Dynamics of Structures

Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Introduction to Dynamics of Structures
Lecture – 04
Equation of Motion

Hello everyone. So, welcome back. One of the most common encountered cases of dynamic loading on a structure is ground excitation due to an earthquake. And as a structural engineer, you will have to design a structure to resist the earthquake forces generated by the ground shaking. So, in today's class, we are going to learn how to set up the equation of motion of any structure, subject to ground excitation, and investigate further into that.

Then, we are going to look into a different type of loading which will demonstrate the effect of dynamic loading. So, we will be going to talk about how to apply a pulse load. So, if a load P_o is suddenly applied then we will see, what would be the dynamic effect on the structure.

And then, we are going to solve some problems in which we are going to simplify our structure to a spring-mass-damper system and then, solve for its response.

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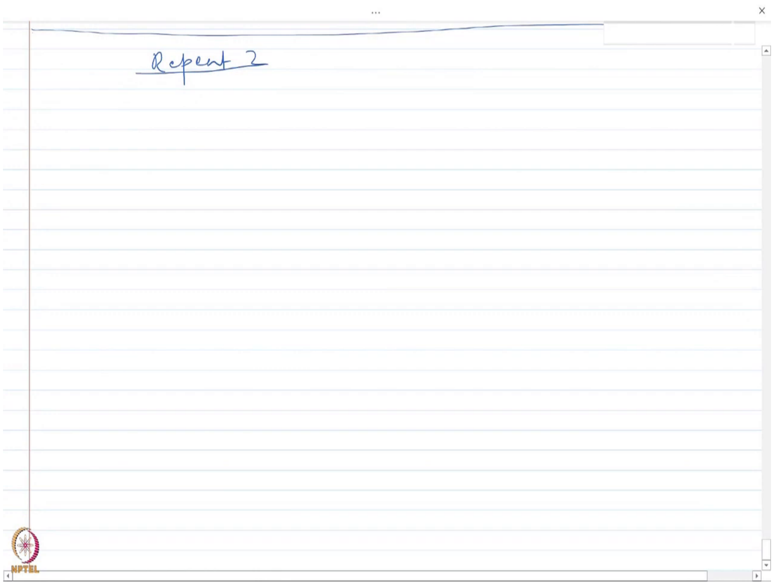

So, we are going to start our lecture today and in today's lecture, we are just going to be reviewing some of the concepts that we did in the last class and then, we are going to be solving some problems here. So, let us start with our equation of motion that we did last class. In the last class, we said that if we have, let us say equation of motion as $f_I + f_D + f_s = p(t)$. This was our equation of motion.

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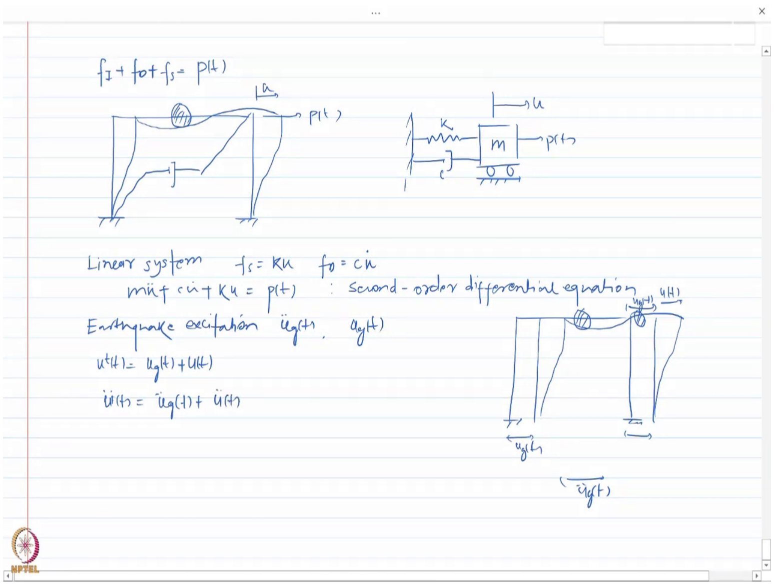

And we consider two representations. The first representation was a frame representation. This representation is better suited for people with a structural engineering background and we assume that the whole mass of the frame is concentrated at a level. And then, under the action of an external load $P(t)$, it deforms by deformation, let us say *u*, and if it is a damped system, then there was also a damper connected in between.

So, this was the first representation and the same representation could also be understood using another representation we discussed which was a spring-mass-damper representation. And a spring-mass-damper representation is used more frequently in mechanical vibration subjects or physics.

So, in this spring-mass representation, again I have the same mass *m* here, its stiffness *k*, damping coefficient *c* and under the action of this applied load here, it undergoes a displacement *u*. So, these two representations we considered had a linear system.

