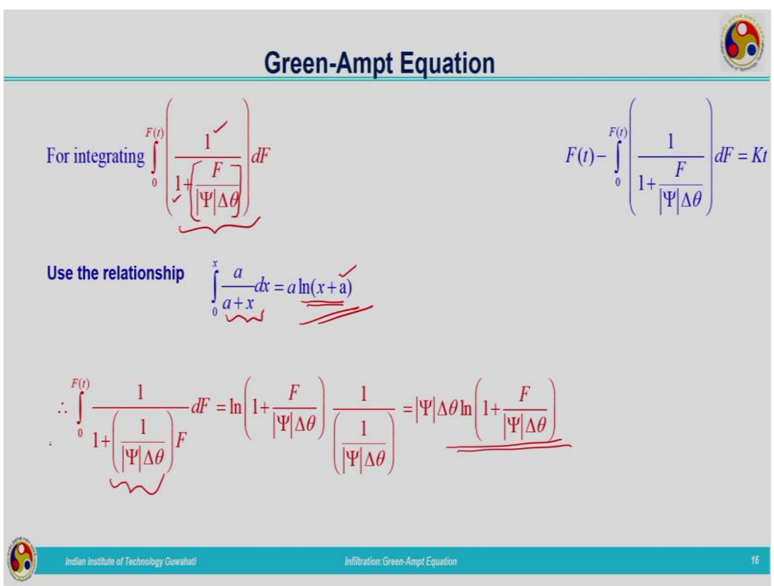

So, this is our expression. We are going to consider the second term separately. For integrating this, we are going back to our basic calculus and making use of one particular integral,

$$\int_0^x \frac{a}{a+x}dx = a\ln(x+\mathrm{a})$$

So, we will be making use of this particular relationship for finding out the solution of this integral. So that can be written as

$$\therefore \int_0^{F(t)} \frac{1}{1+\left(\frac{1}{|\Psi|\Delta\theta}\right)F}dF = \ln\left(1+\frac{F}{|\Psi|\Delta\theta}\right)\frac{1}{\left(\frac{1}{|\Psi|\Delta\theta}\right)} = |\Psi|\Delta\theta\ln\left(1+\frac{F}{|\Psi|\Delta\theta}\right)$$

So, this is the solution for this integral.

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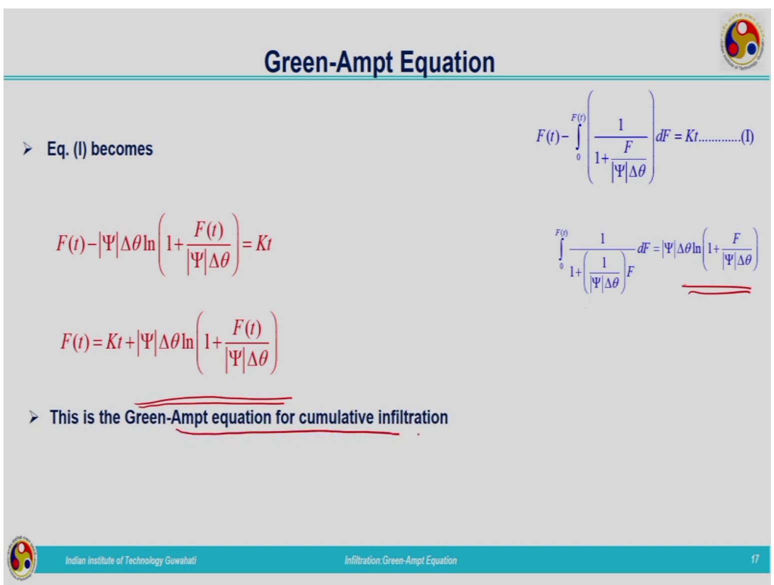

Now, we can substitute this particular solution of this integral in the Equation 1. After substituting, it will be taking the form,

$$F(t) - |\Psi|\Delta\theta \ln\left(1 + \frac{F(t)}{|\Psi|\Delta\theta}\right) = Kt$$

$$F(t) = Kt + |\Psi|\Delta\theta \ln\left(1 + \frac{F(t)}{|\Psi|\Delta\theta}\right)$$

This is the Green-Ampt Equation for cumulative infiltration.

