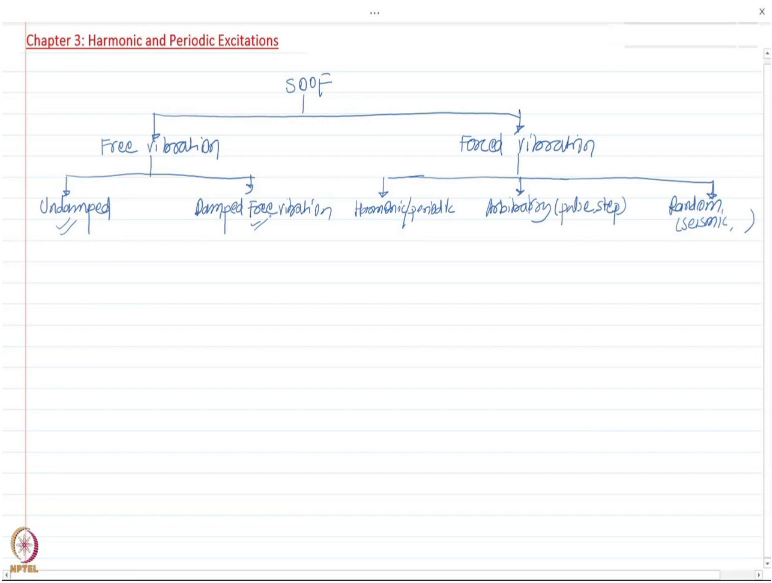
Dynamics of Structures
Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Forced Harmonic Vibrations
Lecture - 08
Undamped Harmonic Excitations

(Refer Slide Time: 00:24)



Welcome back everyone. So, in this lecture we are going to see how to obtain response of a single degree of freedom system subject to Harmonic loading. We will first start with undamped system and then find out the solution of the differential equation and then we are going to look into damped system. So, let us get started.

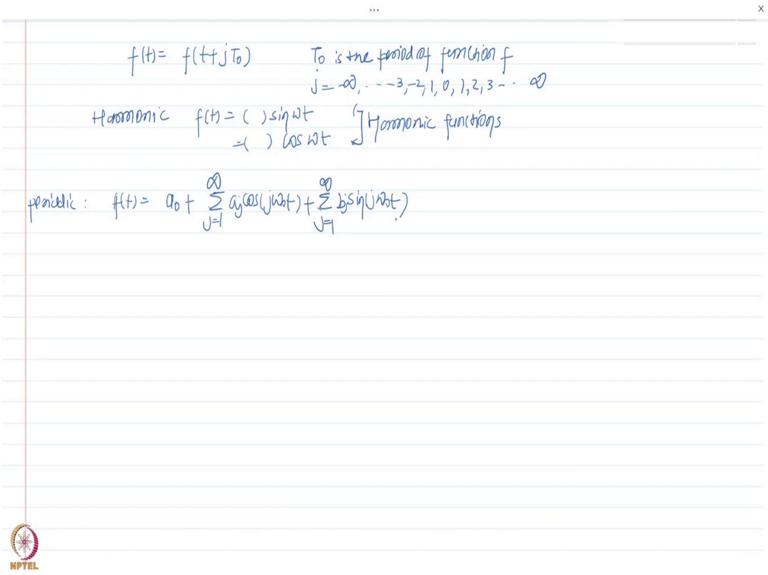
So, till now, basically what we have studied on this response of single degree of freedom system. Let me draw that chart. So, let us say we have a single degree of freedom system right. Now, in terms of the response of a single degree of freedom system we could have free vibration or we could have forced vibration.

Now, in free vibration it can be characterized or categorized as undamped free vibration and then there would be a damped free vibration. For the forced vibration it can be categorized based on the what kind of force that is being applied on the single degree of freedom system.

So, we could have for example, harmonic or periodic excitation alright and then we could have say arbitrary excitation. For example, it could be a pulse. So, pulse excitation like sine pulse or triangular pulse or a step function like that or we could also have random excitation, example could be seismic excitation like you know and wind and others.

Now, we have already studied the free vibration of damped and undamped systems. So, today what we are going to start? We are going to see how to obtain the response of a single degree of freedom system subject to forced vibration and more specifically, in this chapter we are going to focus on harmonic or periodic excitation.

(Refer Slide Time: 03:15)



So, let us start with that. Now, in terms of periodic excitation, any excitation or function can be characterized as periodic if I can write as-

$$f(t) = f(t + jT_0)$$

Where, T_0 is the period of function *f* and *j* is the integer. So, *j* could be from $-\infty$ to ∞ all the integers. So, let us say I write it as $-\infty........-3,-2,-1,0,1,2,3.........\infty$.

So, if any system can be written like this, it would be characterized as a periodic function. And harmonic functions are the functions, which can be written as some constant time sin or cos.

So basically, if I have functions like something times sin or something times cos, these are called harmonic functions. Now, as you would have noticed all harmonic functions are periodic however, not all the periodic functions are harmonic. Now, we will start with harmonic excitation.

So, one might ask a question that what is the uses of studying the response of a single degree of freedom system subject to harmonic excitation. In reality forces are not always harmonic and in reality, forces are not always single degree of freedom system. But the idea is that we are going to start with simple understanding and then we are going to build on that platform to understand more complicated system subject to a more complex excitation.

So, basically the idea is that if you understand the response of single degree of freedom system to harmonic excitation it would assist you in understanding or provide you insight how the system would behave to other type of excitation. So, I mean if you know Fourier transformation, remember from your mathematics class, you know that any periodic function.

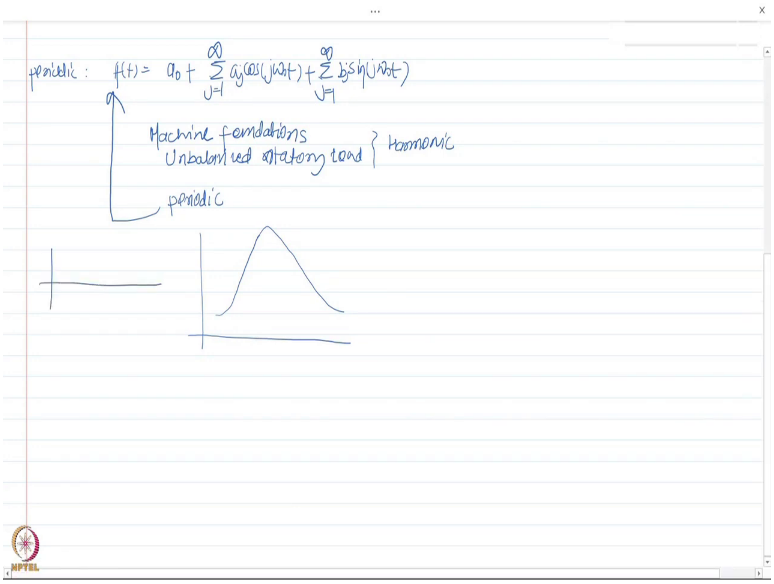
So, let us say I have any function $f(t)$ that I have written above, if it is a periodic function, it can be written as sum of many harmonic excitation. So, we will be going to start with harmonic excitation and once we understand the behaviour subject to harmonic excitation it would also give us insight into understanding to any type of periodic functions.

So, we know that any periodic function $f(t)$ can be expressed as-

$$f(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\omega_n t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_n t)$$

This is from your mathematics class. So, if the excitation in reality is being applied as in harmonic excitation.

