Engineering Hydrology
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Lecture 33
1-D Unsteady Unsaturated Flow Equation

Hello, all. Welcome back. In the previous lecture we have started with the Module 3, related to subsurface water. In that lecture, we have already seen the soil water related concepts. We have defined the pore space, and we have seen what is meant by porous media. And pore space is defined by means of θ, the moisture content, that is the volumetric moisture content, and the porosity, η.

So, we know what is the definition for θ and η, and we have seen the expression also. After that, we have seen driving energy within the porous media related to a flow in an unsaturated medium. And there, we have found that the predominant component is the suction head, compared to velocity head. So, total energy was consisting of the suction head and the datum head. In this lecture, we are going to see the equation for unsaturated flow.

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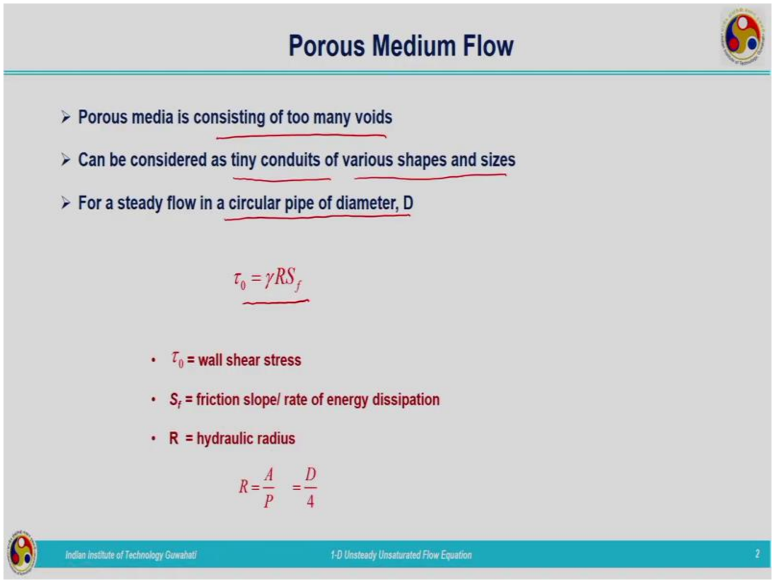

Before starting the derivation of unsaturated flow equation, we just need to have some more basics related to the flow through the porous media. Porous media is consisting of too many voids. We know already, in a porous media, if you are considering a sample, it is consisting of solid soils and also voids. Voids are filled with air and water.

So, this is consisting of too many voids, and these voids can be considered to be connected to each other. If that is the case, it can be considered as tiny conduits of various shapes and sizes. What we are trying to do here is that the flow through the porous media can be considered as a flow through a small diameter pipe.

So, for a steady flow in a circular pipe of diameter *D*, we know already that

$$\tau_0 = \gamma R S_f$$

In this, τ_0 is the wall shear stress, S_f is the friction slope or rate of energy dissipation, *R* is the hydraulic radius.

$$R = \frac{A}{P}$$

A is the wetted area and *P* is the wetted perimeter. So, these concepts are coming from the hydraulics. In a circular pipe, which is having diameter *D*, we can write the wall shear stress in terms of energy dissipation. Energy dissipation is represented by means of friction slope. This is based on the consideration of friction force into account. And we can get the expression for Wall shear stress (τ_0). So, in the case of pipe R can be written as

$$R = \frac{A}{P} = \frac{D}{4}$$

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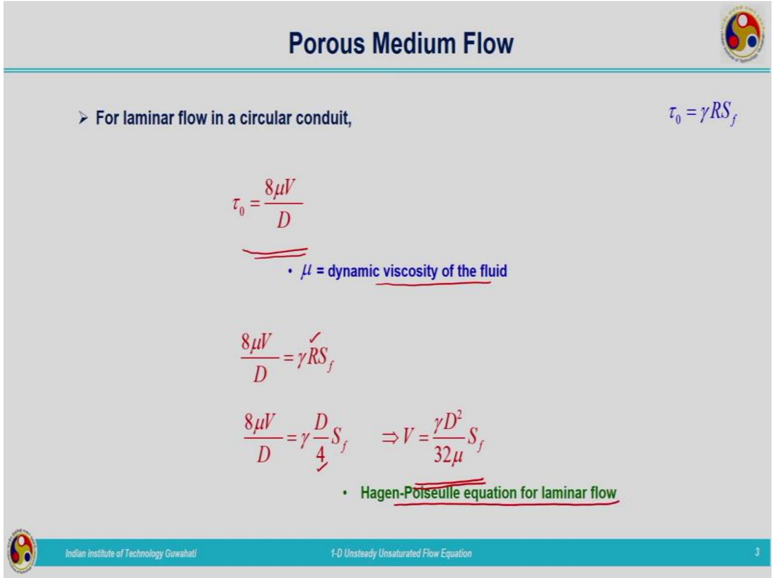

Now, in the case of laminar flow, we are having another expression for the wall shear stress, that is based on the laminar flow criteria. Depending on the Reynold's number we can classify the flows into laminar, turbulent and in transition. This we have studied in the fluid mechanics course. The same criteria we are taking here also. And in the case of pipe flow, which is having laminar flow, we can write

$$\tau_0 = \frac{8\mu V}{D}$$

In this, μ is the dynamic viscosity of the fluid.

