Dynamics of Structures

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Free Vibration
Lecture - 06
Damped Free Vibration Part – 1

Welcome back everyone. In last class, we started free vibration of a single degree of freedom system. In today's class, we are going to study how to obtain response or the Free Vibration response of a Damped System.

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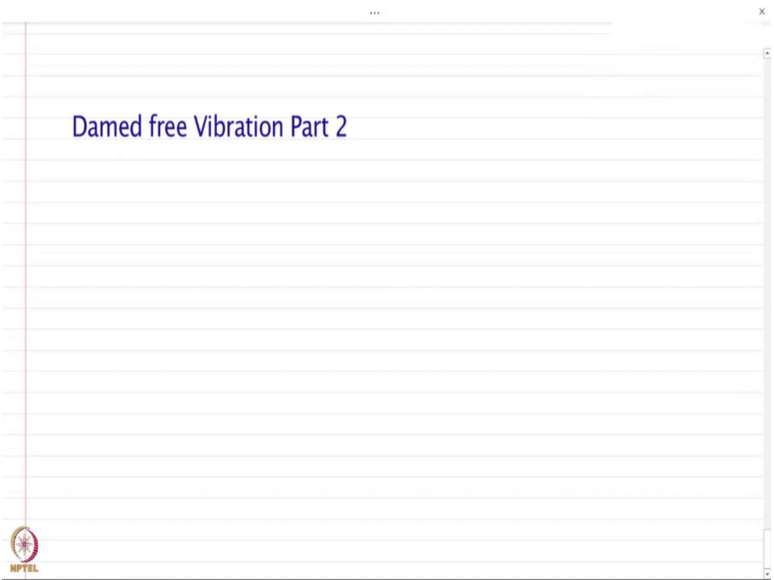

As you know in reality, most of the structure would have some amount of damping and in some cases; we can ignore it if the effect of damping is not much. But realistically, damping need to be considered in the response and we are going to see if we have a damped system, then how do we obtain the expression for the displacement and the amplification factors related to displacement. So, let us get started.

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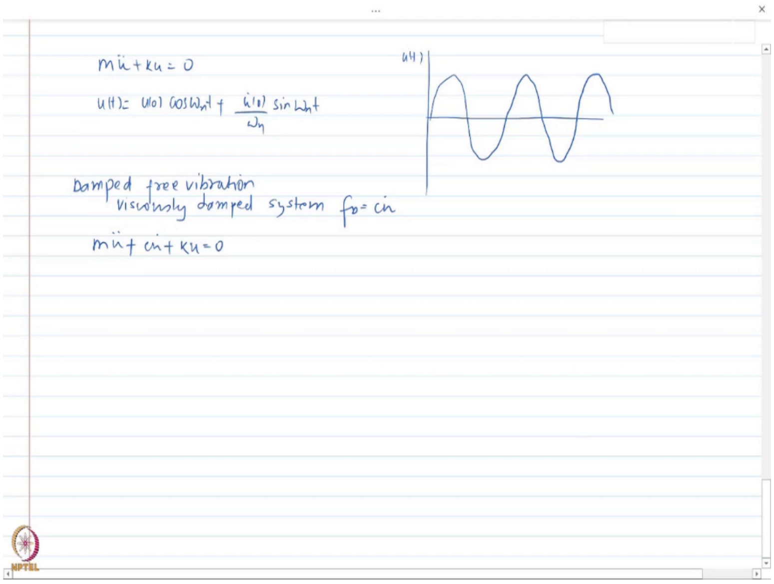

What we are going to study today's class? Last class, we saw that what is the equation of motion and how to solve the equation of motion for an undamped free vibration. So, we said that the equation of motion for undamped free vibration is nothing but this $m\ddot{u} + ku = 0$ and then, we saw that the response of this can be represented in terms of initial conditions as

$$u(t) = u(0)\cos w_n t + \frac{\dot{u}(0)}{w_n}\sin w_n t.$$

And we said that w_n is basically the natural circular frequency of the system right and we try to plot the *u(t)* and then, we try to understand and discuss what is the physical meaning of each of these parameters. Now, in this class today, what we are going to focus on is damped free vibration.