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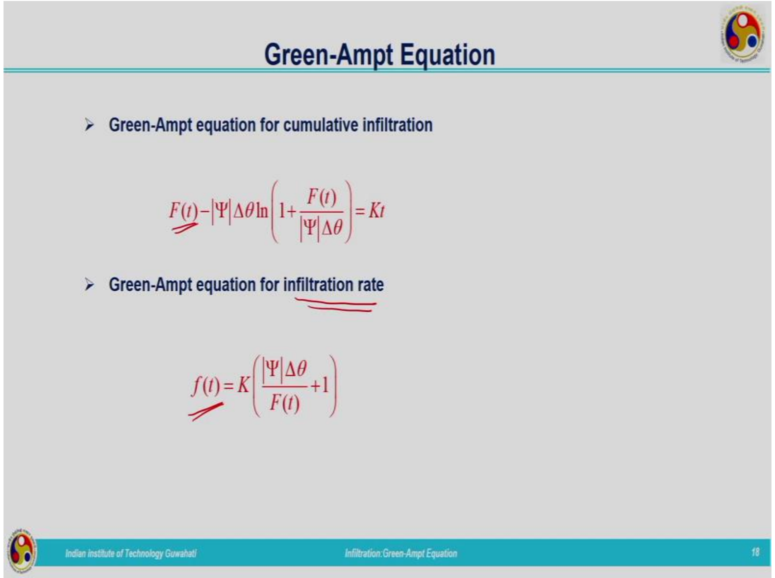

Green-Ampt equation for cumulative infiltration is given by

$$F(t) - |\Psi|\Delta\theta \ln\left(1 + \frac{F(t)}{|\Psi|\Delta\theta}\right) = Kt$$

and Green-Ampt equation for infiltration rate is given by

$$f(t) = K\left(\frac{|\Psi|\Delta\theta}{F(t)} + 1\right)$$

We have made use of the continuity and momentum equations, and momentum equation directly gave the equation corresponding to infiltration rate, and by making use of the momentum equation and continuity equation, we got the equation corresponding to cumulative infiltration.

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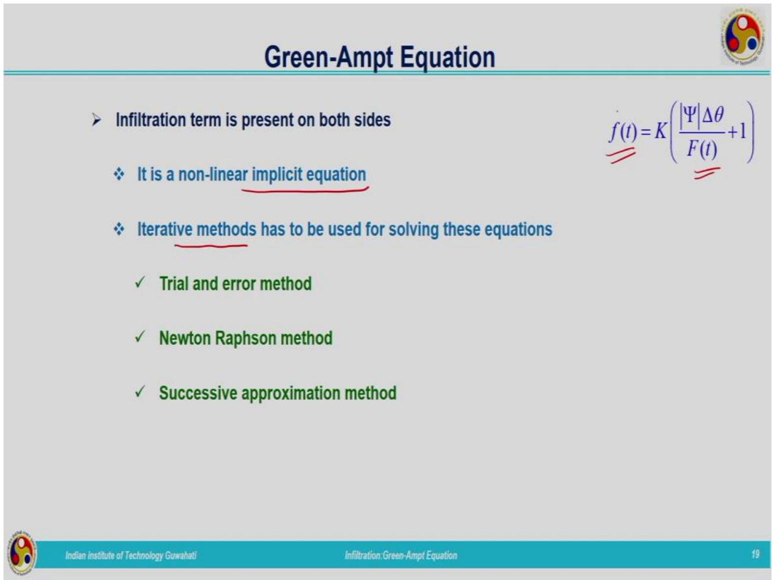

Now, in the expression for $f(t)$, $f(t)$ is there on the left-hand side, $F(t)$ is there on the right-hand side. So, infiltration is present on both the sides. Similarly, in the expression for $F(t)$, infiltration is present on both the sides. So, finding the solution of this equation is not that easy, so it is non-linear implicit equation.