(Refer Slide Time: 07:08)

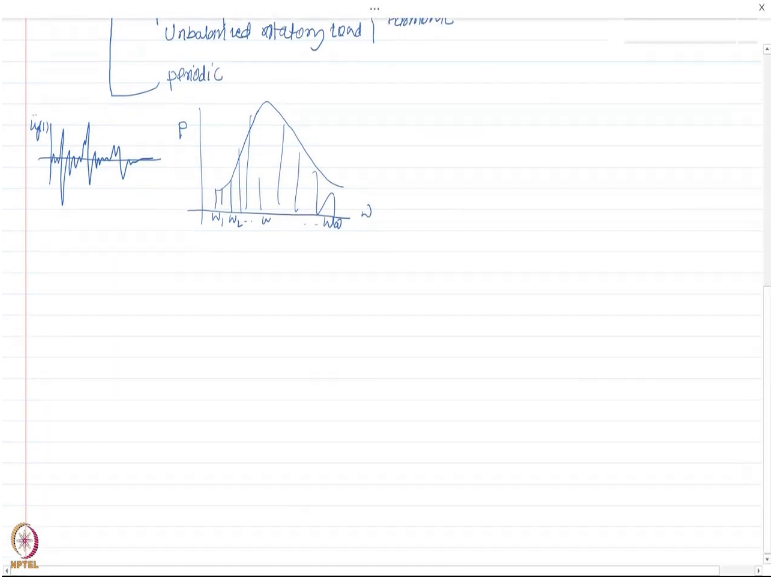


For example, few examples could be like you know machine foundation. So, you could have machine foundation where the load is applied as harmonic excitation or you could have any unbalanced rotatory load and we are going to model that actually in this chapter. These can be directly model as harmonic excitation.

And even if it is a periodic excitation but not harmonic, I could still obtain the response as sum of several harmonic excitation. So, if we understand the response to simple harmonic excitation then I would be equipped with knowledge to interpret response to any other periodic function as well.

And same goes for earthquake as well. So, earthquake in reality might be a random excitation. So, earthquake is composed of several frequencies. You will see that later when we will get into that. If the power spectral density, there is a term called power spectral density which basically defines how much of energy is actually situated at each of these frequencies.

(Refer Slide Time: 08:35)

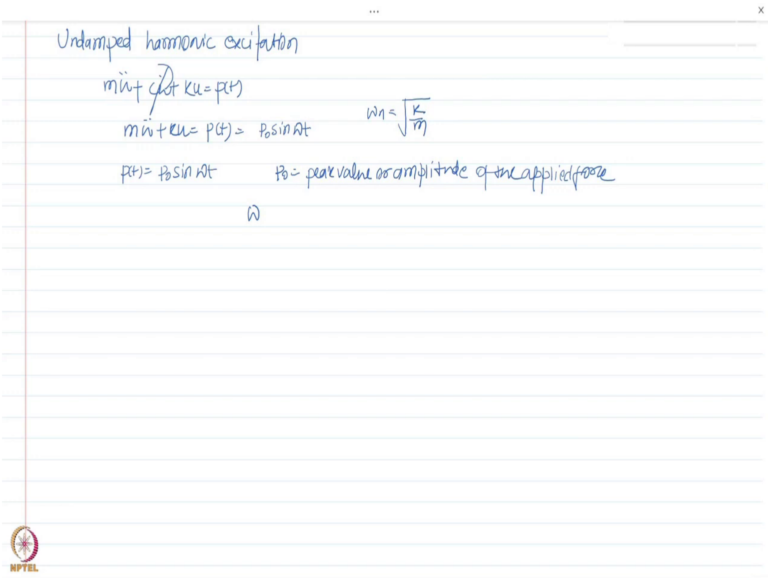


So, let us say this is my ground excitation and I am representing it with some random function which is neither periodic neither harmonic. Even this has basically distribution of these frequencies here and depending upon which frequency is predominant. I could have like this up to infinity.

And this is basically power spectral density and this is frequency. So, if I understand the response of single degree of freedom system to individual excitation frequency. Then it would even help me in analyzing or interpreting the behaviour of the structures subject to earthquake excitation as well.

So, I hope this provides you background that why we are doing response to harmonic excitations first and how it would be useful in subsequent chapters. So, in terms of harmonic excitation we are first going to start with an undamped system and then we are going to a damped system.

(Refer Slide Time: 09:50)



So, first let us do undamped harmonic excitation. So, if it is undamped, I know that my damping term in the equation of motion would be 0 for the single degree of freedom system. So, we know that our equation of motion is general equation of motion is this.

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

Now, if the damping is 0, I can simply delete damping this term and then I would be left with-

$$m\ddot{u} + ku = p(t)$$

Now, any harmonic excitation $p(t)$ can be represented as-

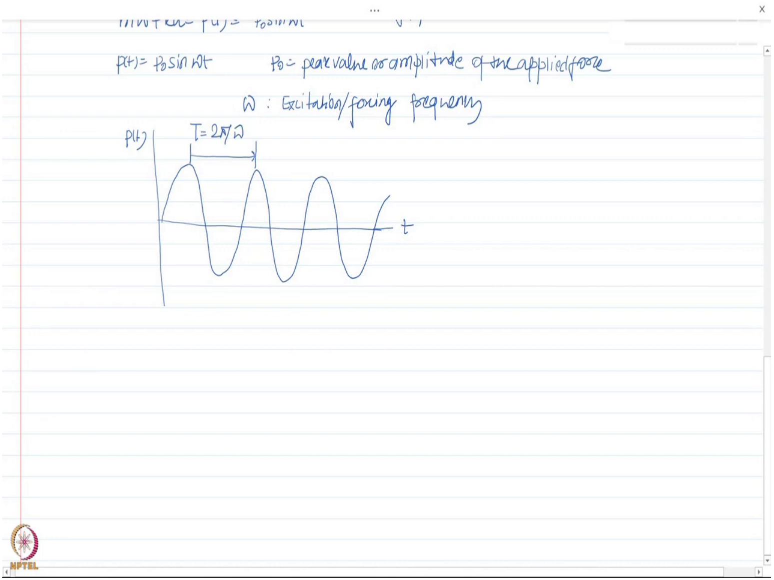
$$p(t) = P_0 \sin(\omega t) \text{ or } P_0 \cos(\omega t)$$

So, first we would be doing $\sin(\omega t)$ and the response subject to $\cos(\omega t)$ would also not defer by much. So, this excitation which I am representing as $p(t) = P_0 \sin(\omega t)$, P_0 is the force amplitude of the force that is being applied.

So, this P_0 is the peak value or amplitude of the applied force. And, note here, now I have an additional frequency ω and this is different from the natural frequency of the system which was ω_n. And we saw that ω_n for this system, I could simply get it as $\sqrt{\frac{k}{m}}$.

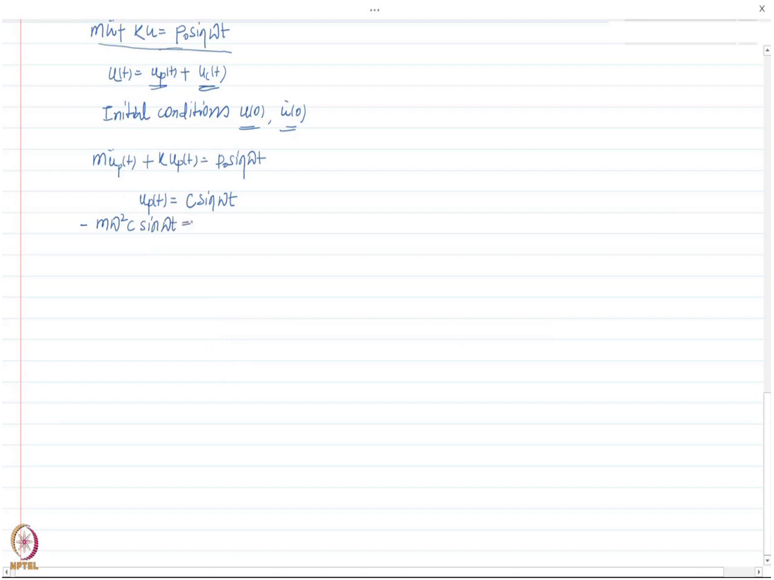
Now, ω is different from this ω_n, ω is basically the applied frequency of the harmonic force that is there and it is called excitation frequency or the forcing frequency. So, we would be referring it further as either excitation frequency or forcing frequency.

(Refer Slide Time: 12:01)



And if you plot this function $P_0 \sin(\omega t)$ it would look like a simply sinusoidal function. Which let us say starts here. So, I can plot it like that. So, this is my applied force and this is time here. This is the forcing time period or excitation time period (T) which I can write it as $\frac{2\pi}{\omega}$.