So, it is still a free vibration so, the external force will be zero. However, for this case there would now be a non-zero value of damping. So, we are going to consider damped free vibration. Now for damped free vibration, we are going to consider a specific type of damping that I have previously stated, that it is viscously damped system.

So, for viscously damped system as I have previously stated the damping force can be represented as a linear function of velocity. So, that my equation of motion that I need to solve it becomes $m\ddot{u}+c\dot{u}+ku=0$.

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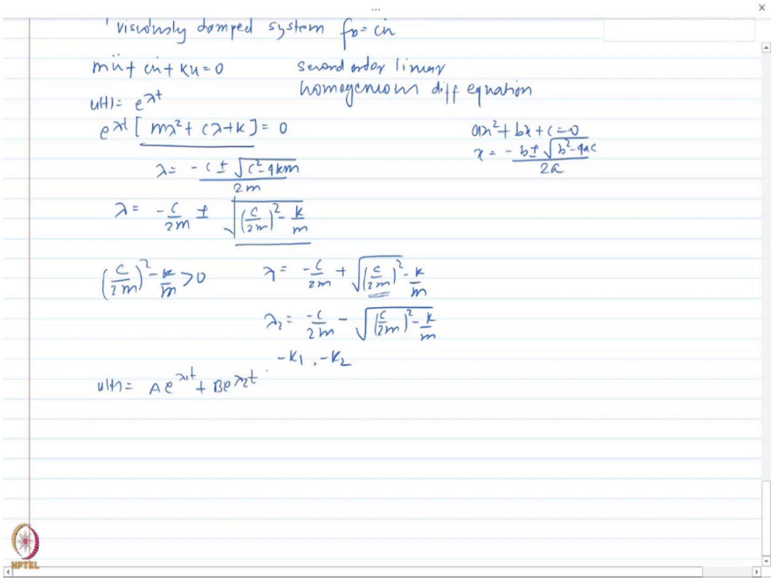

So, now today, we are going to solve this equation that we have here, and we are going to follow the same procedure. So, again it is a second order linear homogeneous differential equation, and the solution would take the same form as the last class. So, I can represent my solution u(t) as an exponential function $e^{\lambda t}$.

So, once I make that substitution back to this equation of motion or damped free vibration, I will get as $e^{\lambda t}(m\lambda^2+c\lambda+k)=0$. Now, $e^{\lambda t}$ cannot be equal to 0, that does not give me any feasible solution.

So, I am going to equate the expression $m\lambda^2+c\lambda+k=0$ and that is a quadratic equation and from your, like you know from your previous knowledge. The solution to this equation can be written as $\lambda=-\dfrac{c}{2m}\pm\sqrt{\left(\dfrac{c}{2m}\right)^2-\dfrac{k}{m}}$.

So, I have got two roots for the lambda corresponding to the positive and negative sign, ok. Now, if you notice here carefully, I have a term inside the root here. Now, this term would be positive, or this term could be negative and that would determine actually what or what kind of motion, resulting motion the system would have. So, let us consider the first case in which we say that, or we consider that

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0$$

So, what I am saying here this term here is actually greater than 0 that would mean λ_1 and λ_2 would be real, I will get two real roots.

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Now, if you look carefully at both of this; i.e., λ_1 and λ_2 are would-be negative values.