So here we are having two expressions for wall shear stress. One which is seen in the previous slide, and the other one is based on the laminar flow criteria. So, these two can be equated.

$$\tau_0 = \frac{8\mu V}{D} = \gamma R S_f$$

$$\frac{8\mu V}{D} = \gamma \frac{D}{4} S_f \Rightarrow V = \frac{\gamma D^2}{32\mu} S_f$$

So, this is the expression which we can write for the velocity of flow through the pipe. But we know the expression for velocity. If the flow is steady, if we are having a discharge of Q passing through the pipe, we can get the velocity to be Q/A, discharge divided by area of cross section.

Here, by considering the flow characteristics, we got the expression $V = \frac{\gamma D^2}{32\mu} S_f$. This equation is termed as Hegen-Poiseulle equation for laminar flow. In detail, characteristics, all these things we have studied in fluid mechanics related to Hegen-Poiseulle flow. Same concepts, we are going to make use in the case of flow through porous media, assuming that the flow which is taking place through the tiny pores within the porous media can be considered as laminar flow. And the same concepts which is applicable to laminar flow in a pipe, closed conduit, is utilized here.

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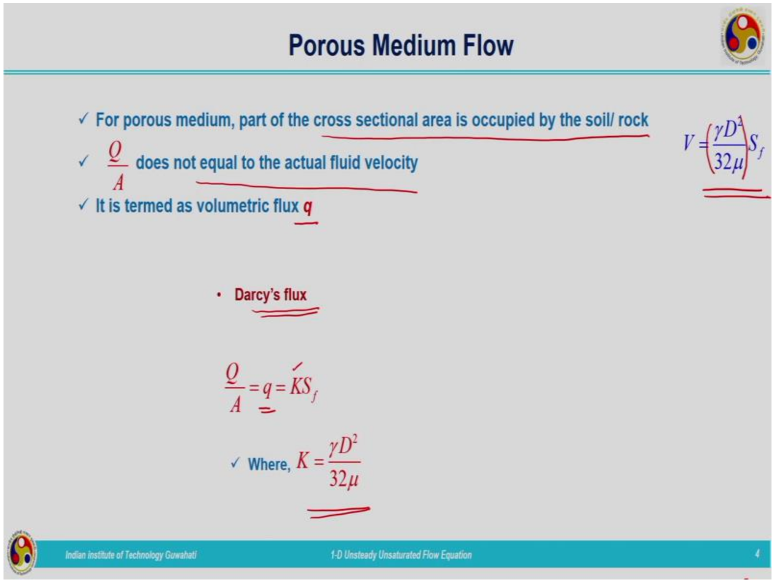

For porous medium, part of the cross-sectional area is occupied by soil or rock, but in the case of pipe flow entire cross section is occupied by the flow. But you consider a sample in the porous media, the sample is consisting of soil, that is, solids, and the voids, voids consist of air and water. So, part of the cross-sectional area is occupied by soil or rock depending on the strata.

So, we cannot exactly write Q/A is representing the velocity of flow. Q/A does not equal to the actual velocity of flow. So here we are representing it by a term volumetric flux q. So, in the case of porous media, we are representing

$$\frac{Q}{A} = q \text{ (Darcy's flux)}$$

This is also termed as Darcy's flux.

We have seen $V = \frac{\gamma D^2}{32\mu} S_f$ and $\frac{Q}{A} = q$

So, we can write, $\frac{Q}{A} = q = KS_f$ where, $K = \frac{\gamma D^2}{32\mu}$

So, this concept, how the flow through the porous media is related to the flow through a pipe, laminar flow through a pipe, that should be very clear to you. Based on the laminar flow in a pipe, we are making use of those concepts here and deriving the expression related to flow through a porous media. So here we have seen q, it is not exactly representing the velocity of the fluid through the soil, but it is represented as the volumetric flux or Darcy's flux.

So that is written as $q = KS_f$ or this Darcy's flux can be written as proportional to S_f, S_f is representing the friction slope or it can be mentioned as the energy dissipation taking place as it moves from one location to another location.