(Refer Slide Time: 12:43)



So, differential equation becomes-

$$m\ddot{u} + ku = P_0 \sin(\omega t)$$

Now I need to solve this differential equation. So, we are going to employ the same techniques that we have used in previous chapters to solve this second order linear differential equation.

So, how did we solve it? Basically, we said that the total response can be represented as sum of particular solution and sum of complementary solution.

$$u(t) = u_p(t) + u_c(t)$$

And this complementary solution had two unknown constants. So, this particular solution is any unique solution that can satisfy this equation and complementary solution was the general solution to the homogeneous part of this equation.

So, if you set the right-hand side equal to 0 and then whatever the solution you got that was basically the complementary or homogeneous solution. It had two unknown constants and those unknown constants were determined using initial conditions. So, let us assume our initial conditions are given in terms of initial displacement $u(0)$ and initial velocity $\dot{u}(0)$.

Now, let us see how do we solve this. So, basically in terms of particular solution here, I would write this as-

$$m\ddot{u}_p(t) + ku_p(t) = P_0 \sin(\omega t)$$

Let us say I want to get any unique solution that satisfy this equation. Now, what I see here on the left-hand side of this differential equation, I have a term which is $u_p(t)$ and then a double differential of term and then sum of that is actually equal to some sin function time with some constraint.

Now, let us say, if I assume-

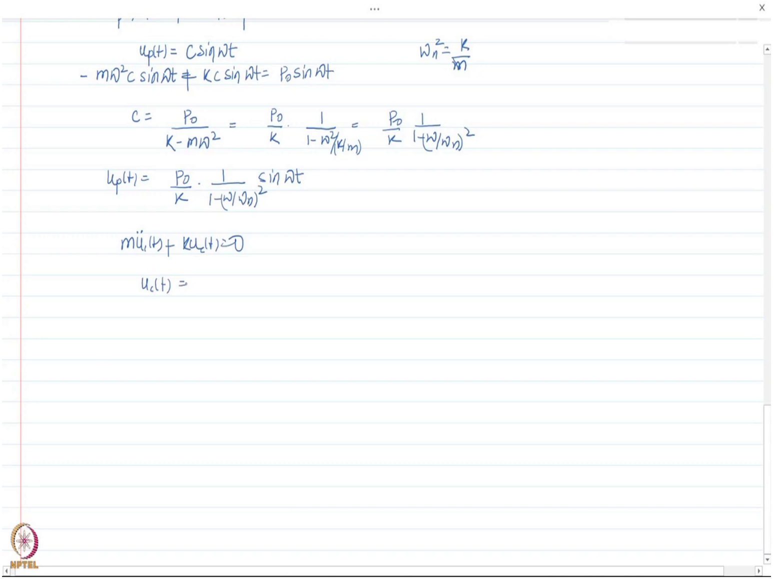
$$u_p(t) = C \sin(\omega t)$$

I know that the sum of term $u_p(t)$ and $\dot{u}_p(t)$ would be again some constant time $\sin(\omega t)$. Keeping that in mind I have selected this as particular solution. The only thing that needs to be found out here that, what is the value of C?

So, if I substitute this here, I can write it as-

$$-m\omega^2 C \sin(\omega t) + kC \sin(\omega t) = P_0 \sin(\omega t)$$

(Refer Slide Time: 15:53)



So, if we compare the coefficient of $\sin(\omega t)$. We get the value of C as-

$$C = \frac{P_0}{k - m\omega^2} = \frac{P_0}{k}\frac{1}{1 - \frac{\omega^2}{k/m}} = \frac{P_0}{k}\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

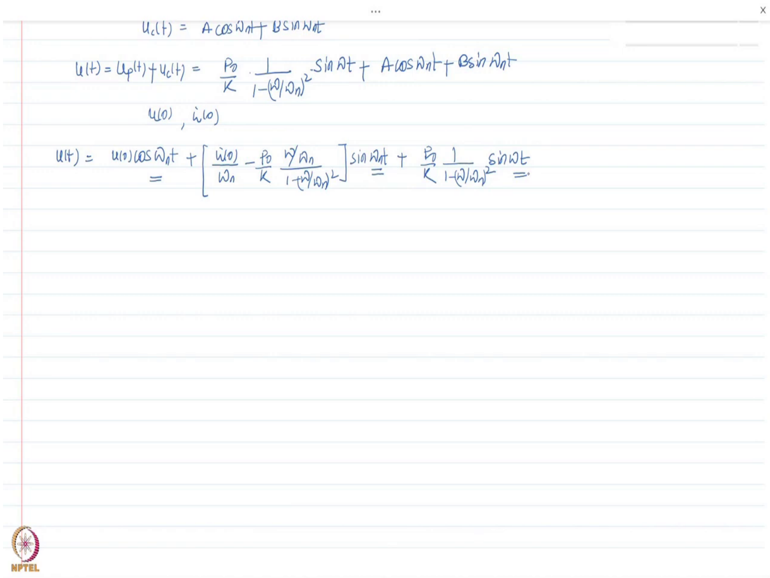
So, the particular solution I have obtained as-

$$u_p(t) = \frac{P_0}{k}\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\sin(\omega t)$$

Now, the second thing that remains is the complementary solution which is nothing but solution to this equation here.

$$m\ddot{u}_c(t) + ku_c(t) = 0$$

(Refer Slide Time: 17:26)



And that we saw in previous chapters can be directly written as-

$$u_c(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

So, I can write the total solution as particular solution plus the complementary solution and this can be written as-

$$u(t) = \frac{P_0}{k}\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\sin(\omega t) + A\cos(\omega_n t) + B\sin(\omega_n t)$$

Now, I have two unknown constants here, A and B which can be found out by substituting at t equal to 0, initial displacement $u(0)$ and the initial velocity $\dot{u}(0)$. So, you will need to differentiate it and then substitute again t equal to 0.

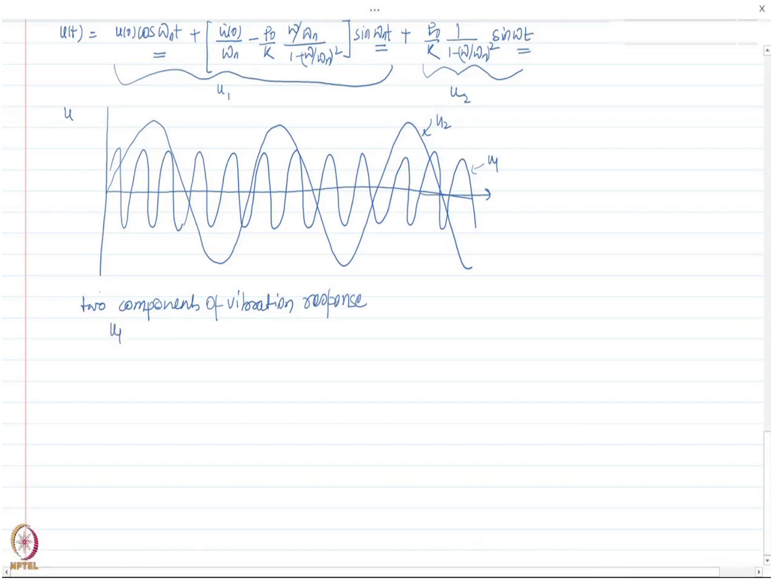
If you do that, you will see after substituting these conditions-

$$u(t) = u(0)\cos(\omega_n t) + \left[\frac{\dot{u}(0)}{\omega_n} - \frac{P_0}{k} \frac{\omega/\omega_n}{1 - \left(\omega/\omega_n\right)^2} \right] \sin(\omega_n t) + \frac{P_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sin(\omega t)$$

So, this is the final solution for undamped vibration subject to harmonic excitation which is $P_0 \sin(\omega t)$. Now, what you see here? There is one thing an important thing to note here we have in the response now two frequencies. Let me just write it here again we have two frequencies first frequency is basically the natural frequency (ω_n) of the system and then the second frequency is actually the applied frequency (ω) of the force or the excitation frequency.