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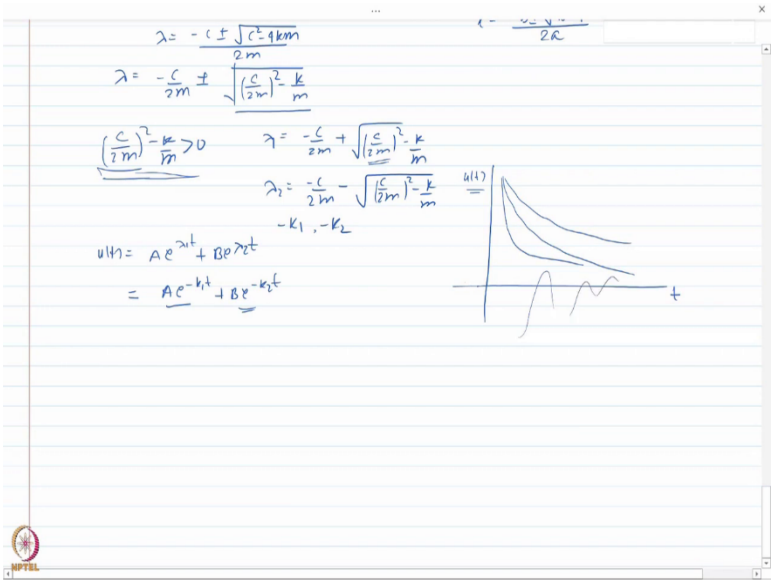

So, let us say that my two roots are actually $-k_1$ and $-k_2$ where k_1 and k_2 are positive numbers. So, the solution as I said for this type of equation is written as a linear combination of both roots. So, I can write it as $u(t) = Ae^{-k_1 t} + Be^{-k_2 t}$.

And if you look at this function both $e^{-k_1 t}$ and $e^{-k_2 t}$ are exponentially decreasing function. So, if I write it here or if I try to plot this function here, I have $u(t)$ and then, I have t, a function would look like something like this and like you know of course, depending upon the value of k_1 and k_2.

If k_1 and k_2 are very large, it would decay very fast if that they are not very large, it would decay much slower. So, the rate of decay actually depends on the value of $k_1$and k_2. However, one thing to notice here is that now, my damped system here, when this condition is there, when under the term under the square root is actually greater than 0, two real negative roots ok, I do not see any vibration so, do you see any oscillation here ok? If you plot $u(t)$ versus t, you see that it is just a monotonically decreasing function that goes to 0 at very large value of t.

However, there is no oscillation about the equilibrium position which let us say in this case was u equal to 0, if it had like you know oscillation, it would be something like this (Refer

Slide Time: 08:01). So, for this condition, I do not see any oscillation. So, let us check the other condition.

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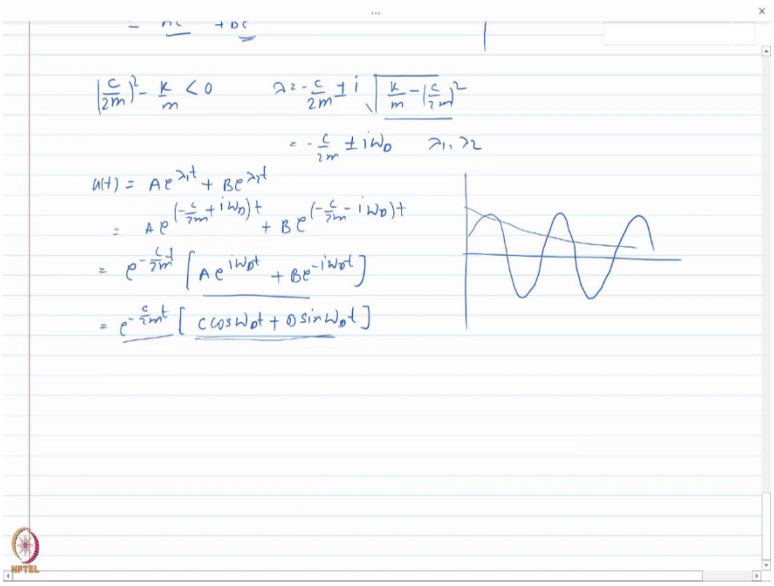

So, in other condition, I am assuming this quantity.