For that, we may have to go for some of the iterative methods to solve these equations. So, some methods are trial and error method, Newton Raphson method and successive approximation method. Any of the methods for solving this type of implicit equation from mathematics can be utilized for getting the solution of these equations.

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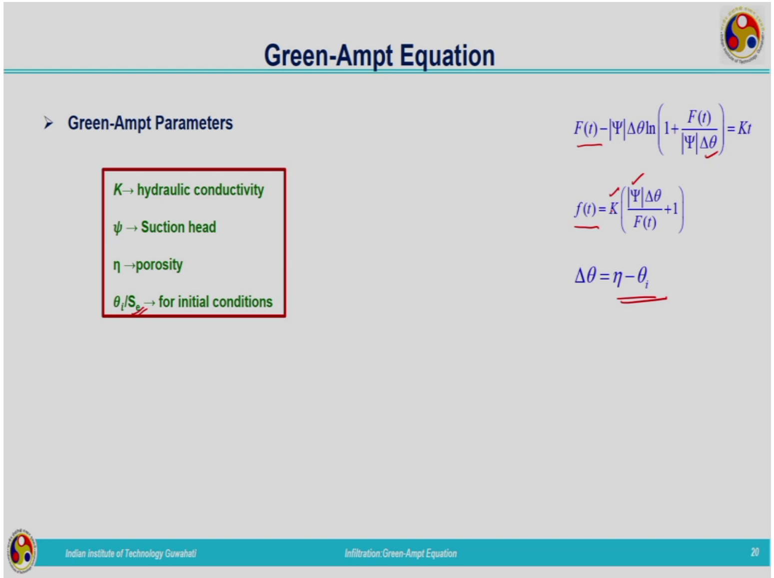

Now, coming to Green-Ampt parameters. Looking at the two equations, cumulative infiltration and infiltration rate equations, we can tabulate the Green-Ampt parameters.

- K→ hydraulic conductivity
- Ψ→ Suction head
- η →porosity
- θ_i/S_e → for initial conditions

Now, we need to understand what is S_e.

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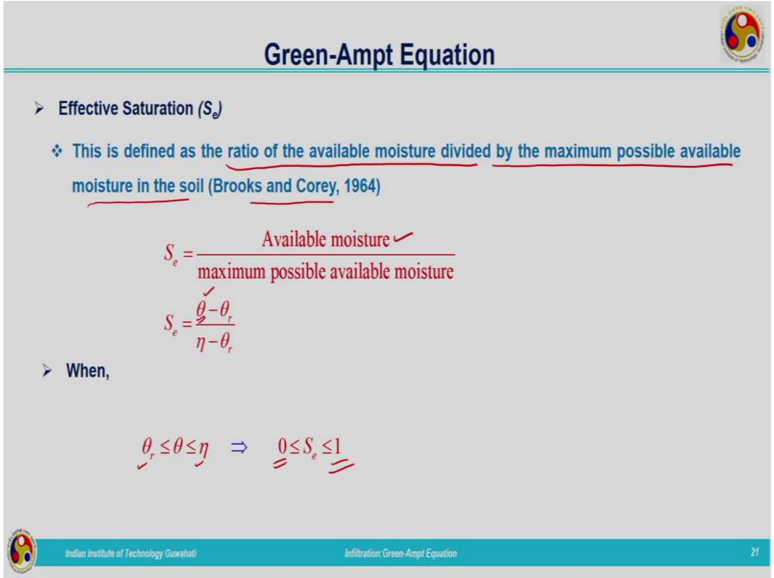

S_e is the effective saturation. It is assumed to be constant for a given instant of time. It is defined as the ratio of the available moisture divided by the maximum possible available moisture in the soil. So, this definition is given by Brooks and Corey in the year 1964. Even though, Green-Ampt equation was originally derived in the year of 1911, the effective saturation concept was taken from Brooks and Corey's theory and used later in Green-Ampt equation.

$$S_e = \frac{\text{Available moisture}}{\text{maximum possible available moisture}}$$

Available moisture content is the moisture content at that particular instant of time, maximum how much the soil can withhold. So that can be written as

$$S_e = \frac{\theta - \theta_r}{\eta - \theta_r}$$

Available moisture content, that is the, at a particular time instant moisture content is θ, and initially itself, we were having a moisture content or residual moisture content θ_r, so at the particular time instant, the available moisture content will be (θ - θ_r). And what will be the maximum possible available moisture content? It will be (η - θ_r). That is what is represented over here in the equation.