So, for a linear system, we said that by spring force or the stiffness force f_s would be simple ku and for linear damper, it would be $f_D = c\dot{u}$. So, that I can write my equation of motion as

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

And this is a second-order differential equation.

So, this was in general, when there is a force $p(t)$ that was being applied to the structure. Now, we also saw that if we had an earthquake excitation in which there was a ground; so for an earthquake excitation, the same equation can be adapted to be written in a different form. So, for an earthquake excitation, with ground acceleration represented by $\dot{u}_g(t)$ and a displacement represented by $u_g(t)$ here.

So, we saw that the total displacement of this frame that we had, there would be a rigid component due to, so rigid motion of the body due to ground displacement. So, this is $u_g(t)$ and then, there would be a further deformation due to the response of the structure. So, in this case, it would be something like this. So, initially, I had a mass here which finally, shifted it here.

So, just remember that this is $u_g(t)$ here and then, the displacement of the structure representing as $u(t)$ (relative displacement of the structure). So, the total displacement of the structure can be written as-

$$u^t(t) = u_g(t) + u(t)$$

and same would be the relationship, if you double differentiate it, you would again get as-

$$\ddot{u}^t(t) = \ddot{u}_g(t) + \ddot{u}(t)$$

So, this is the representation.

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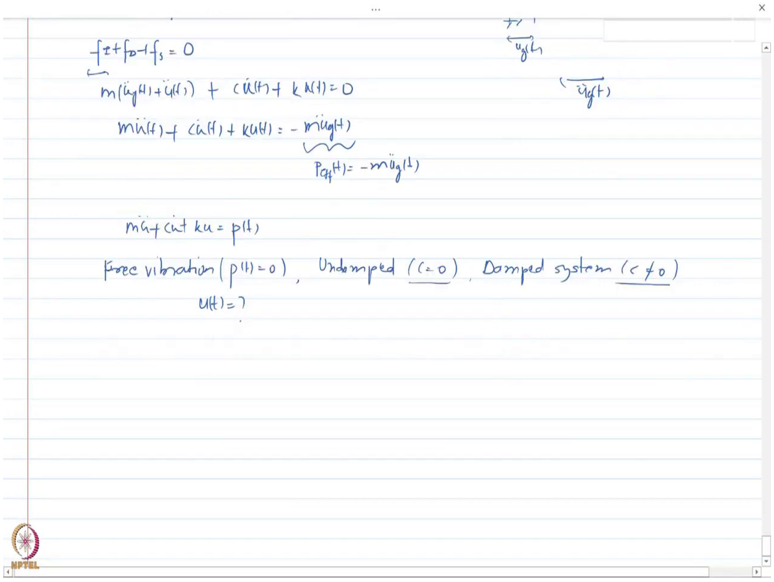

Now, similarly if we consider the free body diagram of this, remember that there is only f_I, f_D and f_S here and there is no external force as such that we had at the top of the structure *p(t)* -

$$f_I + f_D + f_S = 0$$

And we said that inertial force is always due to total acceleration however, the damping and the stiffness force is due to the deformation of the structure.

So, the rigid body motion which is basically $u_g(t)$ which is the rigid offset that is not going to produce any stiffness force in the structure because it does not produce any deformation in the structure. Similarly, for the damping, a damping is contributed by the velocity in the structure respect to its base and for that representation, you could only have a velocity if the structure is deforming and undergoing a let us say a velocity $\dot{u}(t)$.

So, that is why we could write it as-

$$m\left(\dot{u}_g(t)+\ddot{u}(t)\right)+c\dot{u}(t)+ku(t)=0$$

I will bring the $\dot{u}_g(t)$ on the right hand side, then write it as-

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=-m\ddot{u}_g(t)$$

This is basically the same as the equation that we discussed earlier. Except there is a difference that now we have a effective force (p_{eff}) here which is represented by the $-m\ddot{u}_g(t)$ and this is basically the earthquake force that acts on the structure.

So, earthquake force on a structure is acts opposite to the direction of the ground acceleration and it is mass of the structure times the ground acceleration $\left(m\ddot{u}_g(t)\right)$. So, just keep that in mind. So, the next step that comes that if I have a second order differential equation, how do we solve it ?

If I have this equation of motion-

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=p(t)$$

How do I solve this? So, depending upon whether it is a free vibration $\left(p(t)=0\right)$ or whether it is an undamped system $(c=0)$ or a damped system $(c\neq 0)$.

So, I could obtain different expression for $u(t)$. And that we are going to do in the subsequent chapter. When we are going to study free vibration and the response of undamped and the damped system subject to periodic and non-periodic loadings?.