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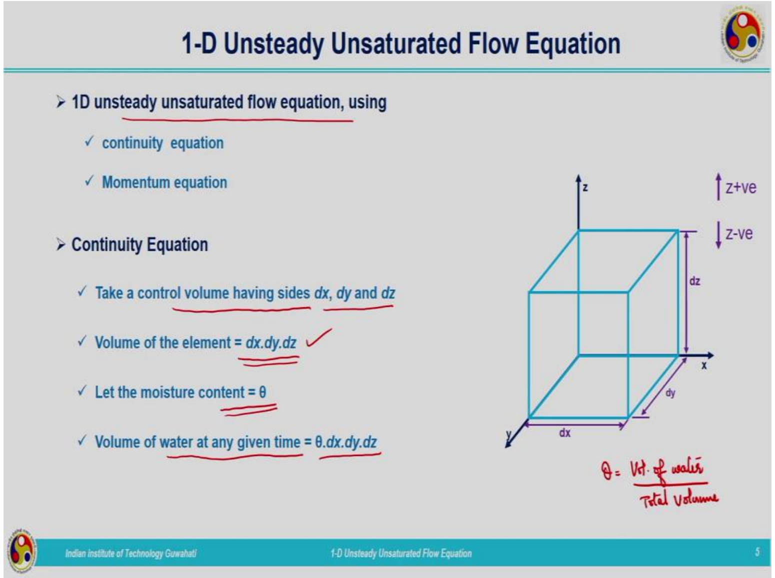

Now, let us move on to the derivation of equation corresponding to the flow through an unsaturated porous media. So here we are going to develop one-dimensional unsteady unsaturated flow equation. Basic laws related to mass conservation and momentum conservation is utilized for deriving the one-dimensional unsteady unsaturated flow equation.

First, we will start with continuity equation. For that, what we are going to consider? We are going to consider the control volume. That means, we will be making use of Reynold's Transport Theorem here and deriving the further expressions. So, we will consider a control volume having sides *dx*, *dy*, and *dz*. So, we can first define the cartesian coordinates, *x*, *y*, and *z* directions.

So, *z* is the vertical direction, it is considered as positive in the upward direction and negative in the downward direction. Now, we can consider the control volume, which is having the dimension, horizontal dimension *dx* and *dy*, and the vertical dimension as *dz*. Now we can write down the expression for the volume of the element. It is nothing but *dxdydz*.

Let the moisture content be θ. So, if this is a control volume corresponding to a porous media. So, this control volume is consisting of soil and also voids. So, voids will be filled with air and moisture. So, we can define the moisture content as θ.

$$\theta = \frac{\text{volume of water}}{\text{total volume of soil sample}}$$

Then volume of water at any given time = $\theta \times dxdydz$

At any given time, I am specifying since this θ will be varying with respect to time when a flow is taking place within the porous media.

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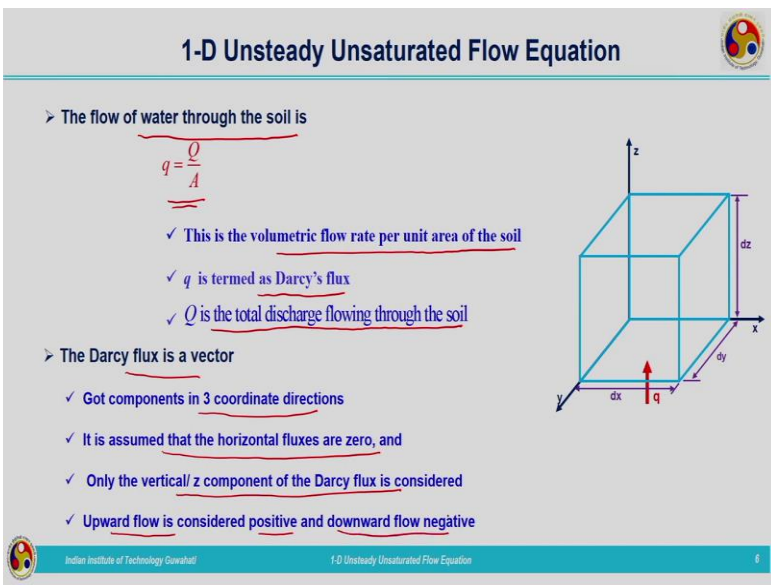

Now, when we consider flow of water through the soil, here at the bottom face, a flux of q is entering the control volume.

$$q = \frac{Q}{A}$$

q is the Darcy's flux which we have seen earlier. Q is the total discharge flowing through the soil. This Darcy's flux q is a vector. Here, we are considering only the one dimension. So, it has got actually three dimensions in the three coordinate directions. But in our problem, it is assumed that the horizontal fluxes are zero, only the vertical z component of the Darcy's flux is considered while deriving the unsteady unsaturated flow equation in this case. Also, you should understand we are considering the upward flow, and that is positive, and downward flow is negative. So, here q is acted up from the bottom face in the upward direction.