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$$

I can write

$$\lambda_1 = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = -\frac{c}{2m} \pm iw_D$$

We will see the physical significance of *wD*. But let us for the right now for the time being, assume that this quantity *(c/2m)* here is represented by another constant *wD* where *wD* is a positive real quantity.

Now, using this, now I can write my solution as for the u(t) remember that it corresponds to λ_2 and λ_2 corresponding to the positive negative sign. So, I can write $Ae^{\lambda_1 t} + Be^{\lambda_2 t}$.

If I substitute it here, I would get

$$u(t) = Ae^{(-\frac{c}{2m}+iw_D)t} + Be^{(-\frac{c}{2m}-iw_D)t}$$

$$u(t) = e^{-\frac{c}{2m}t}[Ae^{iw_Dt} + Be^{-iw_Dt}]$$

And if you remember for from your undamped free vibration, when we tried to derive the solution we get something like this.

$$u(t) = e^{-\frac{c}{2m}t}[c\cos w_Dt + D\sin w_Dt]$$

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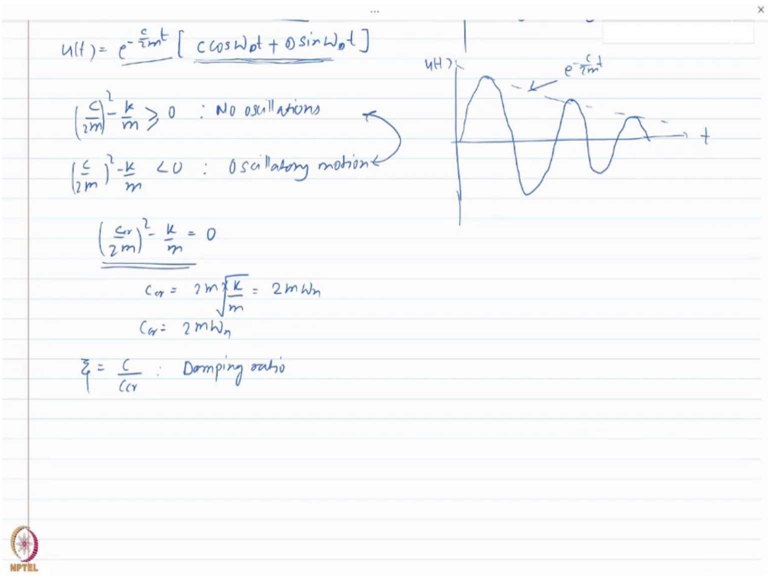

Now, I know that what is the plot of this function it is an exponential decreasing function. So, if I try to plot this function for the second case. I know that this function looks something like this from the solution of the free vibration.

However, it is multiplied with another term which is an exponential decreasing term. So, when you multiply it, the resultant function or the *u(t)* would look like something like this a function which is actually decreasing in the amplitude alright, but it is still sinusoidal or still harmonic should not say sinusoidal, but still harmonic.

Now, compare this to the function I had in the case where the term under the square root was greater than zero. I did not have any kind of oscillation here. However, in this case, I get an oscillation because of this term the *sine* and the *cosine* term that I have here. So, for this condition, when it is satisfied, I get an oscillation and I can represent my function or the response i.e., displacement response as this, where the amplitude is actually decreasing with time which is due to the damping of the system alright?

So, for a damped system, the amplitude is not constant and, the envelope curve, this can be represented as $e^{-\frac{c}{2m}t}$. So, it is decreasing at this rate alright. So, what I found out that the term that I have here

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0$$ no oscillation.

So, if you give initial displacement or initial condition, it would simply come back to its original position after a certain time. However, if this is a smaller than 0,

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$$ Oscillation

then I get oscillatory motion ok. Now, what would happen if this is equal to 0?

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$ No oscillation

Well, if this is equal to 0, I would again get two negative roots here.

