Dynamics of Structures
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Lecture - 05
Free Vibration

Welcome back everyone. In today's lecture, we are going to discuss what happens when a dynamic system undergoes free vibration.

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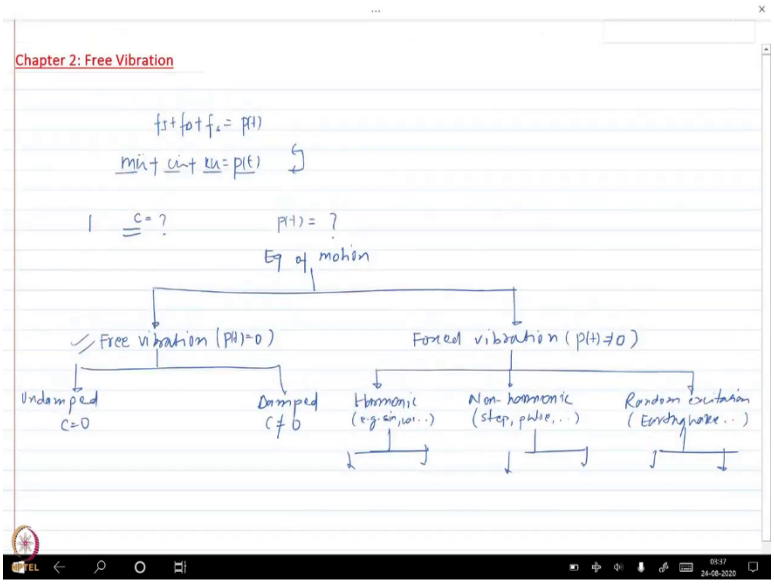

So, in the previous lectures, we have seen that when a structure is subjected to dynamic load, then how to set up the equation of motion. Now, in subsequent classes including today, we are going to discuss some special cases that can be analytically solved and start with the simplest case of dynamic loading which would be Free Vibration. In this case, a structure is assigned some initial condition like displacement and velocity and it is allowed to let vibrate without any external force acting on it. So, let us get started.

Till the lectures that we have done so far, we saw that how to set up the equation of motion of a system under the action of an external force and we derived that the equation of motion can be written as *f(I)+f(d)+f(s)* which is equal to the applied force and for a linear system, we saw that we could write it $m\ddot{u}+c\dot{u}+ku=p(t)$.

So, up to the last chapter, our focus was to derive or to finally come up with an equation of motion which is of the form mentioned above, for different types of systems. In summary, we had different systems acted upon by external loads and we saw that how to derive this equation of motion. Now, what we are going to do? We are going to study special cases of this equation of motion subjected to different types of vibration.

Now, depending upon what is the external load and what my system comprises, we are going to discuss each of these systems one by one and then, see what are the application of these types of systems and then, how do we physically interpret them. So, if you look at this equation of motion, we have three components and whether we have a value of damping zero or non-zero and what is our force of excitation based on these two important parameters, we are going to discuss the different properties of dynamic systems. So, for example, let me characterize what happens for each of these cases.

Referring to the hierarchy in the slide above, we saw that we had an equation of motion. Now when damping is zero, we had free or let us say when the applied force is equal to zero, i.e., p(t) is equal to zero, then, we will have what is something called free vibration. So, this is free vibration and under this free vibration depending upon whether my system has a zero or nonzero value of damping, we can either have undamped free vibration or damped free vibration. So, again, we are going to characterize between undamped free vibration and damped free vibration.

So, remember that this was for p(t) equal to applied force equal to zero and also c equal to zero and the other one is for c not equal to zero. So, this is the one categorization, one tree branch. In the other part, we are going to discuss the non-zero value of the applied force. In that case, we are going to discuss some specific cases. For example, what we are going to start with is, if we have a non-zero value of the applied force we will have forced vibration.

And depending upon the nature of the forcing function, we are going to study a few specific cases followed up by few generalized cases. So, we can have harmonic motion as a forcing function. When I say harmonic I mean something similar to or combination of sine and cosine and then, we could have non-harmonic motion. Non-harmonic could be either step functions or like you know pulse excitations or things like that.

And then, we will have random excitation and in random excitation, we have let us say earthquake excitation or seismic excitation. So, depending upon what is the nature of the forcing function, we are going to study all these topics. And then of course, as you know in each of these topics, we are again going to study the cases of damped and undamped vibrations.

So, again, we will have damped and undamped cases and we are also going to discuss that how to obtain the response of these systems. For example, if you have a harmonic excitation if it is feasible to obtain an analytical solution or a mathematical solution which is a closed-form solution. For non-harmonic cases for very few specific cases, we would be able to obtain analytical solutions. For random excitation, it is not possible like you know because the excitation cannot be represented as a simple, I mean a combination of functions that are amenable for the mathematical solution.