But what I want to show you here, today is that through a simple example, how a load that is applied suddenly or you can call it like you know dynamically. How does it differ from the load that is applied statically and how would actually the response would differ ?.

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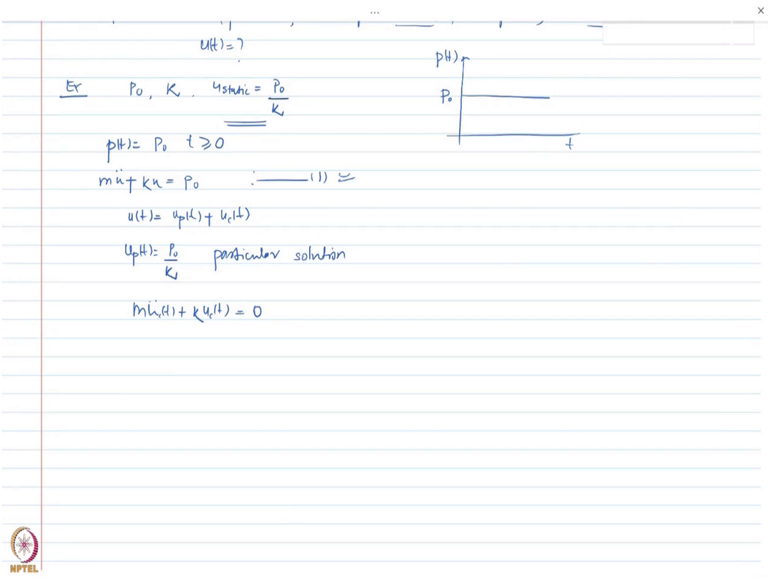

So, let me take an example here. In this example, I am basically applying a step force in which the load P_0 is applied, but it is applied suddenly. So, I have this time variation of the load and then, I need to find out what would be the response of the system. So, I am considering basically an undamped system. So, that $c = 0$, just for the sake of demonstration and I want to find out what is the peak displacement.

Now, if you remember from your undergraduate classes before, if you were told that there is a force P_0 in a spring that has a spring constant as *k*, the deformation it was set and I am going to represent as-

$$u_{static} = \frac{P_0}{k}$$

Now, let us see if I apply the system force suddenly and then, if I have available the time variation of the $P(t)$, then I am going to solve the equation of motion and then, I am going to see whether my dynamic response is different from this value and how much actually is that difference.

So, in this case, I have given the force $P(t)$ which is-

$$P(t) = P_0 \quad t \geq 0$$

It is an undamped system, so equation of motion becomes-

$$m\ddot{u}(t) + ku(t) = P_0$$

So, my goal is to solve this equation and we are going to this in few subsequent chapters, how do we solve this differential equation; I just want to give an overview that how do we solve the second order differential equation.

So, for a second order differential equation, the total solution $u(t)$ is represented by sum of a particular solution $u_P(t)$ and a complementary solution $u_C(t)$. Complementary solution is also called homogeneous solution and particular solution is any solution that satisfies the equation $m\ddot{u}_P(t) + ku_P(t) = P_0$.

So, any solution that satisfies the equation of motion can be used as a particular solution. So, in this case, if I substitute-

$$u_P(t) = \frac{P_0}{k}$$

then the double differential of this is –

$$\dot{u}_P(t) = 0$$

Put these value in the equation- $m\ddot{u}_P(t) + ku_P(t) = P_0$ and then, I get $P_0 = P_0$. So, this could be used as a particular solution and then, I have to find out complementary or homogeneous solution which is basically finding out the solution of this equation $m\ddot{u}_C(t) + ku_C(t) = 0$

So, we said the right hand side equal to 0 and that is why we call it a homogeneous solution. Now, we try to find out the value of for the expression for $u_C(t)$.

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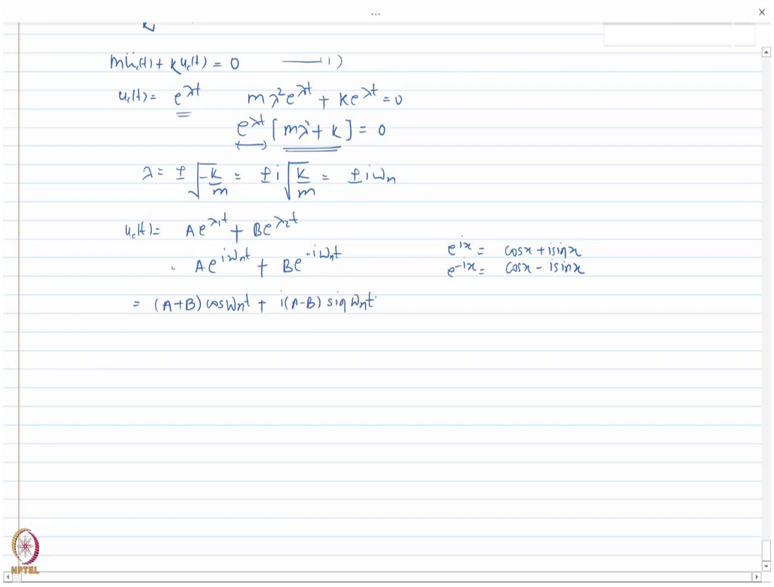