Because this term would be 0 now so, the solution would still be similar to what I have here so, again no oscillation because there would be any complex terms so, there would not be any sine and the cosine term. So, it would not have any kind of oscillation. So what I can do if it is greater than or equal to 0, there would be no oscillation so, it would need to be strictly less than 0 or to have a an oscillatory motion.

Now, for this case, what I am going to do equate this term equal to 0, that the c that represent this condition is actually the damping coefficient right at which there is a transition from oscillatory motion to non-oscillatory motion and this c is actually called c-critical. So, at this critical value of damping, what you will see the transition from an oscillatory to non-oscillatory motion and I need to find out that value of damping coefficient.

So, I am going to equate it to 0 so, that it gives me the value of c critical as $2m\sqrt{\frac{k}{m}}$ and which I can further write it as $2mw_n$ remember that there is a square root term here. So, the critical damping coefficient can be written as $2mw_n$.

Now in reality, the actual damping constant can be smaller than this or it can be greater than c critical. So, I define a ratio ξ which is the actual damping in the system divided by the critical damping coefficient. So, the ratio of $\xi = \frac{c}{c_{critical}}$ is defined as ξ which is called the damping ratio or at many places you will also see that being referred as you know damping as a value of the critical.

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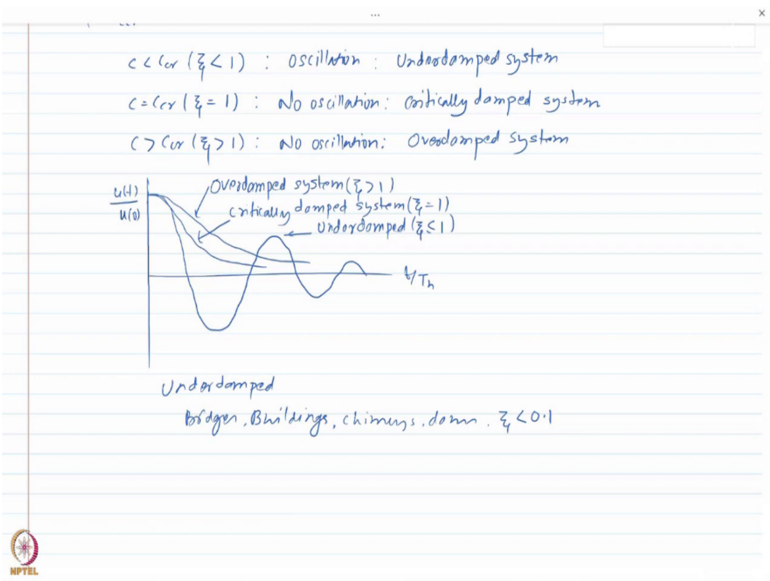

So, as you can see in terms of c critical or ξ, what did we deduce here? We said that if c is less than c critical or ξ is less than equal to 1, I would have oscillation and this is called under-damped system.

When c is equal to c critical or the damping ratio is equal to 1 ok, I would not have any oscillation so, no oscillation is called critically damped system. and when c is greater than c critical or ξ is greater than 1, I would again have no oscillation, and this is called over damped system.

And you know if I try to plot the variation of *u(t)* for these three types of systems. So, now I am plotting *u(t),* but now I am dividing or like you know plotting it as a ratio of the initial displacement that is given to it and this is t as a ratio of the time period of the system. So, what I would see for each of these let me just draw it.

(Refer Slide Time: 18:51) So, as we discussed, this is my under-damped system which provide me; which provides me the oscillatory motion so, under-damped ξ is smaller than 1. This is an over damped system. So, this is an over damped system so, ξ is greater than equal to 1 and this is again a critically damped system with ξ is equal to 1. So, this is the only three situations in terms of the resulting motion of a damped free vibration that are possible.