So, in that case, we are going to study how do we obtain a solution using numerical methods. Now, today, what we are going to do is we are going to focus on this free vibration.

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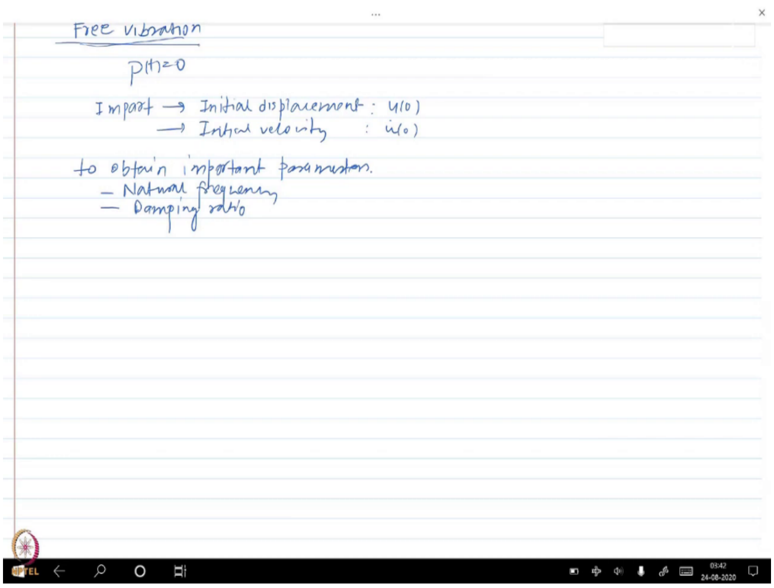

Now, as I said, what do you mean by free vibration? Free vibration happens when you impart an initial velocity or displacement to a system and let it vibrate without any action of external load. So, basically in short, what we will have as p(t) is equal to zero. So, this is the condition for free vibration without any action of external load, we let it vibrate freely.

And we will see that for this type of system, we could have either undamped free vibration or we could have damped vibration and we are going to study both of these cases today. Now, the question becomes that you know, how do we impart or initiate free vibration? so, one of the ways to do this is because we just said that there is no external force; the question is like in how does the motion start?

So, to start a free vibration, we impart either initial displacement to the system or initial velocity. So, initial displacement means u(0) or initial velocity would mean $\dot{u}(0)$, and like you know I mean you can see there is a lot of example for this type of system.

For example, if you take a pendulum and then, you lift it or if you provide initial angular displacement of letting us say θ_0 and let it vibrate. Then, it would go under the action of free vibration. So, that is one example. There are other examples as well.

For example, an initial velocity would be something for eg you take a hammer and then, you hit something, let us say an elastic system with that hammer. And then that would be imparting initial velocity because in that case, the hammer imparts a lot of force in a very small amount of time. So, it does not have time to react. So, it would not have initial displacement as such; but because of a large force in a short duration of time, it would acquire initial velocity and we are going to discuss that is called impulse and how do we impart that. So, these are the two conditions through which we can initiate the vibration in a system.

Now, this type of system or this type of motion in free vibration as I said because it is without the action of any external load. This type of system can be utilized or this kind of motion can be utilized to obtain important parameters of an oscillatory motion.

For example; to obtain important parameters basically, two important parameters can be obtained experimentally. So, two important parameters are basically what is the natural frequency and what is the damping ratio.

So, these two parameters can be obtained using the free vibration of a system. So, it is like you know it becomes very imperative that we obtain an analytical solution for the free vibration of a system. So, the first thing that we are going to do in this chapter would be undamped free vibration and then, we are going to go to and discuss the damped free vibration.

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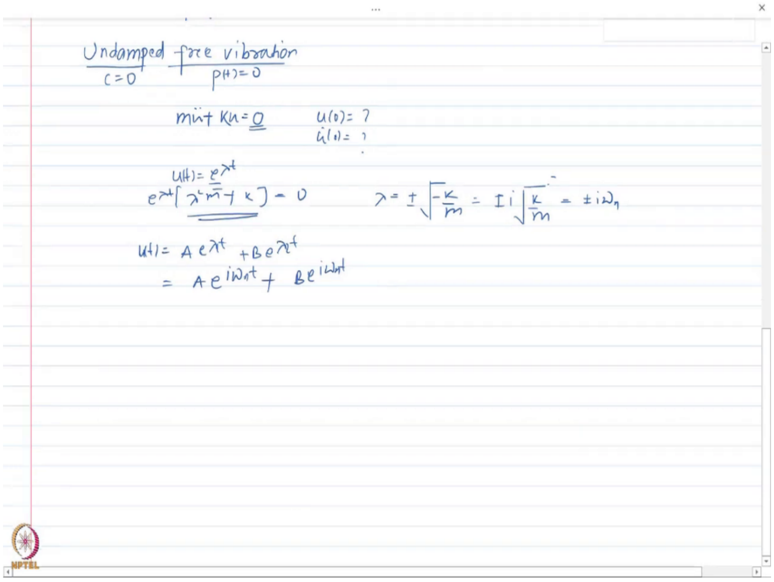

So, the first thing, we are going to discuss is undamped free vibration. So, remember, this is a free vibration and this is undamped. So, free vibration means p(t) is equal to 0; undamped means c is equal to 0. So, our equation of motion becomes what?