So, in this case, I am going to assume $u_C(t) = e^{\lambda t}$ and the double differential of this function is $\dot{u}_C(t) = \lambda^2 e^{\lambda t}$ .Now, I substitute $u_C(t)$ and $\dot{u}_C(t)$ in the above equation-

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0$$

$$e^{\lambda t}\left(m\lambda^2 + k\right) = 0$$

Now, I know that this term $e^{\lambda t}$ cannot be equal to 0. It does not give me any feasible solution. So, what I am left with this expression $\left(m\lambda^2 + k\right)$ here. If I equate that to 0, I get-

$$\lambda^2 = -\frac{k}{m}$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n$$

ω_n is basically the natural frequency of the system that you are considering. We are going to come back to that in a later chapter. Just right now, you like you know interpret as any other constant through which I am representing this quantity $\sqrt{k/m}$.

So, I have two roots here;

$$\lambda_1 = i\omega_n and \lambda_2 = -i\omega_n;$$

So, if I have two roots, the solution of this complementary this equation right here can be written as a linear combination of both roots. So, I would write that as remember that I had assumed this as to be the solution.

So, linear combination of the two roots is-

$$u_C(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

$$u_C(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}$$

And I also remember from my knowledge of complex number that I can write-

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

So, I will do that here and then, rearrange the terms to obtain-

$$u_C(t) = (A+B)\cos\omega_n t + i(A-B)\sin\omega_n t$$

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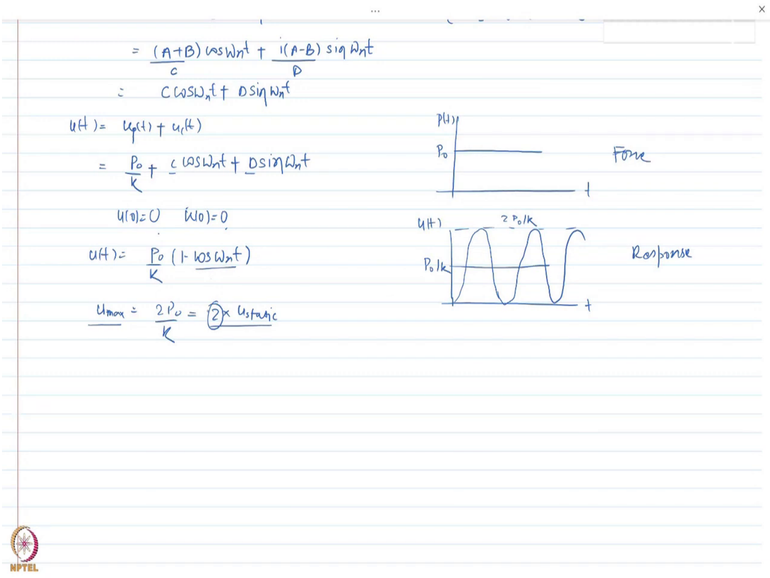

So, these two $(A+B)$ and $i(A-B)$ are unknown coefficients that I need to find out, I am going to represent them with two unknown coefficients here and I can write it as-

$$C = A + B \text{ and } D = i(A-B)$$

So, this is the complementary or the homogeneous solution.

$$u_C(t) = C\cos\omega_n t + D\sin\omega_n t$$

So, the total solution of the initial differential equation, I can write it as-

$$u(t) = u_P(t) + u_C(t)$$

$$u(t) = \frac{P_0}{k} + C\cos\omega_n t + D\sin\omega_n t$$

Now, you see that we consider a second order differential equation and now, we have two unknown constants. So, we need two conditions to actually find out the specific solution to our differential equation and where those two conditions are going to come from?

Well, it comes from the initial conditions for a dynamic problem. So, the initial conditions in a equation of motion would be your initial displacement $u(0)$ and initial velocity $\dot{u}(0)$. And in this case because it is given that initially they were at rest, suddenly a force is applied.