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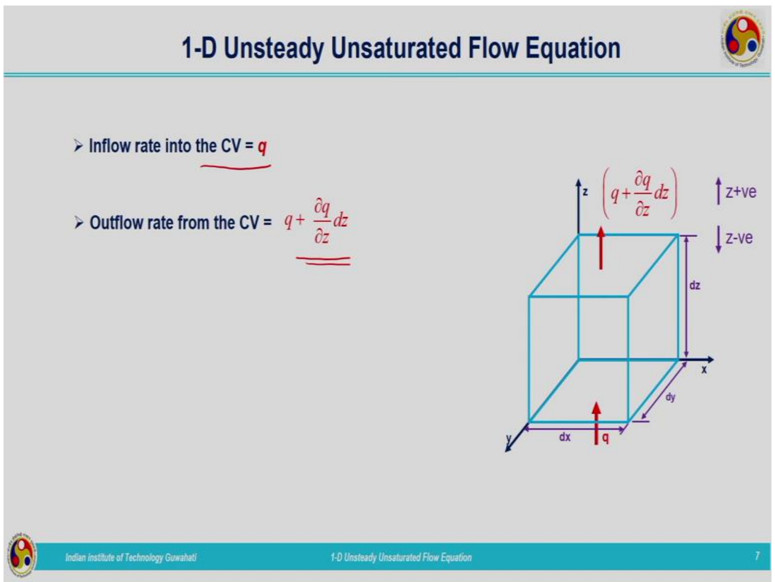

So, we know already, inflow rate is q. Now, we need to find out an expression for outflow taking place from the control volume. So, the outflow rate from the control volume can be written as

$q+\frac{\partial q}{\partial z}dz$ (change in q as it traverses a distance of dz multiplied by dz)

So, depending on the change in storage, that can be plus or minus, but here we are considering in the upward direction it is positive.

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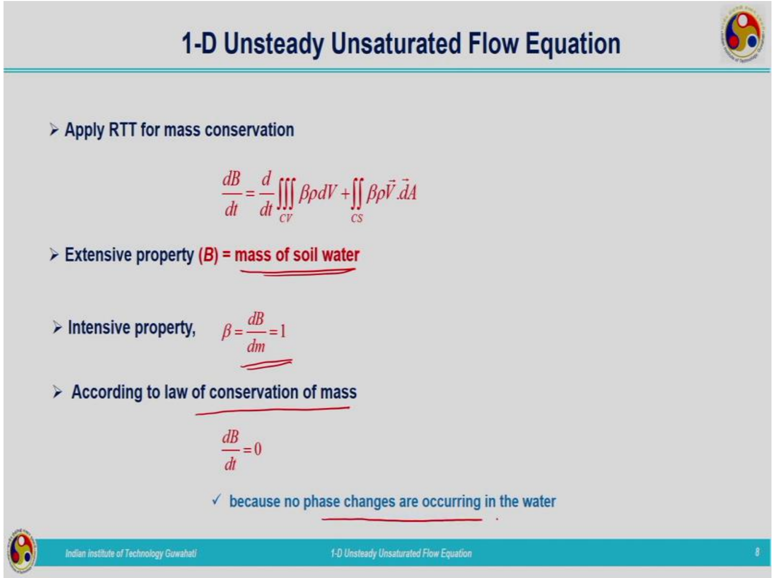

Now, what we are going to do. we are going to derive our mass conservation equation. So here we will be making use of our Reynold's Transport Theorem. Reynold's Transport Theorem expression is very much familiar to you. If we are making use of Reynold's Transport Theorem, we need to define our extensive property and also intensive property.

So, this is something related to flow through the porous medium, and the extensive property will be related to the mass of the fluid. So,

extensive property, B = mass of soil water.

$\beta = \frac{dB}{dm} = 1$ (Since there is no phase change taking place)

Left hand side, we are going to make use of RTT. According to law of conservation of mass, time rate of change of extensive property, that is, the time rate of change of mass of soil water definitely will be equal to 0.

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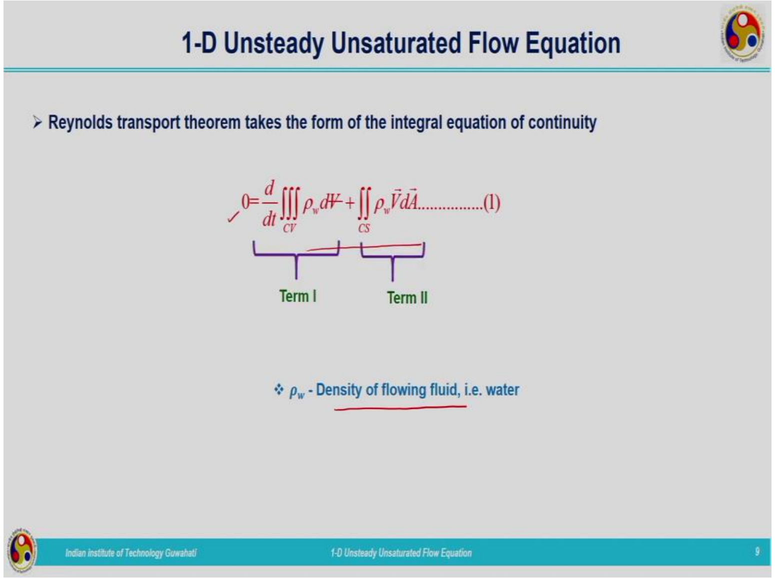

So that, we can substitute in the Reynold’s Transport Theorem. So, left hand side becomes 0. Right hand side, for β we are substituting 1.

$$0=\underbrace{\frac{d}{dt}\iiint_{CV}\rho_w d\cancel{V}}_{\text{Term I}}+\underbrace{\iint_{CS}\beta\rho_w\vec{V}\,\overrightarrow{dA}}_{\text{Term II}}----------(1)$$

This is our expression for integral continuity equation. ρ_w is the density of the flowing fluid, that is, water. Here, we are splitting this into two terms, that is, Term I and Term II. We can separately consider the two terms.