If you substitute these two equations of motion, I get this $m\ddot{u}+ku=0$. Of course, we need as I said, we need to have an initial condition; otherwise, if you solve this differential equation without an initial condition, you would get a trivial solution. So, the initial condition will need to be a non-zero initial condition either the values of u(0) and $\dot{u}(0)$ one of them must be non-zero to initiate the free vibration ok?

So, these two conditions need to be utilized. So, what we are going to do? First, we are going to solve this system and obtain the analytical solution for u(t) and then, we are going to get into the physical interpretation of the response for the free vibration.

So, the solution to a second-order linear homogeneous differential equation, remember that we now do not have any term on the right-hand side of this equation. So, we discussed that how to get the solution of a homogeneous differential equation basically, let me write my general solution; u(t) as $e^{\lambda t}$, and then, we are going to substitute it back to the original equation.

So, what I get after the substitution, should be equal to 0. Now, $e^{\lambda t}$ the term would never be 0; it would not give me any feasible solution. So, the only thing that can be 0 is what I have with the brackets here and that gives me the value of the λ.

So, there will be two roots of this quadratic equation, it would be $\pm\sqrt{-\frac{k}{m}}$, which I can write as $\pm i\sqrt{\frac{k}{m}}$; where, i represent $\sqrt{-1}$ as the complex number i and this is nothing but let me just fix the view.

Yeah, so that I can write this as $\pm i\sqrt{w_n}$ and we are going to discuss the significance of this w_n and this parameter. So, we are going to do that, but let us first substitute our roots to the original equation which is u(t) because it has now two roots, I am going to write my solution as a linear combination of $e^{\lambda_1 t}$ which represents the solution with respect to the first root and then, $e^{\lambda_2 t}$ which is corresponding to the second root.

So, I will get here as $Ae^{iw_n t} + Be^{-iw_n t}$.

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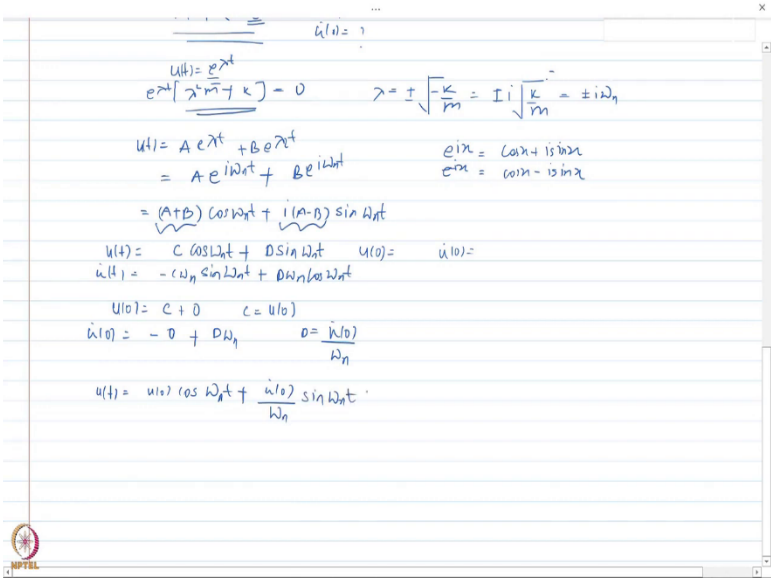

Now, if you remember from your complex numbers theory, what did you have? If you have e^{ix} , you could write it as *cos x+i sin x* and then, e^{-ix} can be written as *cos x-i sin x*. So, if you make that substitution in this equation here and then, again rearrange the term, you will get something like $(A+B)\cos w_n t + i(A-B)\sin w_n t$.

And again, these two parameters can be represented as another constant, the values of which need to be determined to obtain a specific solution to this equation. So, I am going to represent another constant $C\cos w_n t$ and then, $D\sin w_n t$. So, I have obtained the solution to this second-order homogeneous equation, linear homogeneous equation here. Now, I have two unknown constants and that we are going to determined using the provided initial conditions.

So, let us substitute those values. Before that, I need to differentiate it once, so obtain the expression for $\dot{u}$. So, if I do that, I will have as $-Cw_n \sin w_n t + Dw_n \cos w_n t$. So, let us substitute these values. So, u(0) is equal to c, and remember at t equal to 0, it would be 1 plus 0 because $\sin w_n t$ at t equal to 0 would be 0. So, I have obtained the value of c as u(0).