So, one question might arise if you try to plot this system at what frequency would the system vibrate? Would it vibrate at ω_n or would it vibrate at ω you know? And, then the question also becomes that what do these frequency actually mean? What is the physical significance of this frequency? So, let us draw each term that we have here and then see how do they look like.

(Refer Slide Time: 20:52)



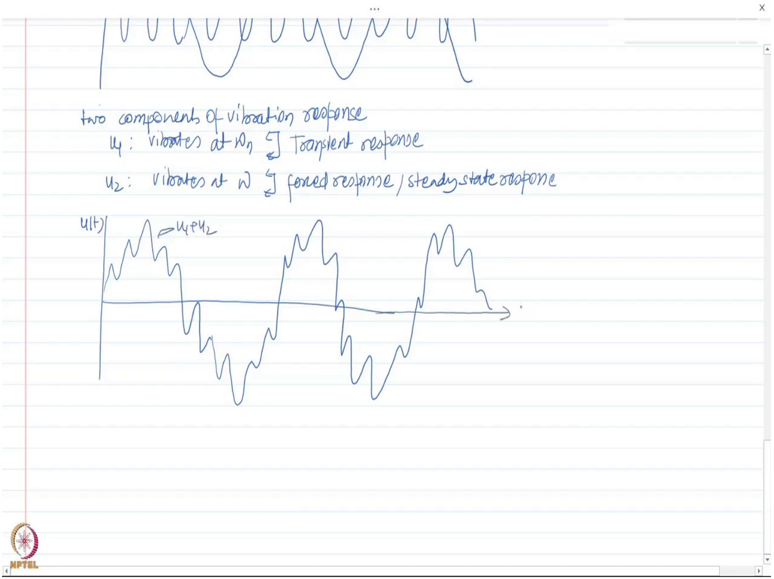
So, I am trying to draw here the displacement. I am not going to say if it is the total displacement, it is like one of these displacements, let me first just say that it is a displacement quantity. Now, first I am going to draw this term here $\left(u_1 = u(0)\cos(\omega_n t) + \left[\frac{\dot{u}(0)}{\omega_n} - \frac{P_0}{k} \frac{\omega/\omega_n}{1-\left(\omega/\omega_n\right)^2} \right] \sin(\omega_n t) \right)$ then I am going to plot this term $\left(u_2 = \frac{P_0}{k} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \sin(\omega t) \right)$ here.

So, if you plot u_1 what you going to see, it vibrates at its own frequency ω_n which is the natural frequency of the system and let us say it looks something like this with some initial conditions. Remember that I do not have any damping so amplitude is going to remain constant here and it will keep on vibrating like this with time period (T_n) equal to $\frac{2\pi}{\omega_n}$.

Now, I have another term, the $\sin(\omega t)$ term and let us plot u_2 and see what do we get. So, let us say it looks something like this. Now, you might argue that why the first u_1 looks like this and why not the other way well it depends on the relative ratio of the ω/ω_n.

So, just for demonstration I am assuming that let us say it looks like this. So, let us say this is my u_1 and this is my u_2. So, these are the two components of the response that we get subject to the harmonic excitation of single degree of freedom system. So, we can say that two components of the vibration response.

(Refer Slide Time: 23:41)



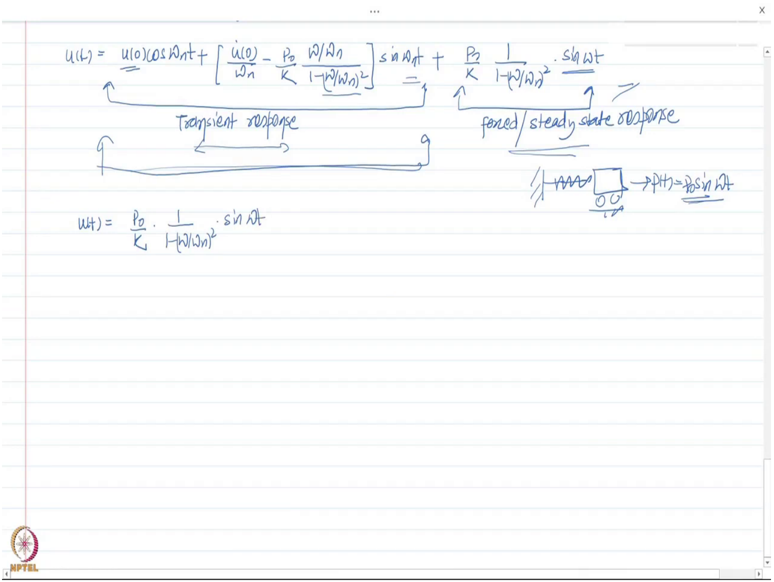
The component 1 (u_1) is basically vibrates at ω_n the natural frequency of the system and component 2 (u_2) basically vibrates with the whatever the applied frequency ω. This response (u_2) is called forced response or steady state response and we are going to come back to that why do we call it steady state response and this is called transient response (u_1).

So, for this I have drawn u_1 and u_2. So, if I try to again draw the total response which is $u = u_1 + u_2$, it would look something like this. Let me just first draw it so, it is going to look something like this. This is not to scale of course.

You can perhaps plot this kind of graph using MATLAB. So, this is the total response. So, this is $u_1 + u_2$ state transit response plus steady state response which is basically some of these two responses. So, what it is basically doing? It is taking the transient response which is u_1 and then it is vibrating along this steady state response.

So, the sum of two function would look like this. This is basically the total response of the system. So, there are certain characteristic of each of these responses. If you look at the transient response.

(Refer Slide Time: 26:00)



So, as I said this is the transient response (u_1) and this is the forced or more commonly it is called steady state response (u_2). And these terms are better explained when I will get into damped forced vibration however, note here, the transient response depends on initial conditions $u(0)$ and $\dot{u}(0)$.

However, force or steady state response does not depend on initial condition and that you can easily imagine. For example, let us draw the spring mass representation. If you apply a force which is harmonic force. The body would have some response whether there is no initial velocity and no initial displacement as we have applied force.

And that is the steady state response due to this force. Now, for the transient response it depends on the initial condition however, even if the initial conditions are 0 this force P_0 would still provide some response to the system that would oscillate at the frequency ω_n. And how much of that response contribute to the total response? Well, it would depend on the ω/ω_n which we would see later.

Now, in reality what happens? The transient response it seems that it would remain constant and it would not decrease with time. Say, if you look at here let me go back to this graph here. So, if you look at u_1 the amplitude actually remains constant and it does not change with time.

However, in reality all the system would have certain amount of damping and what that damping actually does, although not evident from this undamped the equation from the undamped vibration, the damping would reduce this transient response to 0 after sufficient amount of time.

However, the forced response or the steady response due to applied force would still remain and that is why it is called transient. Transient means something that is between a state or a changing state. So, let us say it was with some initial condition and then finally it achieves a steady state which is the forced response.

So, in between whatever the response system has is called the transient response which actually dies down in a damped system, but you cannot evidently see here. So, when we will do the damped system, we will see that this response actually dies down.

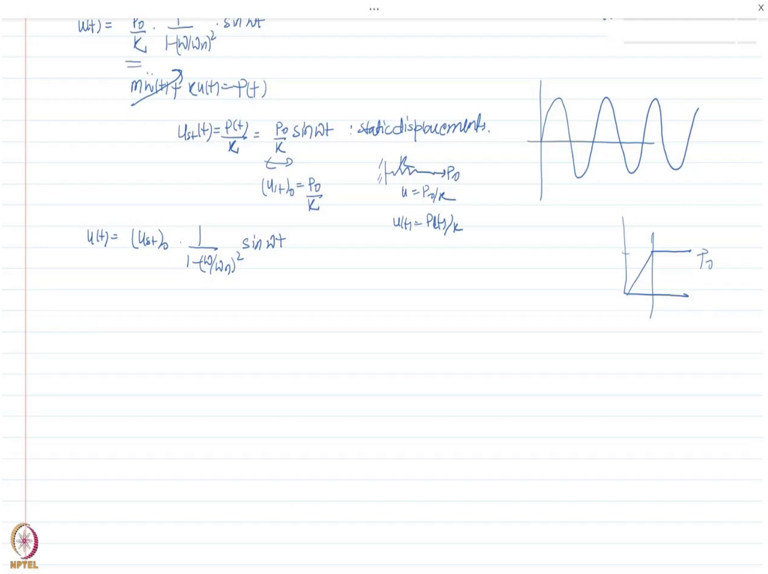
Now, in dynamics mostly we are concerned with the steady state response because in reality all the system would have some damping and the effect of applied force is measured in terms of steady state response.