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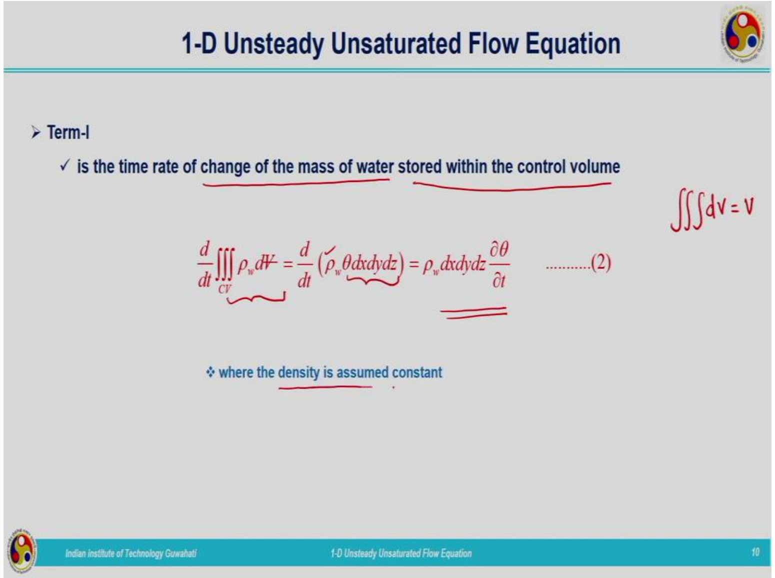

<u>So, consider Term I</u>

It is the time rate of change of mass of water stored within the control volume. So, first term is something related to the volume integral, that is, the time rate of change of extensive properties stored within the control volume. So, the expression is

$$\frac{d}{dt}\iiint_{CV}\rho_w d\forall = \frac{d}{dt}\left(\rho_w\theta dxdydz\right)$$

dxdydz is the dimensions of control volume considered. Here we are considering the fixed control volume so there will not be any change taking place with respect to time. So, it can be taken out of the differential term and only the variation is there for the moisture content, θ. So, we can write the expression to be

$$\frac{d}{dt}\iiint_{CV}\rho_w d\forall = \frac{d}{dt}\left(\rho_w\theta dxdydz\right) = \rho_w dxdydz\frac{\partial\theta}{\partial t} -------(2)$$

Why $\frac{\partial\theta}{\partial t}$? θ is varying with respect to space and also time. This equation can be termed as equation number two. Here, ρ_w, the density is assumed to be constant.

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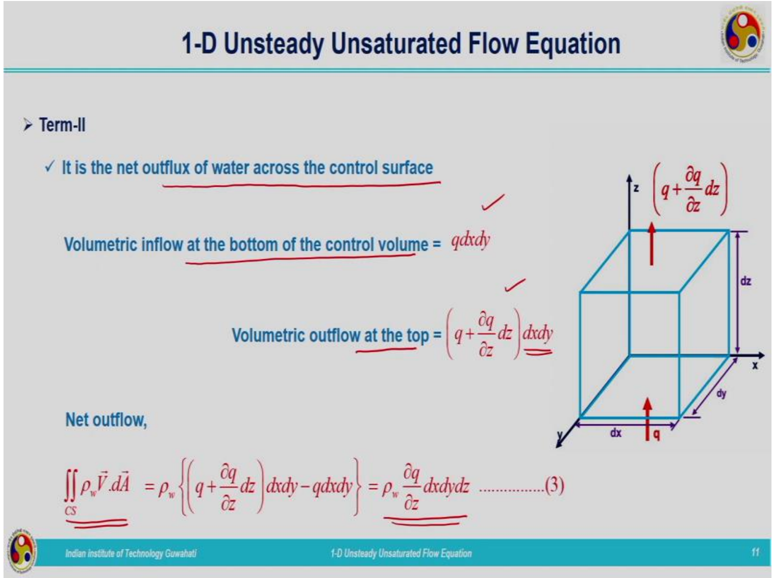

<u>So, consider Term II</u>

Term II is the net out flux of water that is, the outflow minus inflow, across the control surface. So here, we will substitute the terms corresponding to outflow and inflow. Volumetric inflow at the bottom of the control volume = $qdxdy$.