So, I can assume, that $u(0)$ and $\dot{u}(0)$ both to be equal to 0, for my case. So, we can substitute that in this equation here to obtain the value of the constant C and D and you will get the final solution as-

$$u(t) = \frac{P_0}{k}(1 - \cos\omega_n t)$$

So, this is the solution that I have, due to the step force that is being applied to my single degree of freedom system. So, let me just write it again here or redraw it again. So, I have *P(t)* here and then, I have *t* here and this is P_0. Now, remember had it been a static problem, then my u_{static} would have been just P_0/k.

So, if this is my *u(t)* and so, this is the this is the applied force (Refer Time:19.15) and this is the response (Refer Time:19.21). For a static problem, I would have this displacement as P_0/k, but I can see that my dynamic displacement actually is not P_0/k, but it actually fluctuates due to time variation of $\cos\omega_n t$ and if you try to plot that, it would look like something at *t* equal to 0, your displacement is 0 and at $\omega_n t$ equal to π, what you will find? this term would be minus 1 so that we will get $2P_0/k$.

So, if I draw this is a time displacement, this value is here is $2P_0/k$ which is the maximum dynamic displacement. Which would occur when $\cos\omega_n t = -1$. So, I can write $u_{max} = \frac{2P_0}{k}$ which is 2 times the u_{static} that I had determined.

So, you can see that if I had applied the load P_0 statically or for a static problem, I would get a displacement which is equal to P_0/k. However, if I apply the same load suddenly like a step force, then I am getting a peak displacement which is 2 times the static displacement.

So, this is the effect of the structural dynamics that is coming into the play here and this 2 is as you would study later it is called dynamic amplification of the response and there are different factors to define it.

So, this example was just to demonstrate that if a load P_0 is applied statically or if it is applied dynamically, then your response or the peak responses are actually different and for this specific case, it was 2 times the static displacement.

So, I hope this problem give you a better idea of what the dynamic nature of problem could do to the response of a single degree of freedom system. Alright, if that is clear to you, what we are going to do now? We are going to solve some examples related to this type of equation of motion.

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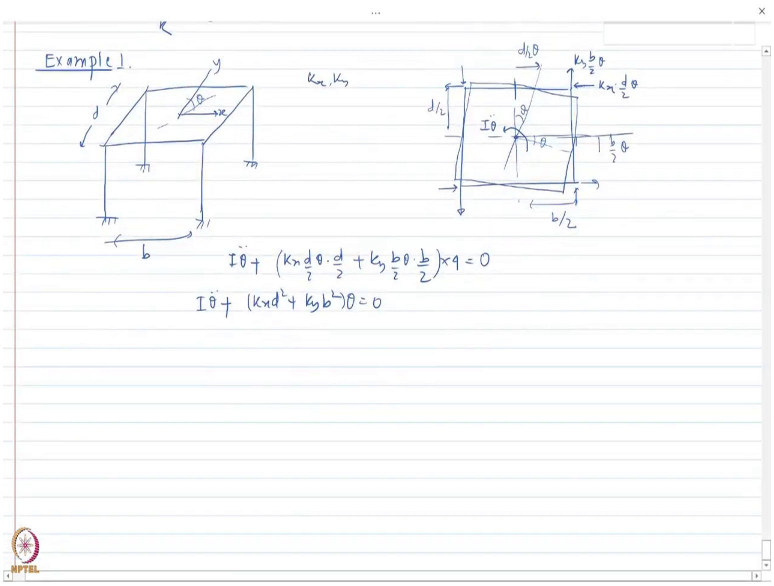

Then, see how do we tackle different problems. So, in example one here, what do I have here is actually a slab that is supported on 4 columns. The dimension of this slab is b in this direction and d along this direction.

And it is given that these four columns with respect to these axis is x and y have its stiffnesses k_x and the lateral stiffness k_y in the two perpendicular direction.

So, each of these columns would have these stiffnesses in two direction and what do you need to do? For a small angular deformation of the slab in plane, so let us say slab is actually rotating like this. In plane, you need to set up the equation of motion for this slab here alright. So, I would like you to pause this and then try to solve this problem alright.

Let us discuss the solution of this problem. So, let us consider the top view of this slab that I have here, slab let me just draw it like this. So, this is the slab that I have and if it has an angular rotation, let me again draw the deflected position of the slab (Refer Slide Time 23.33) which would be I just redraw it here, would be something like this. Sorry for the bad drawing. I mean figuratively like you know it does the job.

So, what is happening here, it is rotating by θ here. So, that at all points, it is rotating by θ with respect to its original position. Now, remember as it rotates at any corner, the horizontal displacement is the same displacement that I have here which is represented by the half of this length which is $\frac{d}{2}$ times the angle (θ).

So, this horizontal displacement is $\frac{d}{2}\theta$. Similarly, the vertical displacement at this point is nothing but the half of this length which is $\frac{b}{2}$ times this angular displacement (θ), which is $\frac{b}{2}\theta$ and same would be the displacement at all the 4 corners.