Now, $\dot{u}(0)$ would be equal to this term again would be equal to 0 plus Dw_n, and then, the value of $\cos w_n t$ would be 1. So, that gives me the value of D as $\dot{u}(0)$. If I substitute this to the equation that I had the expression that I had for u(t), I would get it as u(t)=

$$u(0)\cos w_n t + \frac{\dot{u}(0)}{w_n}\sin w_n t$$

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So, this is the solution of free vibration for the initial condition u(0) and $\dot{u}(0)$. And you know, of course, depending upon whatever the values of u(0) and $\dot{u}(0)$ you have, you can obtain the specific or like a numerical value of u(t).

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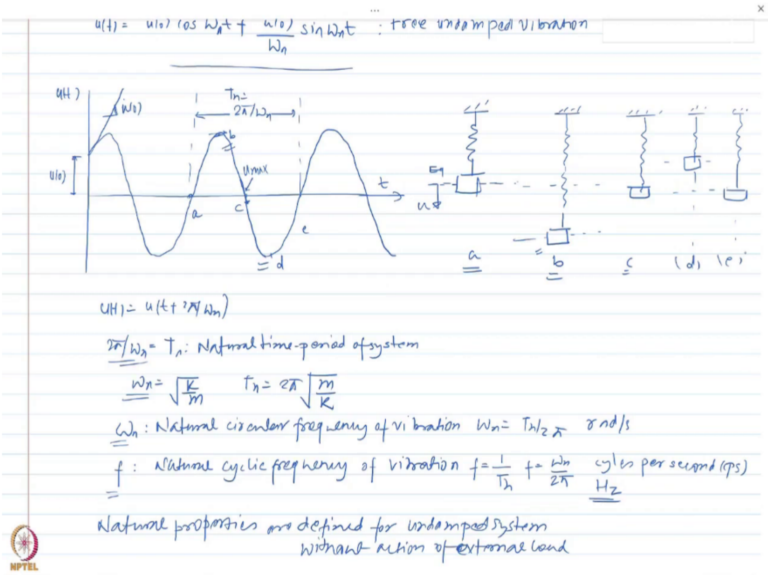

So, this is free undamped vibration. So, as you can see I have the expression for u(t), and let us just try to plot this function and see how does it look like. So, I have a cosine function and then, I have a sine function. So, if you plot it, if you plot this expression what I have here this and then, this is u(t). So, I can write it as, so if you plot it here, it would look like something like this (refer to the above slide).

So, what do I see here my resultant function is an expression for u(t), which is a harmonic function and also, a periodic function which repeats itself. So, it repeats after a specific time. Now, if you look at this graph here, I know that whatever the initial displacement at time t equal to 0 was u(0) and the slope, initial slope of this u(t) is the velocity which I am going to write it as a $\dot{u}(0)$.

Now, in this case, if you look at this or let me just write it like this. So, it has some kind of periodicity, this function here and this periodicity is actually if you saw this function, if you see the function here, it repeats itself after a or after $\frac{2\pi}{w_n}$ seconds.

And that you can prove by substituting or finding out the value of u(t) at time equal to t as well as $t+\frac{2\pi}{w_n}$ ok? You can find that out from this and you can prove that the values are still the same. Now, this quantity here $\frac{2\pi}{w_n}$ and then, all the time after which it repeats itself is called the natural time period of the system.

And remember that the value of w_n that we have obtained is $\sqrt{\frac{k}{m}}$. So, now, the value of T_n would become using this relationship here $2\pi\sqrt{\frac{m}{k}}$. So, as I said T_n is called the natural time period of the system and w_n is called the natural circular frequency of vibration.

So, it is called the natural circular frequency of vibration and as you can see from this relationship, w_n is related to T_n using some relationships discussed above. And the unit of

w_n is radian per second. And then, there is another parameter which is called the natural cyclic frequency of vibration and we represent it using f and it is called the natural cyclic frequency of vibration. And the unit of this f is first to let me just write down the relationship between f and w_n.

So, they are related as you can see f is first let me write in terms of the time period i.e. 1/T ok? So, this is the cyclic frequency of vibration and it is also related to w_n using this relationship here. And the unit of f is cycles, in many places you will see it is written as cycles per second or in a short CPS, but more commonly you will see that Hertz is being used as the unit of the natural cyclic frequency of vibration ok.

Now, we are using the word natural quite a lot. So, you just have to keep in mind, the significance of the word natural is basically that this is the natural means that it is without any action of external load and it is for an undamped system. So, these natural properties are defined for the undamped system, without the action of external load.

Now, it is very important to understand you know the physical significance of these parameters and what are the typical values that you would see for these parameters. Now, time period as I said, is the time required to complete a single cycle of motion.