Volumetric outflow at the top of the control volume = $\left(q+\frac{\partial q}{\partial z}dz\right)dxdy$

Now, net outflow is outflow minus inflow.

$$\iint_{CS}\rho_w\vec{V}.\vec{dA} = \rho_w\left\{\left(q+\frac{\partial q}{\partial z}dz\right)dxdy - qdxdy\right\} = \rho_w\frac{\partial q}{\partial z}dxdydz -----(3)$$

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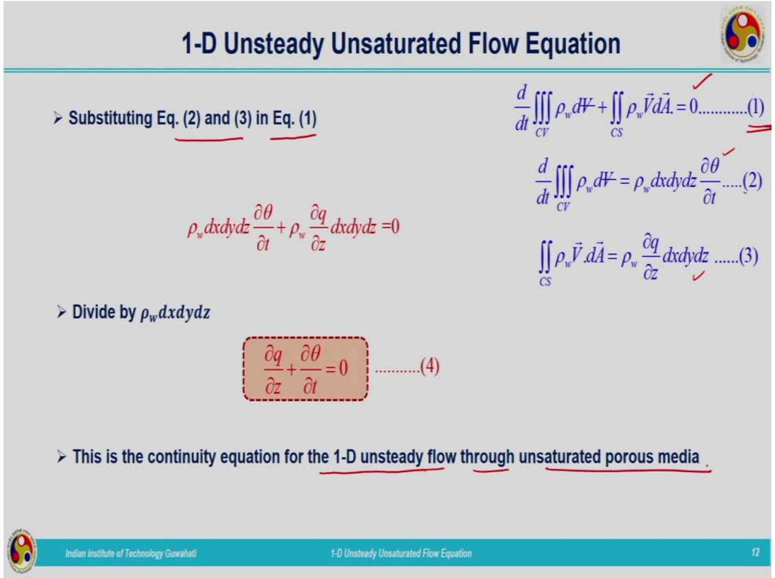

Now what we are going to do? We are going to substitute Equation 2 and 3 in Equation 1.

$$\rho_w dxdydz \frac{\partial \theta}{\partial t} + \rho_w \frac{\partial q}{\partial z} dxdydz = 0$$

Now dividing all the terms by $\rho_w dxdydz$ we will get the equation to be

$$\frac{\partial q}{\partial z} + \frac{\partial \theta}{\partial t} = 0 --------(4)$$

And this is our continuity equation for one dimensional unsteady flow through unsaturated porous media. So, we have considered a control volume, and inflow, the volumetric flux was taken as small q (Darcy's flux), and related to that we were having some outflow, and based on the Reynold's Transport Theorem, we have considered our extensive and intensive properties, and after substituting in the Reynold's Transport Theorem, we derived our continuity equation for one-dimensional unsteady unsaturated flow.

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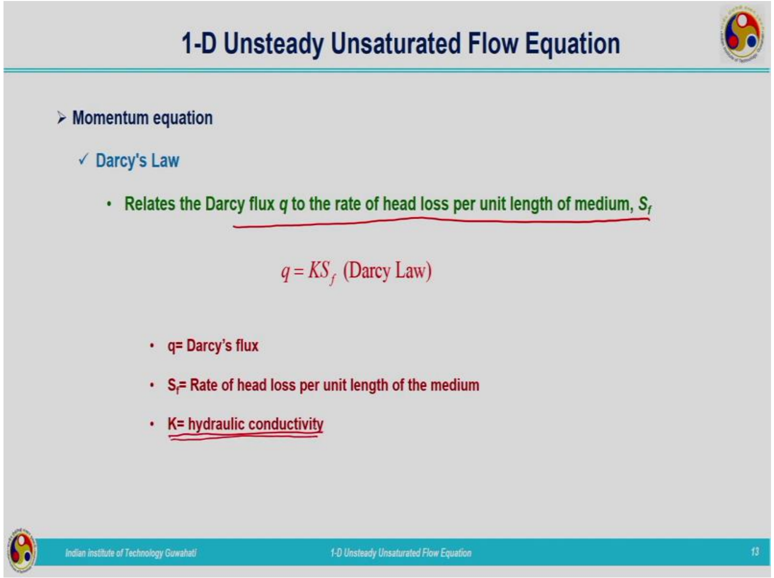

Now, what we are going to do? We are going to derive the momentum equation based on Darcy's Law. It relates the Darcy's flux q to the rate of weight loss per unit length of the medium, S_f.

$$q = KS_f \quad \text{(Darcy Law)}$$

q is the Darcy's flux, S_f is the rate of head loss per unit length of the medium, and K is the hydraulic conductivity.