So, it could. So, if you have a damped free vibration, it would either be if it is an over damp system or if it is a critically damped system, there would not be any oscillation if you provided initial displacement. It would just come back to rest ok from an initial position or from the initial displacement.

However, if it is an under damped system ok, then it would oscillate and then, slowly or gradually it would the amplitude would decrease and then, come down to 0 and the rate at which the amplitude actually decreases depends on the value of ξ or the damping ratio. So, I hope that is clear to you.

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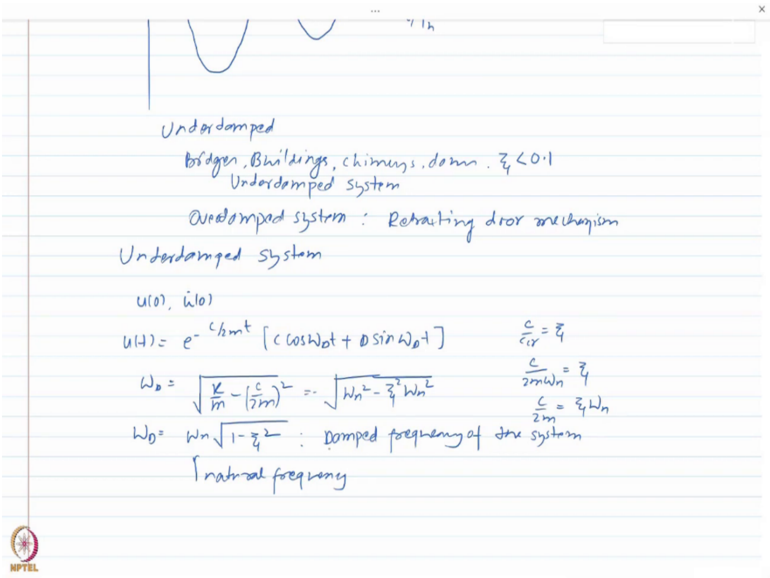

Now, in reality, most of the structure, the systems that are relevant to structural engineers like or the structures that are relevant to structural engineers like bridges, buildings you know most of the systems are actually under damped system.

So, let us say bridges, buildings ok, other type of structure like chimneys, dams etc. most of these structures have value of ξ which is actually less than 0.1 or 10 percent ok and those system can be categorized as under damped system and that would be the focus of study, in this chapter that we are we would be primarily dealing with under damned system.

But you could I mean you know in many cases; you can see the example an over damped system so, where an over damp system would be useful? If you think about it, any system that you do not want to vibrate, but you still want it to be flexible however, you do not want it to vibrate so, it would have stiffness, but it should not vibrate after you give its initial displacement.

So, one of the examples would be a retracting door mechanism. So, I don't know whether you have observed in many places in the classrooms or halls you have actually a piston kind of thing that is connected at the top of the door which when you push the door, it allows basically the door not to vibrate a lot because there are a lot of people coming like you know following you so, to make it safer, what do they have I mean that is one of the purpose to make it safer for other people coming from behind you or the other purpose is also to bring it back slowly to its original position.

So, that piston actually provides so much of damping that the system actually work as an over damped system because we do not want it to vibrate back and forth because once you push it, you want it to slowly come down to its original position. So, an example of over damped system would be retracting door mechanism.

And you know in similar scenarios, wherever you have something that you do not want to vibrate. But simply you want it to come back to rest after giving its initial displacement of velocity, or you would ideally like it to be an over damped system.

So, like you know I mean there are application of over damped system especially it would be more relevant to mechanical engineers or aerospace engineers. But for structural engineer for most cases like building bridges we are mostly focused on under-damped system and that we that is what we are going to study. So, we are going to study under damped system.

Now, we had obtained here the solution, the solution for *u(t)* here, but you can see there are two unknown constants so, those constants are c and d. So, again the question becomes how do I find that these constant? and that again I would utilize the initial conditions *u(0)* and $\dot{u}(0)$.