Now, if I have this graph here and if I had to represent it, let us say for a system the spring-mass system that we have discussed. So, let me say this is *a* this is *b* and I can take that for like you know for any section of this vibration cycle that I see here. It would be *c*, *d,* and *e*. So, the motion is always considered about the equilibrium position, and for many cases like you know equilibrium position is also the initial position or the un-deformed position which is fine.

For example, when we considered the motion of a hanging the spring-mass system, hanging from in the vertical direction. For that, we said that let us consider the motion from this position. So, the initial position was equilibrium, and then, what we do either impart an initial displacement or initial velocity or a combination of both.

So, if I am measuring from the equilibrium, starts from a value of 2 equal to whatever the initial it is provided. So, that would be let us say case *a* here. Then, what happens? As u is increasing, it attains or goes to a maximum say in this case somewhere around here (at maximum amplitude). And this is the maximum displacement of the system due to initial velocity and displacement and this is the position *b* that is shown here.

So, this positions *a*, this is position *b* ok? So, it will go to this maximum position, and then, again it will start coming back to the initial equilibrium position. So, this would be again *c* here and then, it will go; when it goes to a maximum value of the displacement right here, u would be maximum; but the velocity would be 0 and that you can see from this cycle here, as it goes from *a* to *b*, you can see the displacement is maximum.

However, this curve is flat here which means that the slope of this expression is 0, where the slope is the velocity of this spring-mass system. So, the velocity is 0 alright and then, it starts coming back to its original position. And then, it passes to its equilibrium position at which its displacement again becomes 0.

However, it passes at the maximum velocity. So, at this point when your velocity is maximum and that velocity carries this system in the opposite direction. So, the negative u direction as you can see here. So, I am going to represent that let me say here. So, this is the state *d* which corresponds to the negative maximum, and then again, after it attains this negative maximum and starts coming back to its equilibrium position at which it completes a cycle of motion and this you know this keeps on repeating. So, this is the basic like you know the breakdown of free vibration of an undamped system.

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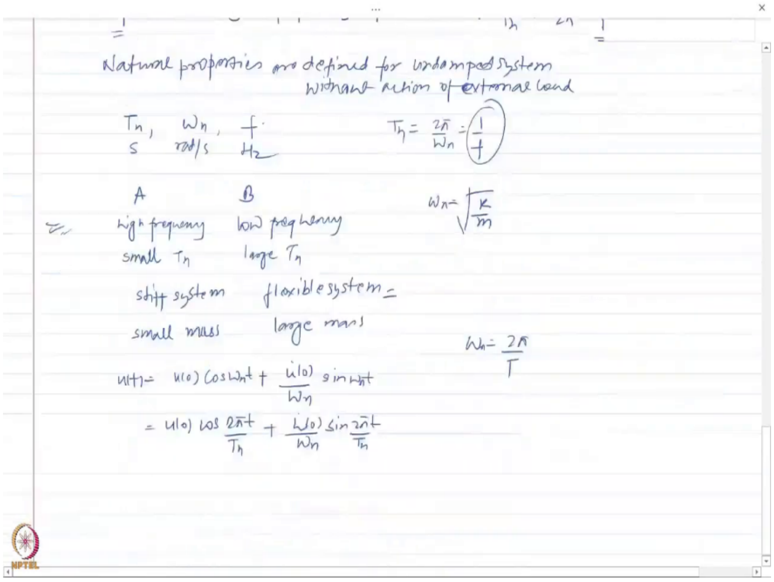

Now, let us come back to these three parameters that we talked about; the time period, the circular frequency, and then, the cyclic frequency and as I said that units would be second, here it would be radiance per second and here it would be Hertz. So, now, what do you think, if I have two systems a system A and a system B?

Now, system A, if I am saying to you that to see system A has a very high frequency; of course you know this would be in comparison with system B here. What are the things you can deduce from this information that has been provided to you?

If it has a system, A is a high frequency compared to system B, there could be multiple possibilities; one could be or like you know let us just list down those possibilities. If the frequency is high, remember that how are the frequencies, how is the time period connected to the frequencies? Time period is either $\frac{2\pi}{w_n}$ or it is also $\frac{1}{f}$. So, it is inversely proportional to the frequency.

So, that means, the time period would be small; small T_n ok, large T_n ok? What other things you can deduce? Remember that my w_n is nothing but $\sqrt{\frac{k}{m}}$. So, if the frequency is higher;

that means, it could be a stiff system compared to a flexible system and I mean this makes sense; is not it? If you consider a flexible system, can you physically imagine that a flexible system would take more time to come back to its or like you know finish the cycle of motion compared to a stiff system?

The stiff system if you give initial displacement, it vibrates at a very high frequency. High frequency means that there are a lot of or there like you know the higher number of cycles in a single second of motion, which is consistent with its definition here $\frac{1}{f}$.