And like you know, if you want you can derive the same thing by considering angular displacement by plotting a line and then, doing it like this. This is just a simple way of considering that whatever the horizontal displacement at this point here would be the same horizontal displacement at this point and the vertical displacement of this point would be the vertical displacement of the slab at horizontal position.

Now, when I tries to rotate it like that, what will happen is there the column would apply a force in this direction which would be k_x times the horizontal displacement which is $\frac{d}{2}\theta$ and in this direction which would be k_y times $\frac{b}{2}\theta$ and same would be the case at other corners as well.

For example (Refer Slide Time: 25.58), here it would apply in this direction and in this direction and at this point, it is going to apply in this direction and in this direction and at this point, it is going to apply in this direction and in this direction. And the one thing that is common that at each corner, each pair of these forces are actually creating clockwise moment about this point the center of this slab.

So, if I want to set up the equation of motion, I need to sum up the moment due to all these forces that are being applied by the column on the slab at all of its corners and then, there is another force or the moment basically, the inertial moment due to rotation of the slab which would be opposite to the direction of rotation and it would be $I\ddot{\theta}$. Then, take the moment of these column forces that are being applied on the slab.

So, it would be-

$$I\ddot{\theta} + \left(k_x \frac{d}{2}\theta \times \frac{d}{2} + k_y \frac{b}{2}\theta \times \frac{b}{2} \right) \times 4 = 0$$

Where, force in x and y direction is $k_x \frac{d}{2}\theta$ and $k_y \frac{b}{2}\theta$ respectively. Lever are in x and y direction is $\frac{b}{2}$ and $\frac{d}{2}$.

As you can see it is producing the same moment at all the 4 corners. So, I will multiply it with 4 to get the final equation. So, I can write it as final equation of motion as

$$I\ddot{\theta} + \left(k_x d^2 + k_y b^2\right)\theta = 0$$

So, this is my final equation of motion for this given problem alright. Once that is clear, let us discuss another example.

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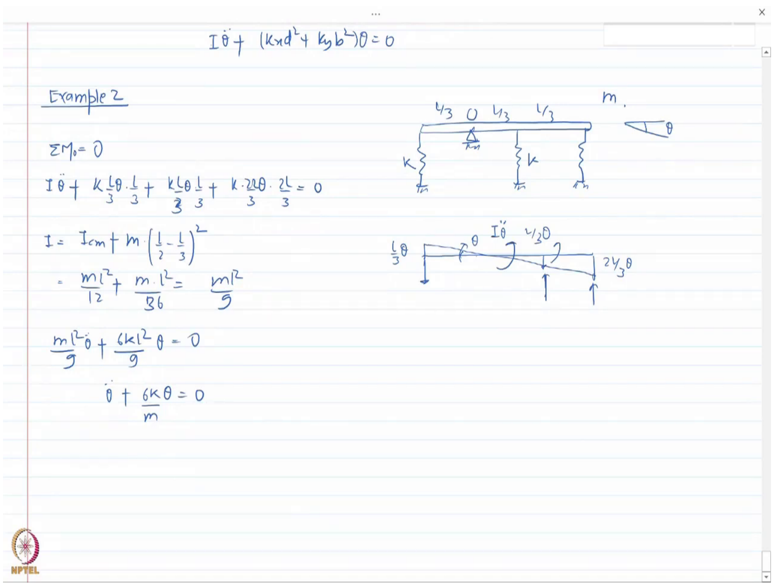

Example 2 here. In this example what do I have? I actually have a rigid bar here. So, let me draw a rigid bar. It is connected to a pin connection, at one-third of its length . And there are three springs each of its stiffness k connected to this rigid bar at length $L/3$, $L/3$ and $L/3$ from each other.

And what I need to do again is to set up the equation of motion for an angular rotation (θ) of this bar and the mass of the bar is m and as I said the total length is L. So, take some time and solve this problem. Let us discuss the solution to this problem. First let me draw the initial position and the deform position for an angular rotation (θ) .

So, let us say it rotates in clockwise direction by an angle θ. Due to this rotation, the springs at the left most end would be stretched by a deformation and I can find that out as $\frac{L}{3}\theta$ and this second spring is going to compressed by $\frac{L}{3}\theta$ and the third spring would be $\frac{2L}{3}\theta$.

So, in terms of forces, this spring is going to apply downward force and these two springs are going to apply upward forces like this. And if you consider the free body diagram due to clockwise rotation, there would be an inertial moment which would be applied anti clockwise. And that would be the moment of inertia of this rod about the point of rotation (I) times $\ddot{\theta}$.