So, we are going to neglect this (transient response), not neglect this but we are going to turn our focus to steady state response because as we will see for the damped system from their mathematical equation for the $u(t)$. This transient response actually dies down. So, let us look at the steady state response. And I am going to write this although as $u(t)$, but remember that I am only considering the steady state response part of total response, because this is what would be important.

However, you have to keep in mind, that initially the system would have some transient response and steady state response. So, look at what is the steady state response. I can write steady state response as-

$$u(t) = \frac{P_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sin(\omega t)$$

(Refer Slide Time: 31:37)



Now, if you remember from this equation of motion let us say this was the original equation of motion correct.

$$m\ddot{u} + ku = P_0 \sin(\omega t)$$

Let us say, I ignore the dynamic effect in the system, so I basically ignored this term $(m\ddot{u})$ here. Can I say the static displacement subject to a force $p(t)$ which is varying with time?

$$u_{st}(t) = \frac{p(t)}{k} = \frac{P_0 \sin(\omega t)}{k}$$

Now, you have to understand what do I mean by this? It simply means that there is no mass in the system so that is why there is no dynamic effect. If I am applying a sinusoidal force the deformation is basically whatever the force is applied divided by *k*.

Now, let us say if a spring is given to you and said like a force P_0 is applied. You would simply find out that the displacement as ${P_0}/{k}$, but if the force is varying with time then at each and every instant displacement would be simply ${p(t)}/{k}$, if no dynamic effect is considered.

So, this is static displacement here is basically, whatever the sinusoidal force you are applying the here, if that is applied without any dynamic effect of the mass then what is the displacement? So, it is different from the ramp loading in which we said that we are applying a force P_0 and the final displacement was basically ${P_0}/{k}$, if this (rising time) was large enough with respect to the time period of the system.

In this sinusoidal variation of the force, we are saying that mass term is 0, so the resultant displacement is ${p(t)}/{k}$. Now the significance of this is that if there was no mass in the system and if I had applied a sinusoidal load, the peak displacement of that is-

$$(u_{st})_0 = \frac{P_0}{k}$$

This is same ${P_0}/{k}$ that I have here.

So, I am going to write the dynamic displacement as-

$$u(t) = \frac{P_0}{k} . \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \sin(\omega t)$$

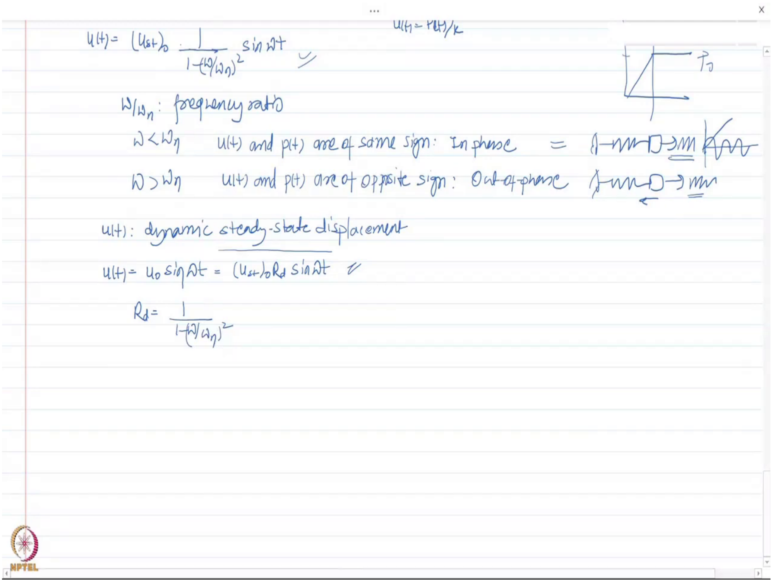
$$u(t) = (u_{st})_0 \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \sin(\omega t)$$

So, you need to understand what is this $(u_{st})_0$. $(u_{st})_0$ is basically the maximum value of the static displacement for a time varying load. Otherwise, you would argue that what do you mean by the maximum displacement of the static load.

Because, if you apply a load which is varying with time and there is no mass, then at each instant of time you can divide that load by the stiffness k to get the displacement at time t, but that is still the static displacement because there is no mass in the system.

And this $(u_{st})_0$ represents that peak static displacement subject to the sinusoidal or the harmonic excitation. So, I hope that is clear to you. Now if I have this term here ok.

(Refer Slide Time: 35:51)



Let us see what does it mean. For example, you see here that the term ω/ω_n or the frequency ratio. Depending upon the value of frequency ratio $\left(\omega/\omega_n\right)$ the value of $u(t)$ would change. For example, let us consider two extreme cases. Let us say when ω or the applied frequency is smaller than ω_n $(\omega < \omega_n)$, then what will happen? This denominator $\left(1-\left(\omega/\omega_n\right)^2\right)$

term would be positive and $u(t)$ would have same sign as $\sin(\omega t)$. So, $u(t)$ vary accordingly to whatever the variation of the right-hand term.

So, $u(t)$ vary as same as whatever the value of applied force $p(t)$ in the same direction. So, if this $u(t)$ and $p(t)$ are of same sign. It means that they are In phase. However, let us consider the second case when $\omega > \omega_n$, then you will see that denominator $\left(1-\left(\omega/\omega_n\right)^2\right)$ is negative and then $u(t)$ varies as negative of $\sin(\omega t)$ which is again negative of $p(t)$.

So, in that case $u(t)$ and $p(t)$ are of opposite sign and it is said out of phase. Physically what it means? Basically, that you apply a force in the first case. If you apply a force to the right then your system also moves to the right. In the second case, if you apply the force to the right the system moves to the left. You might have a doubt well how is that physically possible?

Remember that we are not considering here monotonic load. We are considering here a sinusoidal load. So, depending upon the frequency ratio $\left(\omega/\omega_n\right)$, once the motion is started it might not be in the same direction at any time instead as the applied force. So, if my force is positive, it might be moving to the negative direction and that depends on the ratio of ω/ω_n.

So, let us see if we can formulate some mathematical expression to actually represent dynamic displacement. So, remember, $u(t)$ is my dynamic steady state displacement. And although in further lectures when we discuss I am not going to always refer it as a steady state displacement, but it should be understood. You should understand that I am talking about a steady state displacement by just looking at the expression of $u(t)$.