Other possibilities for the given scenario would be instead of saying the stiff system, flexible system, for a given the same stiffness, this could also be possible if the mass is small here. So, let us write down a small mass, and here, it is a large mass. So, these are the few of the possibilities that might result in a high-frequency system; a system A with a high frequency compared to a system with B with low frequency.

So, this you have to keep in mind. If a system is stiff, that means that it would have high frequency; if a system has a very large mass, that means that it would have a low frequency. Or I can also say that if a system has a very large mass, that means or flexible system, then it would have a large time period of oscillation compared to a system with a small mass a nested system alright ok?

Once that is there, let me come back to our expression for u(t) again which we had obtained as $u(0)\cos w_n t + \frac{\dot{u}(0)}{w_n}\sin w_n t$. And remember that, we said that $w_n t$ I can write this similarly $\frac{2\pi}{T}$. Now, remember that we know from this graph, what is u(0), what is $\dot{u}(0)$, we have also found out what is the time period of vibration or the oscillation.

There is one more thing that is remaining to find out, remaining to be found out and that is the amplitude of the oscillation here. Let us represent this using u_0 ok. So, the amplitude would be basically whatever the maximum value of this expression that I have here.

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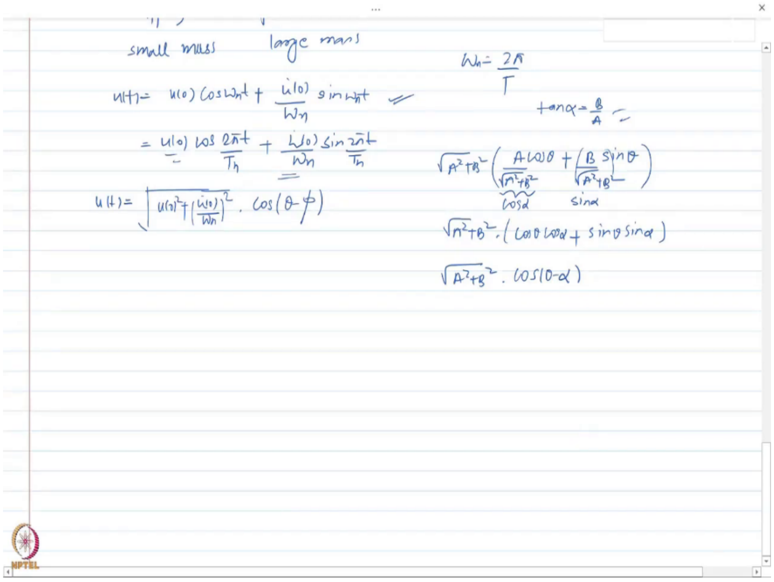

So, I need to find a maximum value of u(t). Let us see how do we get that. If you consider any function of type $A\cos\theta + B\cos\theta$. If you remember your trigonometric identities, what I can do here is let me multiply and divide by $\sqrt{A^2+B^2}$ and if you look at this carefully, if I consider $\tan\alpha = \frac{B}{A}$, then this expression is nothing but $\sin\alpha$ and this expression is nothing but $\cos\alpha$.

So, I can rewrite this further as $\sqrt{A^2+B^2}$ and this would become $\cos\theta\cos\alpha + \sin\theta\sin\alpha$, and from your trigonometric identities, I can write this as $\cos(\theta-\alpha)$. And that we can like you know substitute it back here. So, in this case, remember my A is u(0); B is $\dot{u}(0)$. So, I can write this whole expression as u(0) and $\cos(\theta-\phi)$ say I represent this instead of I will say this is phi ok?

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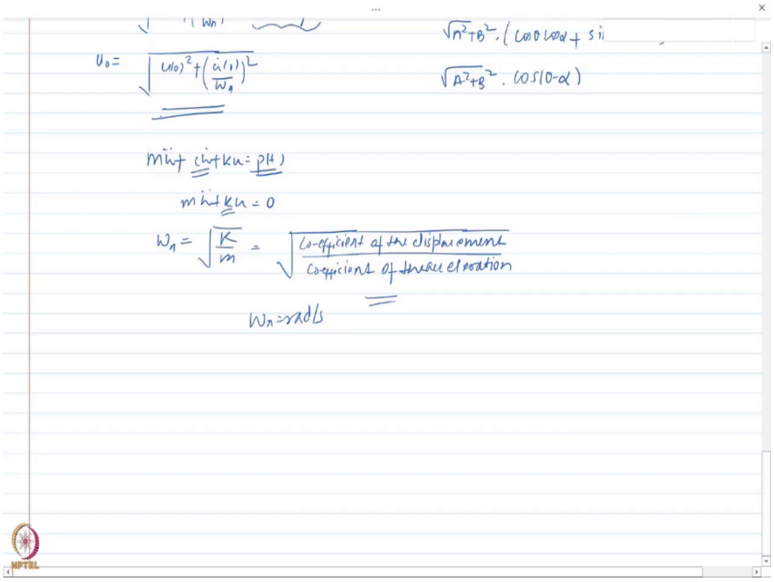

Now, I know that the maximum value of cos function is +1 so that the maximum value of u(t) would be let me write it as u_0 with this. So, the free vibration with initial displacement and velocity happens around its static equilibrium position with an amplitude which is represented by this here. Once that is clear, let us see how do we find out the important dynamic properties of the system.