So, if I take an moment of all these forces and equate it to 0 about the point of rotation O which is the pin support. Let us see what do I get.

The moment of all these three spring forces are in the same direction, the counter clockwise direction. So, it would be $k\frac{L}{3}\theta$ which is the spring force times the lever arm about the point of rotation is $\frac{L}{3}$.

Again, for the second spring $k\frac{L}{3}\theta$ times $\frac{L}{3}$ and then, the right most spring would be $k\frac{2L}{3}\theta$ times $\frac{2L}{3}$ and that should be equal to 0.

$$\sum M_o = 0$$

$$I\ddot{\theta} + k\frac{L}{3}\theta \times \frac{L}{3} + k\frac{L}{3}\theta \times \frac{L}{3} + k\frac{2L}{3}\theta \times \frac{2L}{3} = 0$$

Now, the I of this bar about its about the point of rotation would be-

$$I_{rotation} = I_{center} + m\left(\frac{L}{2} - \frac{L}{3}\right)^2$$

$$I_{rotation} = \frac{mL^2}{12} + \frac{mL^2}{36} = \frac{mL^2}{9}$$

So, I will substitute that in the equation-

$$\frac{mL^2}{9}\ddot{\theta} + k\frac{L}{3}\theta \times \frac{L}{3} + k\frac{L}{3}\theta \times \frac{L}{3} + k\frac{2L}{3}\theta \times \frac{2L}{3} = 0$$

$$\frac{mL^2}{9}\ddot{\theta} + \frac{6kL^2}{9}\theta = 0$$

$$\ddot{\theta} + \frac{6k}{m}\theta = 0$$

So, this is the final equation of motion for this system that is shown here.

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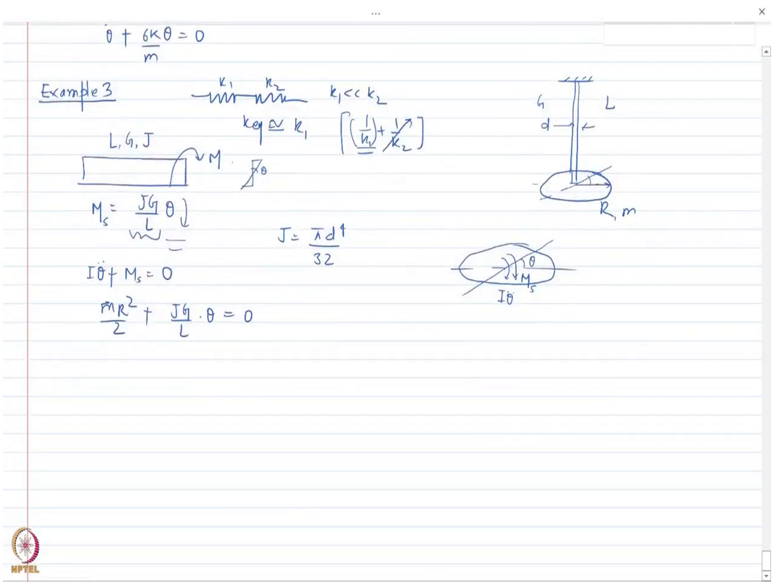

Let us go ahead and do example 3. So, in example 3, what do I have? It is not a very difficult problem, but I just want to demonstrate you the concept of rotation and twisting. So, in this case, what do I have? I have a thin shaft here, which is rigidly connected to a disc. The polar moment of inertia of this shaft can be calculated from its diameter *d*.

The radius of this disc is *R*, the mass of this disc is *m*, the shear modulus of the shaft is actually *G*. Let me see do you need anything else? Yeah, you need the length (*L*) of the shaft. Now, can you imagine if I try to provide twist to this disc? What is going to happen? Remember that disc in its own plane is very rigid compared to the twisting stiffness of the shaft.

So, the all the twisting is going to happen in the shaft and if you remember, if I have 2 spring in series, let us say k_1 and k_2. Let us say k_1 is less than k_2. So, k_{eq} is actually decided by the most flexible spring. In this case, it would be approximately k_1. And that you can find out from the expression $\frac{1}{k_1}+\frac{1}{k_2}$.

If k_1 is very very small than k_2, this quantity $\left(\frac{1}{k_1}\right)$ would be very very large to this quantity $\left(\frac{1}{k_2}\right)$ so that you can neglect it. So, that when you invert it, k_{eq} becomes equal to k_1. So, in this case, disc own stiffness torsional stiffness is much higher than the shaft stiffness and that is why I am just going to consider the whole resistance is coming from the shaft.