When, $\theta_r \leq \theta \leq \eta \qquad \Rightarrow 0 \leq S_e \leq 1$

So, S_e can vary between 0 and 1. When θ = η, S_e or saturation will be 100 %, all the pores are completely filled with water.

So now we need to have an idea about effective porosity θ_e.

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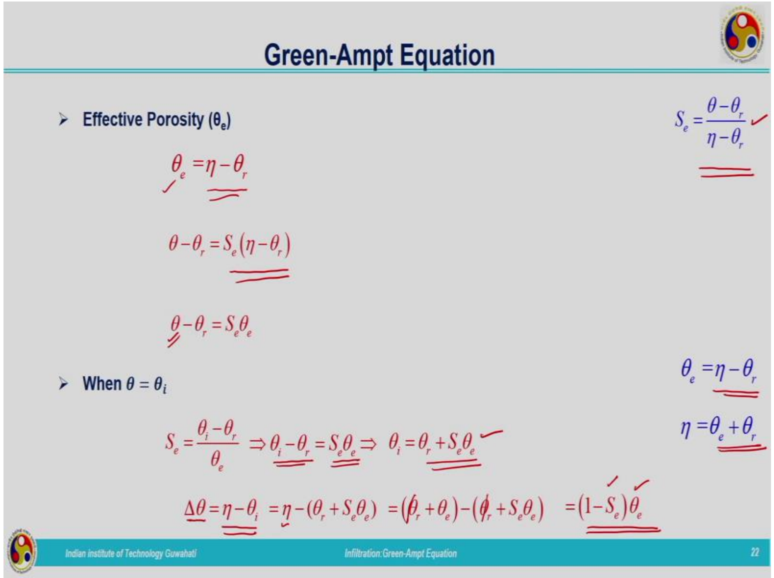

So, θ_e is given by $\theta_e = \eta - \theta_r$

$$\theta - \theta_r = S_e\left(\eta - \theta_r\right)$$

$$\theta - \theta_r = S_e \theta_e$$

Now, we are going to make use of the initial condition that is when $\theta = \theta_i$,

$$S_e = \frac{\theta_i - \theta_r}{\theta_e}$$

$$\Rightarrow \theta_i - \theta_r = S_e\theta_e \Rightarrow \theta_i = \theta_r + S_e\theta_e$$

Because for a soil, at a particular time, θ_r is known to us, effective saturation is a constant value, and effective porosity that is, η - θ_r, that also will be known to us. From that, we can calculate the initial moisture content. Otherwise, there are certain relationships between soil moisture content and the soil suction, that is, represented by soil hydraulic parameters.

So, I am not going deep into that particular topic. You will be learning that in soil mechanics or unsaturated flow related topics coming under geotechnical engineering. So, then you will be able to correlate these things. So here, I am just explaining that

$$\theta_i = \theta_r + S_e\theta_e$$

Now, we were having a term $\Delta\theta$, that is, the soil moisture deficit in the Green-Ampt equation.

$$\Delta\theta = \eta - \theta_i$$

$$\Delta\theta = \eta - \theta_i = \eta - (\theta_r + S_e\theta_e) = \left(\theta_r + \theta_e\right) - \left(\theta_r + S_e\theta_e\right)$$

Even though we don't know the residual moisture content, we can go ahead for finding out $\Delta\theta$. So, we want to discard this θ_r from this expression.

$$\Delta\theta = \left(1 - S_e\right)\theta_e$$

So, if these values are known to us, we can calculate how much is the soil moisture deficit, $\Delta\theta$.

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So here, I am winding up this lecture. The reference related to this particular topic is Applied Hydrology by Ven Te Chow. Thank you very much.