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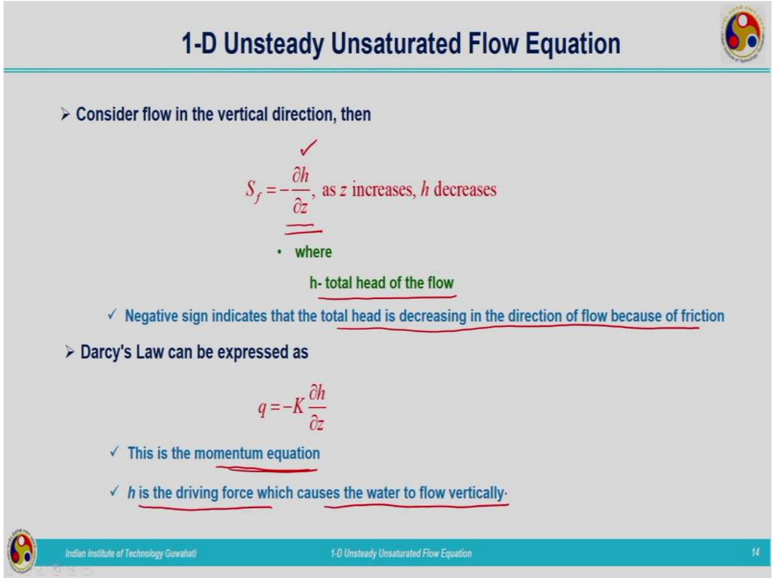

Now, consider a flow in a vertical direction. Then we need to have the expression for S_f, energy dissipation. S_f is the friction slope.

$$S_f = -\frac{\partial h}{\partial z}$$

That is, change in head as the distance travelled from one location to another location. Why there is a negative sign? Negative sign is there because as the distance increases, head decreases. h is the total head of the flow, head causing flow. Negative sign indicates that total head is decreasing in the direction of flow because of friction.

Now Darcy's Law can be written,

$$q = -K\frac{\partial h}{\partial z}$$

This is the Darcy's Law, and this is the momentum equation for unsteady unsaturated flow. h is the driving force which causes water to flow vertically.

Now we have to go back to previous lecture again. h is the driving force. There, we have seen what are the components of h. We have seen the unsaturated soil medium, and what are the different forces acting on that, and finally we had come up with an expression

$h = \psi + z$

Ψ is the suction head and z is the datum head. Same relationship we will use here, because it is something related to the flow through unsaturated porous media.

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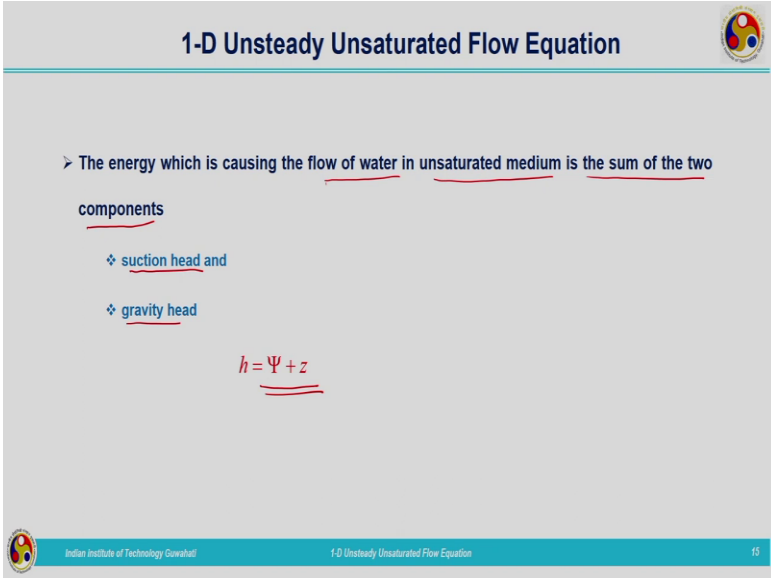

So, for h, we can write, energy which is causing the flow of water in unsaturated medium is the sum of the two components $(h = \psi + z)$. Ψ is the suction head and z is the datum head or the gravity head. We will substitute the expression for h in the Darcy's equation.

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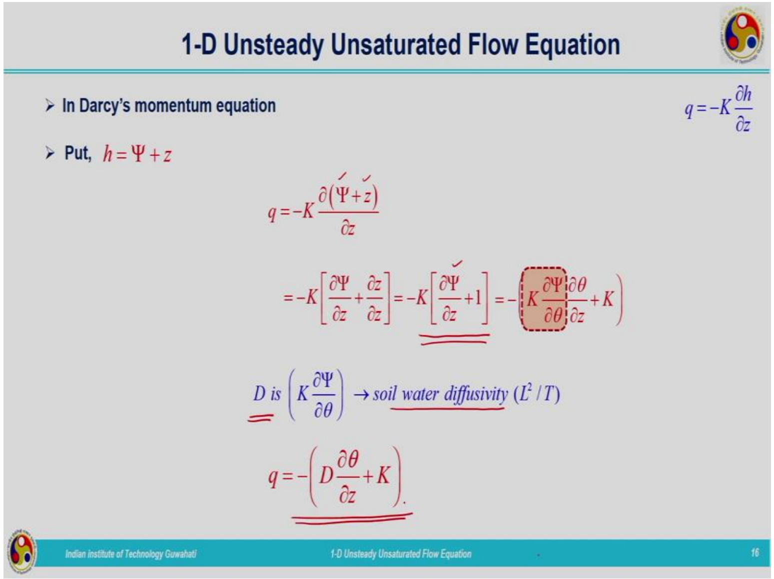