So, let me just write

$$u(t) = e^{-\frac{c}{2m}t}[c\cos w_D t + D\sin w_D t]$$

So, I need to now solve this equation for the initial condition. But before that, remember I said my w_D I am representing it as

$$w_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$w_D = \sqrt{w_n^2 - \left(\frac{c}{2m}\right)^2} = w_n\sqrt{1-\xi^2}$$

w_D is called the damped frequency of the system and this is different from the natural frequency of the system.

Note that I am omitting the term circular frequency and that you know at many places we will do that, we are I am not going to every time pronounce the whole damp natural basically, the natural circular frequency. So, if I say frequency in a refer, you just need to understand it like that. So, w_D is related to w_n, the damped frequency is related to natural frequency using this damping ratio.

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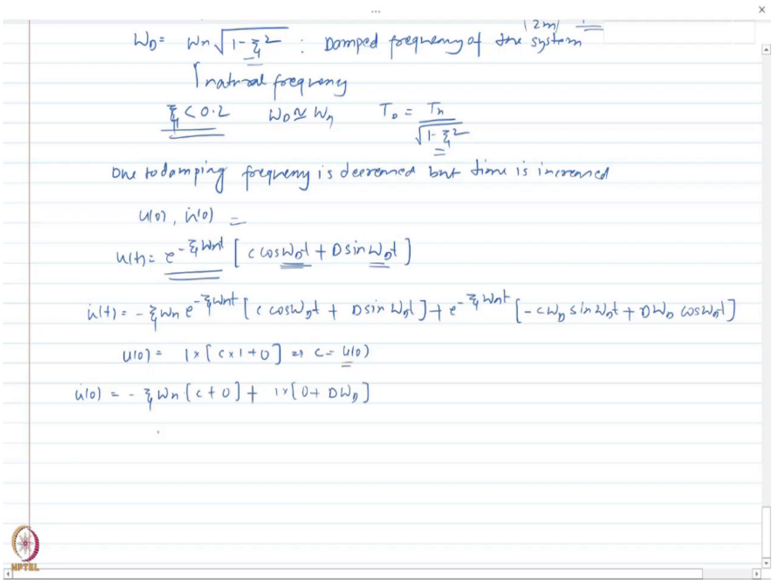

Now, for the under damped system, if ξ is less than 0.2 remember ξ^2 would be 0.04, w_D is approximately equal to w_n alright, keep that in mind it is approximately and I can also write that using this relationship, my T_D or the damped time period as

$$T_D = \frac{T_n}{\sqrt{1-\xi^2}}$$

So, if you look at these expressions for any non-zero value of damping, what I will have? The frequency is actually decreased, but the time periods is actually increased. So, due to damping, my frequency is decreased, but time period is increased and you know by whatever small amount it may be because as I said if ξ is smaller than 0.2.

The damped frequency is actually approximately equal to the undamped natural frequency, but theoretically it would actually lengthen the time period or increase the time period, but decrease the frequency. So, let us now plot the damped response versus undamped response and that we can do after I have solved this equation, the equation that I have here for the unknown constant.

So, let us do that I have these two unknown constants *u(0)* and $\dot{u}(0)$ and I am again going to write now, know that *c/2m* now can be written as ξ w_n. So, I am going to write

$$u(t) = e^{-\xi w_n t}[c\cos w_D t + D\sin w_D t]$$

Remember that the damped system is actually vibrating with $w_D t$, this frequency because sine and cose terms have w_D frequency. So, it is actually vibrating with w_D not w_n. However, the term here outside, the exponential term it is w_n.

So, many times, the students while writing the equation get confused that is why I am doing this derivation so that you do not have to remember anything, you can pretty much derive all the equation by yourself if you tend to forget it. Now, let us utilize this equation for that, first I need to do I need to differentiate this equation once.