Till now, we were deriving our equation of motion as $m\ddot{u}+c\dot{u}+ku=p(t)$. We said that for free vibration, this is equal to 0; for an undamped free vibration, this is equal to 0. So, a simple problem on this would be a system an undamped system would be given to you and you could be asked to find out what is the value of w_n which you have to find as $\sqrt{\frac{k}{m}}$.

Remember that there are multiple ways to obtain like the equation of motion in terms of different degrees of freedom as well. However, whatever you do, the w_n should be the same because remember w_n depends on the stiffness and the mass of the system.

It does not depend on the applied loads; it is a property of a system, it does not depend on the applied load. So, that you have to keep in mind. So, once I write this expression $w_n = \sqrt{\frac{k}{m}}$, you can obtain the equation of motion differently with different coefficients.

The point is what I am trying to make here is whatever the way you obtain your equation of motion, your w_n should be in the simplest form coefficient of the stiffness term or stiffness force. So, here k/m which would be the again coefficient of instead of saying the stiffness force let me just write it as the coefficient of the displacement.

The second term I can write as coefficients of the acceleration. So, using this, I should be able to obtain my w_n which is of ok as I said, it is in that radians per second and then, I can find out the time period of the system as well as the frequency of the system ok alright. I hope up to this point it is clear to you ok?

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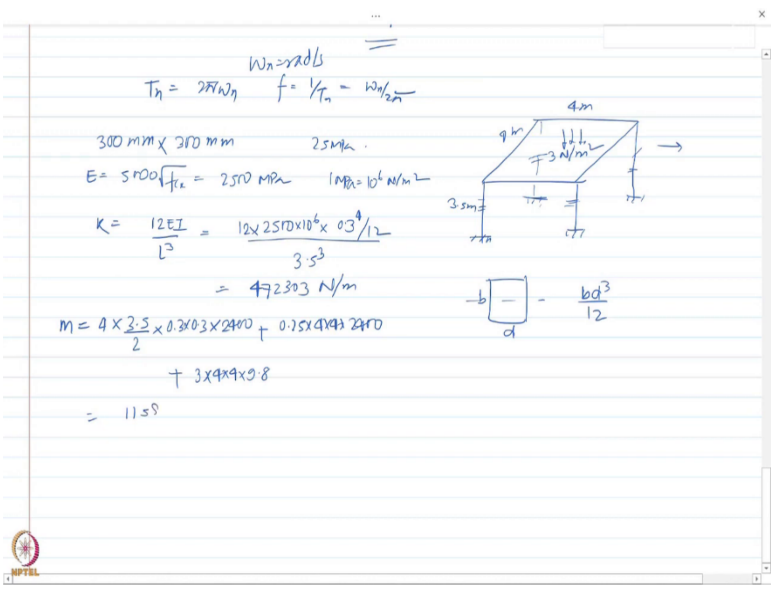

Now, what we are going to do? We are going to do an example, ok and this is not a problem from a book. What we are going to do? We are going to take typical properties of the values of a system of a building let us say with a column and like you know beam and then, see what is the value of the time period that we get ok? So, let us say a typical single-story building is what do you think is the typical size of a room would be?

It might be let us say 10 feet by 10 feet. Of course, if you live in Mumbai, then it might even be smaller; is not it? Anyway, so let us say I have this and I want to find out the time period of the lateral motion. There was a problem, remember in which we had considered torsional motion; but right now, we want to consider only the lateral time period of the system. So, let us see I mean there are 4 columns. A typical column could be 300 mm × 300 mm and there are 4 of those remember.

E of concrete let us say it is 25 MPa concrete. So, let us say it is 25 MPa concrete column. Remember that E is nothing but $5000\sqrt{f_{ck}}$. So, I would get as E as 2500 ok?; and if I consider these not to be actually let me just again write it or let us just say this is the fixed-fixed connection.

Although in reality, it is not like you know always fixed; like you know many times, it would be pin because to provide fixity, we need to do additional detailing and like you know provide rigid support to the ground. But anyway just for the sake of this calculation, let us assume it is a fixed rate connection ok?

So, remember that what was our stiffness in that case was $\frac{12EI}{L^3}$ and a typical story height you can assume as let us say 3.5 meters. So, if I substitute it here, I will get 12E which is 2500 MPa and I can write in 1 MP. Remember 1 MPa is nothing but 10^6 N/m^2. And what would be I of a section?