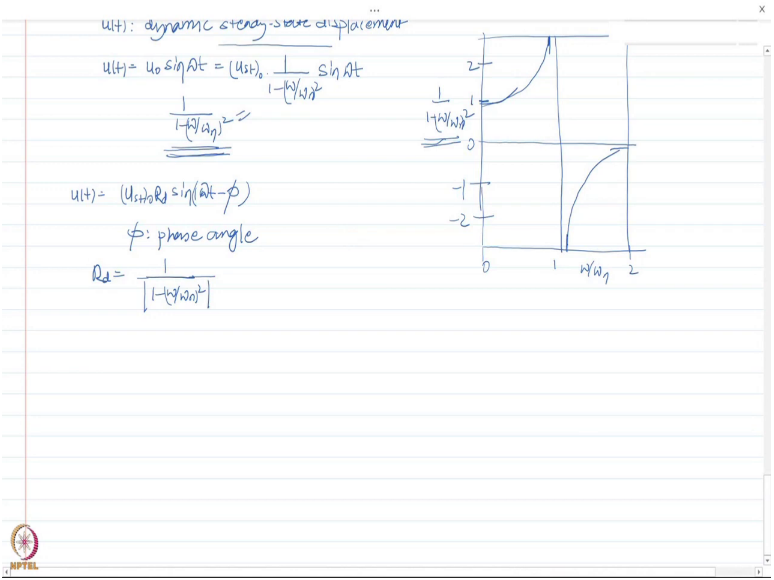
So, this is the dynamic steady state displacement. Now can you imagine if I again have a dynamic steady state displacement which is varying as a function of $\sin(\omega t)$. It would again have some peak value, so it would have some amplitude. Can I write-

$$u(t) = u_0 \sin(\omega t) = (u_{st})_0 R_d \sin(\omega t)$$

Where, u_0 is the dynamic amplitude or the dynamic displacement amplitude. Now, what is the R_d here? If you compare this equation here and this equation here basically R_d is-

$$R_d = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

(Refer Slide Time: 40:43)



Now, if you plot this expression R_d , how does it look like ok? We talked about the phase, let us see how does it look like. First let me draw here. This is my frequency ratio $\left(\frac{\omega}{\omega_n}\right)$. So, horizontal axis is frequency ratio and the vertical axis is $\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$.

Now, I know that at very small value of $\frac{\omega}{\omega_n}$ this $\frac{1}{1-\left(\omega/\omega_n\right)^2}=1$. So, $\frac{1}{1-\left(\omega/\omega_n\right)^2}$ start from 1.

As I increase the value of $\frac{\omega}{\omega_n}$, so what happens? The value of the ratio $\frac{1}{1-\left(\omega/\omega_n\right)^2}$ actually increases because, as you increase the $\frac{\omega}{\omega_n}$, the denominator $\left(1-\left(\omega/\omega_n\right)^2\right)$ decreases. So, the overall R_d which is 1 inverse of that value would increase. So, it increases something like that. I do not know how it would increase.

Other extreme is that $\frac{\omega}{\omega_n}$ is a very large value. So, it is infinity. In that case $\frac{1}{1-\left(\omega/\omega_n\right)^2}=0$. So, it will start somewhere from here and as you decrease the value of $\frac{\omega}{\omega_n}$ then your denominator $\left(1-\left(\omega/\omega_n\right)^2\right)$ would actually increase, but in the negative direction.

And can I say when $\frac{\omega}{\omega_n} = 1$ then the value becomes unbounded $\left(\frac{1}{1-\left(\omega / \omega_n \right)^2} = \infty \right)$. So, $\frac{1}{1-\left(\omega / \omega_n \right)^2}$ becomes unbounded here on positive side and unbounded here on the negative side. So, the graph would look something like this. The $\frac{1}{1-\left(\omega / \omega_n \right)^2}$ actually depends on the frequency ratio.

So, it depends on the frequency ratio $\frac{\omega}{\omega_n}$. And whether $\frac{1}{1-\left(\omega / \omega_n \right)^2}$ is positive or negative that also depends on $\frac{\omega}{\omega_n}$. So, what I would do? I do not need that to plot this R_d. Now, I am just trying to plot here-

$$u(t) = (u_{st})_0 \, R_d \sin(\omega t - \varphi)$$

I am adding a new term φ here which called phase angle and R_d is nothing, but the absolute magnitude. So, I have I had originally this term, but R_d is actually the absolute magnitude of this term here.

$$R_d = \frac{1}{\left| 1-\left(\omega / \omega_n \right)^2 \right|}$$

So, R_d would always be positive. This is the mode of this function here. If that is there then you would ask me that how does the $\sin(\omega t)$ is actually taken into consideration because initially it was coming due to this R_d. Precisely for that I have this angle φ here. Remember we said that when $\frac{\omega}{\omega_n} < 1$ then $u(t)$ varies as the same positive as $\sin(\omega t)$.

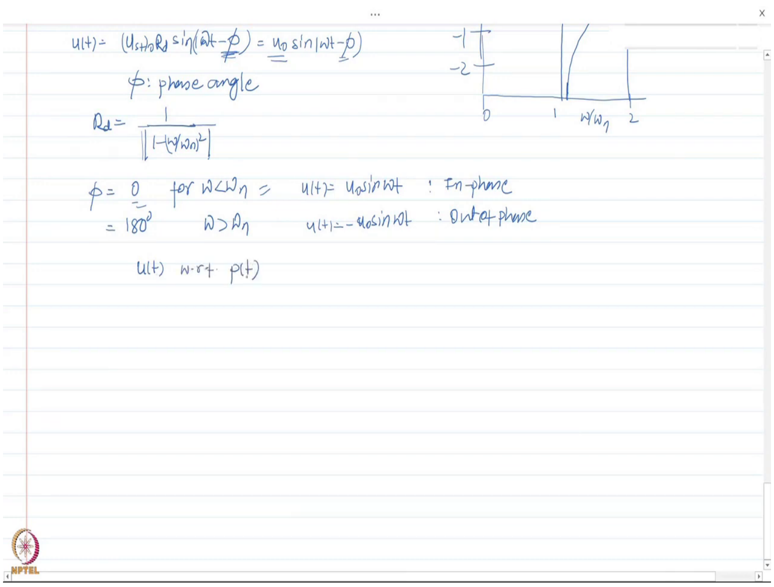
If $\frac{\omega}{\omega_n} > 1$ then $u(t)$ varies as negative of that. So, to take that sign into consideration. I am writing this term here-

$$u(t) = (u_{st})_0 R_d \sin(\omega t - \varphi)$$

Where, R_d I have taken magnitude of that- $R_d = \frac{1}{\left|1 - \left(\omega / \omega_n \right)^2 \right|}$

Now all the sign is now being taken care of by this φ or the phase angle.

(Refer Slide Time: 45:12)



So, this is this term here and the phase angle now I am defining-

$$\varphi = \begin{cases} 0^\circ & \omega < \omega_n \\ 180^\circ & \omega > \omega_n \end{cases}$$

Which makes sense. If you look at it here, I would again have the same expression.

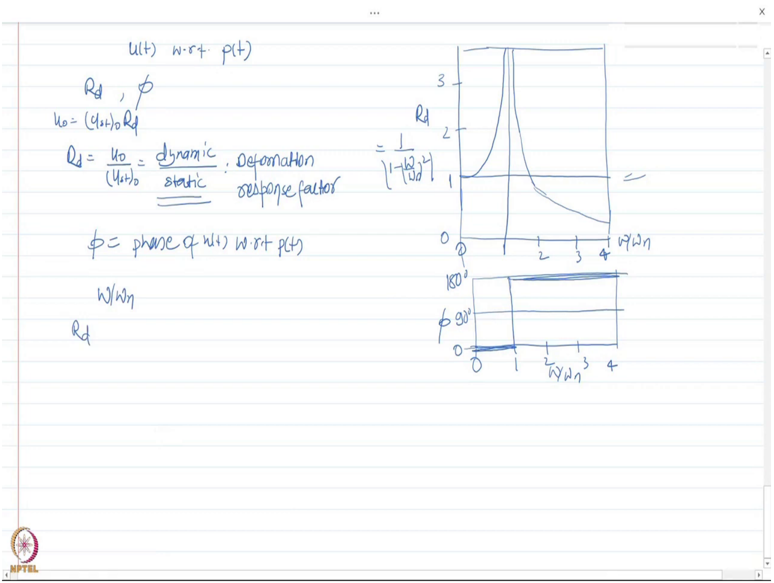
$$u(t) = (u_{st})_0 R_d \sin(\omega t - \varphi) = u_0 \sin(\omega t - \varphi)$$

I remember that now this R_d is always positive, because already I am considering as magnitude. Now, if-

$$u(t) = \begin{cases} u_0 \sin(\omega t) & \omega < \omega_n \quad \varphi = 0^\circ \\ -u_0 \sin(\omega t) & \omega > \omega_n \quad \varphi = 180^\circ \end{cases}$$

So, this is $(u_0 \sin(\omega t))$ in In-phase and this is $(-u_0 \sin(\omega t))$ out of phase. When I say In-phase and out of phase is basically the displacement $u(t)$ with respect to the applied force $p(t)$. Once I have this information let us plot.