So, I will; I am going to use differentiation by parts, I am first going to differentiate the exponential term, this term would still be the same and then, I am going to differentiate the parts that I have inside the brackets which would give me

$$u(t) = -\xi w_n e^{-\xi w_n t}[c\cos w_D t + D\sin w_D t] + e^{-\xi w_n t}[-cw_D \sin w_D t + Dw_D \cos w_D t]$$

So, let us substitute the value of the initial condition. So,

$u(0) = c$

$\dot{u}(0) = -\xi w_n [c + 0] + D w_D$

Hence, $D = \dfrac{\dot{u}(0) + \xi w_n u(0)}{w_D}$

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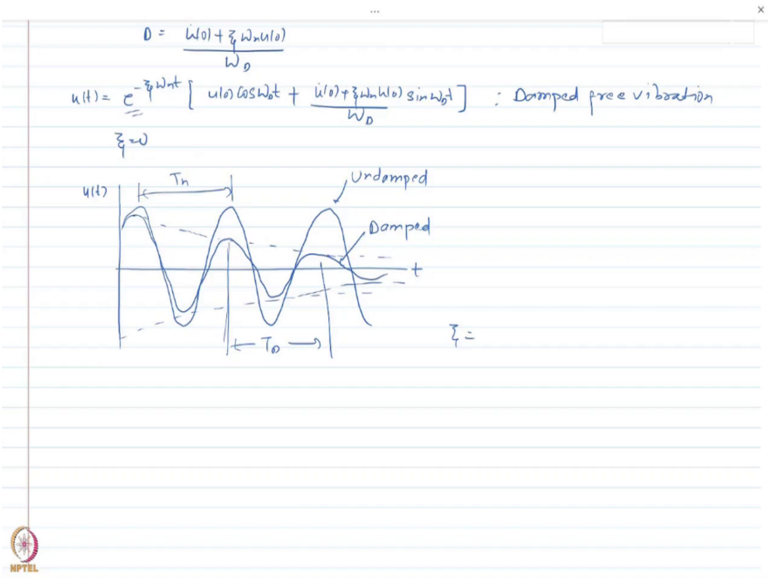

So, now I obtain both constant. So, finally, the expression for the displacement response for the damped free vibration can be written as

$$u(t) = e^{-\xi w_n t}[u(0)\cos w_D t + \frac{\dot{u}(0) + \xi w_n u(0)}{w_D}\sin w_D t]$$

So, this is the expression for displacement response of a damped free vibration.

And I mean you can look at; look at from here that if ξ is equal to 0 so, if you substitute the value of ξ equal to 0, this system actually transforms to three undamped free vibration right because this term would become equal to 1 and then, the second term would be equal to 0 and w_D would be basically w_n because ξ is 0 ok. So, that I can just verify from here.

But you know for undamped free vibration, there is no damping so, amplitude is actually constant, it is not changing. However, I have an exponential decreasing term which would be non-zero if the value of damping is non-zero so that amplitude of damped free vibration actually decreasing with the time.

So, if you plot it undamped free vibration or damped free vibration, it would look like something like this (Refer Slide Time: 33:03) let me just try to plot it here. So, let us say this is undamped free vibration so, constant amplitude and for the damped free vibration, it would be something like this.

So, if you see the amplitude is actually decreasing. So, this is undamped and this one is damped. One thing that you can observe from this plot that the time period T_n which for undamped let us said described it like this and for a damped, I will describe it using. Let me just describe it here these two peaks for the damp system here. Depending upon value of the ξ, you know they might be close, but they might be different that all depends on the value of ξ.

Now, comparing the response, we can see that if a system has very high value of ξ or very high damping, the amplitude would decrease much faster and it would come to the original position ok much faster. So, the time taking to come to the 0 position would be much smaller and if you keep on increasing that would reach a point where it would become critically damped and it would not vibrate at all.

So, once you understand this, you can obtain the response of the system, you can obtain the response of the system using the expression that we have derived. So, we are going to conclude this class at this point of time.

Thank you.