In Darcy's momentum equation after substituting we will get the expression

$$q=-K\frac{\partial(\psi+z)}{\partial z}$$
$$=-K\left[\frac{\partial\psi}{\partial z}+\frac{\partial z}{\partial z}\right]=-K\left[\frac{\partial\psi}{\partial z}+1\right]$$

Ψ is the suction head. It depends upon the dryness or the unsaturation within the soil. So, it is a function of moisture content. So, we can apply the chain rule here, and we can rewrite the term as

$$q=-\left[K\frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z}+K\right]$$

$K\frac{\partial\psi}{\partial\theta}$ is defined as soil water diffusivity (D) (L^2/T).

$$q=-\left[D\frac{\partial\theta}{\partial z}+K\right]$$

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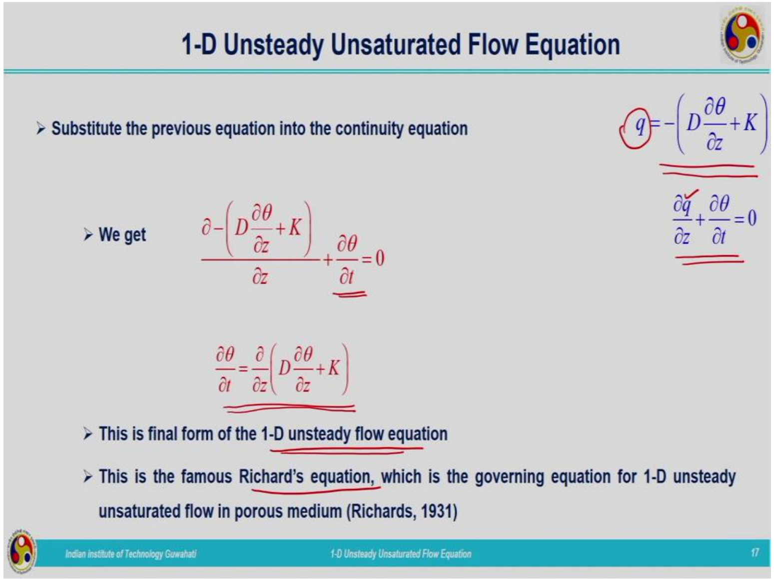

Now, what we can do? We can substitute the expression for q in the continuity equation. Now our continuity equation will be taking the form

$$\frac{\partial - \left[D\frac{\partial\theta}{\partial z} + K\right]}{\partial z} + \frac{\partial\theta}{\partial t} = 0$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z} + K\right)$$

This is the final form of the one-dimensional unsteady flow equation. This equation is known as Richard’s equation (Richards 1931) which is representing the flow through unsaturated porous media.

So here, what we have done? For deriving the one-dimensional unsteady unsaturated flow equation, we have considered continuity equation and also momentum equation. And combination of these continuity and momentum equations have been considered to derive the final form.

So, this is the well-known Richard’s equation which can be used for studying or modeling the flow through unsaturated porous media. This is the equation which is used for deriving many of the infiltration equation. Infiltration is the hydrologic process. So, for studying infiltration or for deriving infiltration equations, we need to have understanding about the Richard’s equation. That is what we have derived over here in this lecture.

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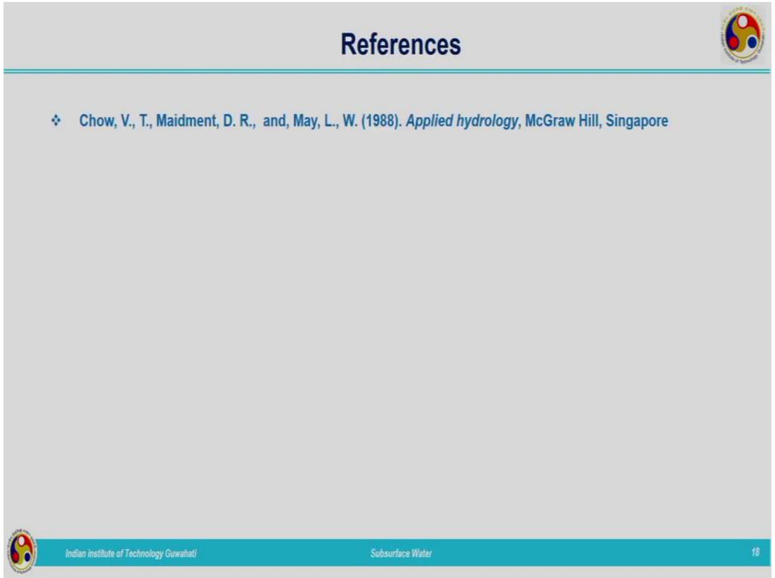

So, the reference related to this topic is the textbook by Ven T Chow and others, The Applied hydrology textbook. So here, I am winding up this lecture. Thank you very much.