(Refer Slide Time: 46:52)



Let us plot both R_d and φ the phase angle and this then see how do they look like. I know that both of them depends on the frequency ratio $\left(\omega/\omega_n\right)$. So, R_d is nothing but, it is the magnitude or the absolute modulus of this function. So that would mean that this function here (In the previous graph for $\frac{\omega}{\omega_n} > 1$) would simply invert to the positive side of the axis.

So, let us try to draw that. So, I have this R_d on vertical axis. I am plotting R_d here and then below this I am also plotting phase angle which is φ and both these x axes are basically ω/ω_n.

The vertical axis in this case for the φ is 0°, 90° and 180° and here (vertical axis of R_d) it is 0, 1, 2, 3 and so on. So, if I simply take the mod of this function, I again R_d starts at 1 for a very small value of ω/ω_n. Let me just write down the expression anyway for your convenience.

So, $\omega/\omega_n << 1$, $R_d = 1$ and it is unbounded at value of $\omega/\omega_n = 1$. So, it will start at 1 and then it will go like this. In the second case, what happens as you increase the value of ω/ω_n it starts from here it will start at value of $\omega/\omega_n = 1$ approaching to infinity and then it would decrease as you start and at some point it is going to cross the x axis and then go to 0 for a very large value of ω/ω_n.

Now, compared to that I have the phase angle φ which is, if $\omega/\omega_n \le 1$, it is 0° and if it $\omega/\omega_n > 1$ it is 180°. So, we have talked a lot about R_d and φ, let us see what is the physical significance of these two quantities. Now, as we said-

$$u_0 = \left(u_{st}\right)_0 R_d$$

$$R_d = \frac{u_0}{\left(u_{st}\right)_0}$$

R_d is nothing but the dynamic amplitude divided by the static amplitude.

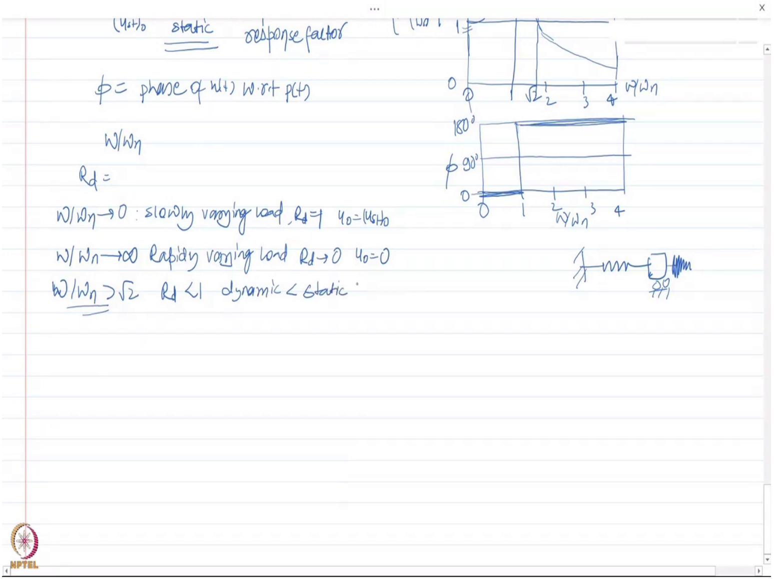
So, if the force had been applied statically, whatever the amplitude of that function is let us say P_0/k and if the system had mass (m) and then the force is being applied, whatever the displacement that you get is dynamic displacement. Then the ratio of both two (dynamic and static displacement) is defined as R_d and it is called displacement or deformation response factor.

And we will see later that there are different type of response factors. Right now, we are only dealing with deformation response factor, which represents what is the amplification in the response. Amplification or reduction whatever happens with respect to the static system, R_d is a measure of that amplification or reduction.

And phi (φ) represents that if the force $p(t)$ is applied, then with respect to the applied force $p(t)$ whether the response or the $u(t)$ is in with In-phase or whether it is out of phase. So, it represents phase of $u(t)$ with respect to applied force. So, if my force $p(t)$ is moving right whether the displacement is also in the right direction or whether it is opposite to the applied force in the left direction, so the phi represents that.

Now, this plot of R_d or the displacement response factor here it gives several useful information in terms of dynamic response, because that is what we are trying to study here right. So basically, if I have a system and a harmonic load is applied on that, based on the frequency ratio ω/ω_n, I would see what happens to R_d which is the displacement response factor.

(Refer Slide Time: 52:27)



So, let us come to this graph here. If the value of $\omega/\omega_n << 1$, I do not see any amplification right, $R_d = 1$ that means $u_0 = (u_{st})_0$, so it means dynamic is equal to static. So, let us consider first case; that $\omega/\omega_n \to 0$.

It means that ω is very small. The applied frequency is very small. When the applied frequency is very small it means that it is a slowly varying load. In that case $R_d = 1$ or dynamic displacement is equal to static displacement.

So, if the load is applied very slowly, we do not see any amplification in the system with respect to the static displacement, and that should be intuitive to you is not it. I mean if you are applying load at very very small frequency then the system would move in tandem with whatever the force you are applying without having the much dynamic effect of the mass. And that is why your dynamic displacement is equal to a static displacement.

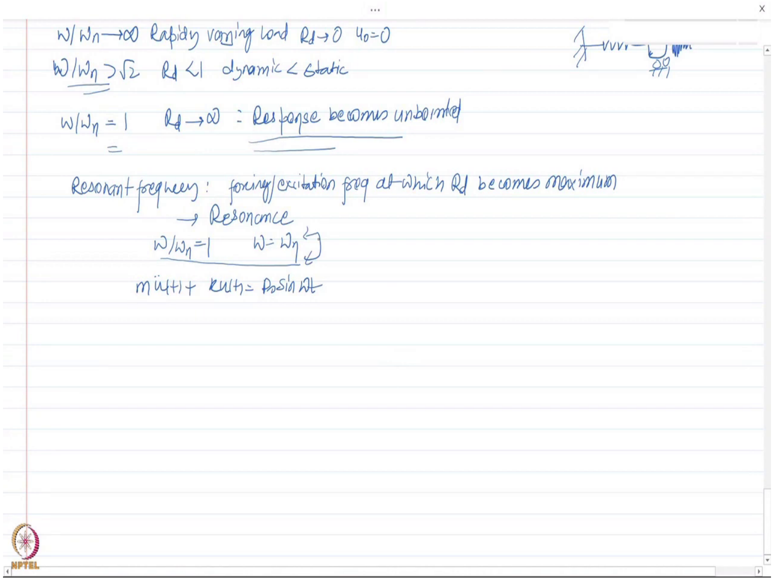
Now, in the second case when ω/ω_n is very large $\left(\omega/\omega_n \to \infty\right)$ that means, a rapidly applying force or rapidly varying force. In that case $R_d \to 0$. That means, dynamic displacement is equal to 0. It implies that for a rapidly varying force, you do not see any response in the system.

So, if you try to imagine this, basically you have a system on which you are applying a very fast-moving load. This system would not even actually respond to this. That is what basically it means, the displacement would be equal to 0. So, these were the two extreme cases. Now, if third case if $\omega/\omega_n > \sqrt{2}$, which basically corresponds (in the graph) to let us say this point here $\sqrt{2}$ then $R_d < 1$.

So, I do not get any amplification in the response. So, the dynamic behaviour actually does not lead to amplification, but it actually leads to reduction in the response. As we will see later this is very useful in design of some systems.

For example, let us I have a system, a static system a building just sitting there. Would I not be happy that if I apply some force or the frequency on my structure, and the response become actually less than the response it would have if the load had been applied statically. I would be very happy. So, that depends on the applied frequency and this actually properties utilized in vibration isolation, we will come back at later. So, my for $\omega/\omega_n > \sqrt{2}$ dynamic R_d is actually less than static.