So, if I have a section like this bd, remember if it is bending about this axis, then it would be I would be $\frac{bd^3}{12}$. Now, in this case, both are the same square cross-section. So, I can it would be just $\frac{0.3^4}{12}$, that is my EI and the length cube is 3.5^3. And this needs to be found out. So, you can calculate this value. So, I get it as 472303 ok? and that would be let us say N/m.

Now, in terms of mass, what I could do? I could lump all the masses here ok. So, half of the mass of all this and remember this is for 1 column sorry. So, what I need to do is use 4 times which I let us say I will do in the final calculation. Let us first talk about the mass.

Mass, you can lump it at the roof level by assuming that half of the mass of these columns is lumped to the roof slab here. And then, the dead weight of the roof slab you can add it to that and then, you can also add the live load and all those would come to contribute to the total mass of this system.

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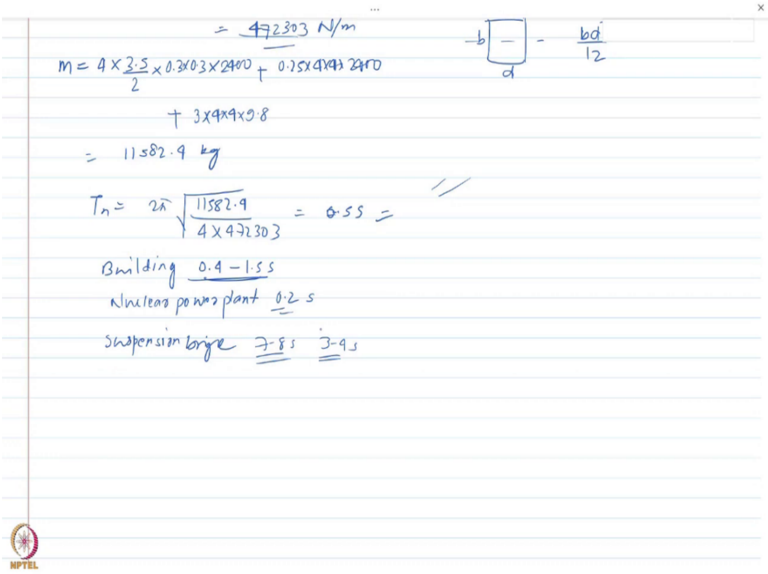

So, for the mass of the system, let us have four-column and I would assume that half of this is lumped. Now, remember that cross-section area is 0.3 × 0.3 and then, I would have to multiply this with 2400 which I am assuming as the density of the concrete. This is for the column; for the slab, let me assume like you know typical slab of 250 mm ok. So, in that case, the total would be, remember that the slab dimensions I can assume this has to be 4 m × 4 m.

So, it would be 0.25 thickness of slab times 4 × 4 and then, again 2400. So, this is the deadweight; but in reality, there is other type of things as well. For example, floor finishing is there, there might be like you know other things that are installed on top of it because I do not have information regarding that I am going to neglect that.

However, what I am going to do? I am going to assume a live load to this slab here ok, which I would assume as live load acting as 3 N/m^2 which is very typical. So, I will write it as 3 N/m^2 times the area, which I have here, and remember that I need to convert this to into mass unit system here.

So, this is Newton, so to convert into kg, I will multiply this with 9.8 ok? So, let us see how much I get? 4. So, I am getting as kgs. So, let us say if I use this, what do I get as time period be this is the mass and then, I have 4 of these columns here. So, I get it approximately as 0.5 second and as you like you know do more and more problems and calculate the time period, you would see that 0.5 seconds is very typical for a low to medium-rise building.

As the height of the building increases, the time period would increase; but using these typical values, we have obtained the time period to be approximately 0.5 seconds. So, depending upon what kind of structure it is, I will give you some typical values of the time period of different types of structure.

So, you know buildings depending upon whether it is a low and like you know low rise building or a high rise building, it could be between let us say 0.4 to 1.5 second actually for a low rise building. If it is a very stiff structure like a nuclear power plant or something like that, then it is a very stiff structure.

So, the time period would be very small; it is around 0.2 seconds. Your typical values are between 0.2 to 0.3 seconds or you could have a very flexible structure like a suspension bridge. And which could have like you know depending upon which direction, you are considering, the time period even like you know 7 to 8 second in 1 direction or like you know 3 to 4 seconds.

So, depending upon the direction, you could have like you know different time periods which are like of typical values like you know varying around 6-7 second or 3-4 seconds. So, you know it is a good practice to keep the values, these values in mind to reflect like you know what are the typical values of different types of systems for their time period and frequencies, just to get a feel of these numbers and these parameters alright? So, with this, I would like to conclude the lecture today.

Thank you.