(Refer Slide Time: 56:29)



And of course, then the case where $\omega/\omega_n = 1$ then $R_d \to \infty$, so the response becomes unbounded. So, these are the four typical cases which you might experience in real life based on how the frequency of the applied force varies with respect to frequency of the system and that decides what is the response of the system.

Now, we talked about, this case here when $\omega/\omega_n = 1$, the response becomes unbounded. We are going to define the frequency or the applied frequency which is called resonance frequency.

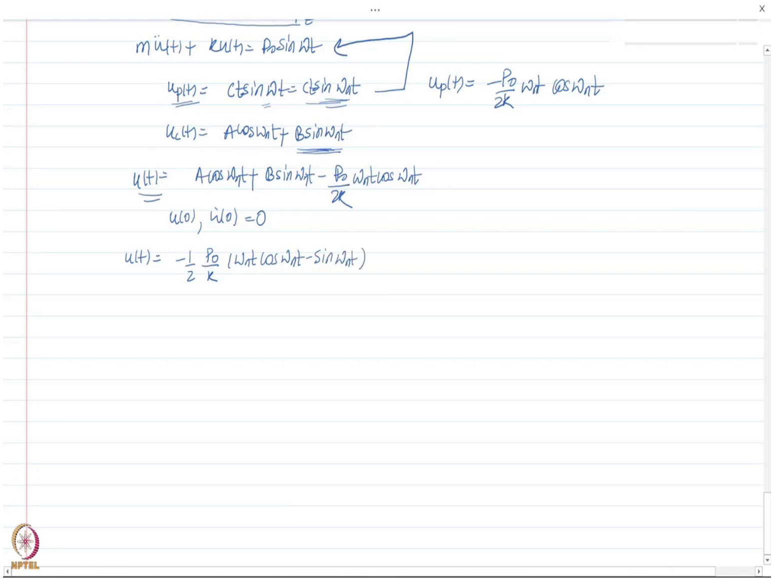
So, this would be the forcing frequency or excitation frequency at which the R_d of the displacement response factor becomes maximum. So, whatever the value of applied frequency at which it becomes maximum is called the resonant frequency and this phenomenon is actually called a resonance. Many of you might be aware with this phenomenon.

Now, in this case what happens? So, the resonance is basically maximum, it happens when $\omega/\omega_n = 1$, so applied frequency is actually equal to the natural frequency of the system, so my

R_d become maximum. Now, when this happens at this point actually the solution that we have obtained for $u(t)$ is not valid anymore and we actually obtain a new solution.

So basically, what I am saying, in this case the solution that we had obtained for our equation of motion this $\left(m\ddot{u}+ku=P_0\sin(\omega t)\right)$ is not valid any more. Only at this $(\omega=\omega_n)$ point. It was valid at all other part. So, we developed a new solution, when this happens. For this case what do we do?

(Refer Slide Time: 59:28)



Basically, remember our particular solution initially we had assumed $u_p(t)=C\sin(\omega t)$. Now, $\omega=\omega_n$. So, this is $u_p(t)=C\sin(\omega_n t)$. And our homogeneous solution or the complementary solution was $u_c(t)=A\cos(\omega_n t)+B\sin(\omega_n t)$.

Now if you look at this $u_p(t)$ carefully, this is now not a unique solution anymore, because this $\sin(\omega t)$ term because your $\sin(\omega_n t)$, now it is a part $(u_c(t))$ of this term here.

Initially, they were different $\omega \neq \omega_n$. This $(C\sin(\omega t))$ was a unique solution, but now because I have an $\sin(\omega_n t)$ term and then it is part of this complementary solution.

So, for these types of cases, what we do? The solution is obtained by assuming-

$$u_p(t) = Ct\sin(\omega t) = Ct\sin(\omega_n t)$$

If you substitute this in this equation you will obtain as $u_p(t)$ particular solution and you can do that calculation yourself. You can obtain that as-

$$u_p(t) = \frac{-P_0}{2k}\omega_n t\cos(\omega_n t)$$

The total solution you can write it as-

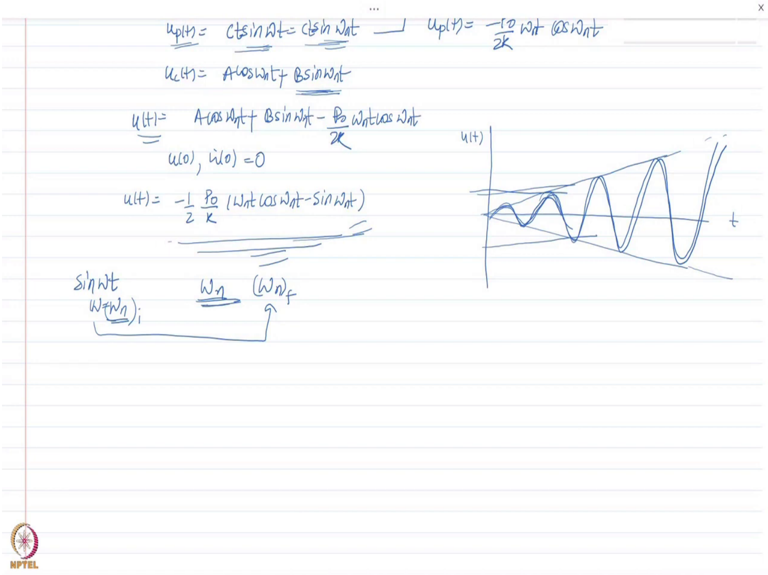
$$u(t) = A\cos(\omega_n t) + B\sin(\omega_n t) - \frac{P_0}{2k}\omega_n t\cos(\omega_n t)$$

So, you can see where $\omega = \omega_n$, the solution or the response does not suddenly become unbounded. If you obtain the response for the initial condition $u(0)$ and $\dot{u}(0)$. You will get some expression. And what I want to do? Let us say both these parameters $(u(0),\dot{u}(0))$. You can get the response as-

$$u(t) = -\frac{1}{2}.\frac{P_0}{k}\left(\omega_n t\cos(\omega_n t) - \sin(\omega_n t)\right)$$

And if you try to plot this function. You can use MATLAB to plot it, it would look like something like this.

(Refer Slide Time: 1:02:18)



Let me just plot it here it would be easier to do. So, my response does become unbounded, but I cannot represent my solution using the previous equation I have to derive the new solution for $\omega = \omega_n$, because of the reason that we just discussed here that particular solution is not a unique solution anymore.

And what happens over the time, the amplitude keep on increasing. If you plot it, the amplitude will keep on increasing and it would become unbounded but it does not become unbounded suddenly. Now try to imagine this scenario in reality. In reality assuming that, it is an undamped system. Which is in self would might not be true, all the system would have some amount of damping. But let us say I do have a system which has very small value of damping that can be neglected.

Even given that fact, what will happen as the real system will starts to vibrate and the response start to become unbounded. What will happen if the displacement exceeds the certain value? For example, if it is a brittle structure like concrete, it will start to break.

Concrete will start to get damage and it will start to break at certain value. So, you do not get actually infinite response, but what happens for the brittle system as the concrete gets damage, its stiffness changes. So, your ω_n actually changes. So, although initially you had applied a response $\sin \omega t$, for which initially applied frequency ω become initial value of ω_n. It started the resonance and the system starts to get damage.

So, this ω_n changes. So let us say this is ω_n after some time $(\omega_n)_f$. So now this $(\omega_n)_f$ is different from the ω_n national frequency of the system. So basically, this system would come out of the resonance, because the frequency of the system itself changes due to unbounded deformation once it starts to develop.

Other solution could be if it is not brittle, but let us say it is very ductile for example, a steel building or some steel structure. Then it would yield at certain point, so again the stiffness would change and the changed frequency again would not be equal to the applied frequency and then it would come out of resonance and again we can go back to the original solution.

So, I hope this discussion give you some idea what happens in an undamped harmonic motion of a single degree freedom system where basically undamped system is an assumption, in reality all the system would have some damping.

So, in next lecture we are going to see what happens when the system has damping. We are going to look into mathematical formulation and then the physical interpretation of the result. So, this lecture we are going to conclude it here.

Thank you.