And if you remember from your solid mechanics class, if I have a shaft of length *L* t, shear modulus *G* and polar moment of inertia J. If I apply a twisting moment (M_s) on this shaft, the relationship between this twisting moment and the twist in the shaft. So, if I project it here, let us say this is deforming by θ. This I can write it as-

$$M_s = \frac{JG}{L}\theta$$

Which is the rotational stiffness or the twisting stiffness of this shaft here.

So, if you consider the free body diagram of this disc and then, assume that it is rotating by θ, remember the same would be the rotation in the shaft. So, shaft is going to apply and equal and opposite moment which I have here which would be M_s of this shaft.

Let us say it is M_s and because I am rotating it by θ, what will happen? I need to again include in the free body diagram a rotational inertia which would be whatever the mass moment of inertia (I) of this disc times $\ddot{\theta}$. So, if I take moment of all the forces about center, I can write it as

$$I\ddot{\theta} + M_s = 0$$

$$\frac{mR^2}{2}\ddot{\theta} + \frac{JG}{L}\theta = 0$$

If you remember, the polar moment of inertia of a shaft is given by $\frac{\pi d^4}{32}$. So, that you can use here alright. So, this was our third example.

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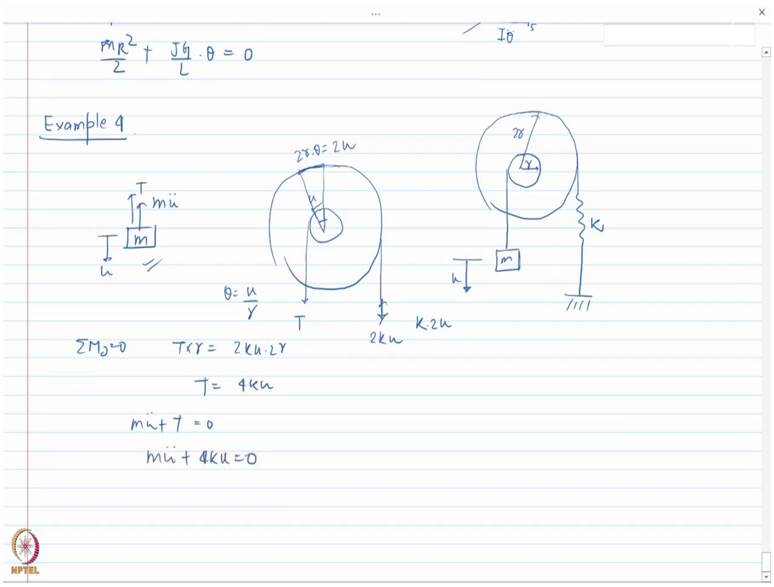

Let us go through another example. In this example, what do I have? I have two concentric discs which are rigidly connected to each other. So, I am going to draw the figure here and there is a string which is connected to this internal disc and there is a mass hanging from this disc and there is a spring connected to the outer disc. The stiffness of this spring is k, the internal radius is r, the external radius is $2r$, and the mass here is m.

So, you need to set up the equation of motion for this system. So, pause the video here and then, take some time to attempt this problem and see what do you get as final answer. Let us discuss the problem. Now, for this setup that I have here, let us consider the motion from the initial equilibrium position. Remember, it is not the undeformed position, but the equilibrium position that I am considering.

And if you remember, gravity load need not to be consider if I am considering the motion from the equilibrium position and if initially, the gravity load it was being balanced by deformation in the spring which is the case here. So, I do not need to consider that gravity load in the equation of motion or in the free body diagram.

So, if I consider the free body diagram of the mass here, what are the forces acting on this mass? The mass (m) is going down. So, there would be inertial force which is acting upwards which would be $m\ddot{u}$ and then, there is a tension (T) in this string here and I am saying that this is coming down by u.

Now, can I say if this string comes down by u, the circumferential rotation of this inner disc would also be u so that the angular rotation of the inner disc is θ. So, that this theta that I get would be how much?

This theta would be

$$\theta = \frac{u}{r}$$

And same would be the rotation of the outer disc as well. However, the circumferential rotation distance would be actually $2r\theta$ which if I substitute the value of θ, it would be $2u$ and the same would be the extension in this spring here.

So, the force in the spring would be actually $k2u$ which is the deformation in the string and that would be the force that would be applied on this outer disc by this spring. So, if I consider the free body diagram of the whole disc arrangement. I would have force $2ku$ acting here; I have force T which is acting here. Remember that this disc do not have any mass them self.

So, if I take the moment of all the forces about the center of the disc, I get-

$$T \times r = 2ku \times 2r$$

$$T = 4ku$$

So, now I know that what is force in the string, I can write down the equation of motion of this mass, which would-

$$m\ddot{u} + T = 0$$

$$m\ddot{u} + 4ku = 0$$

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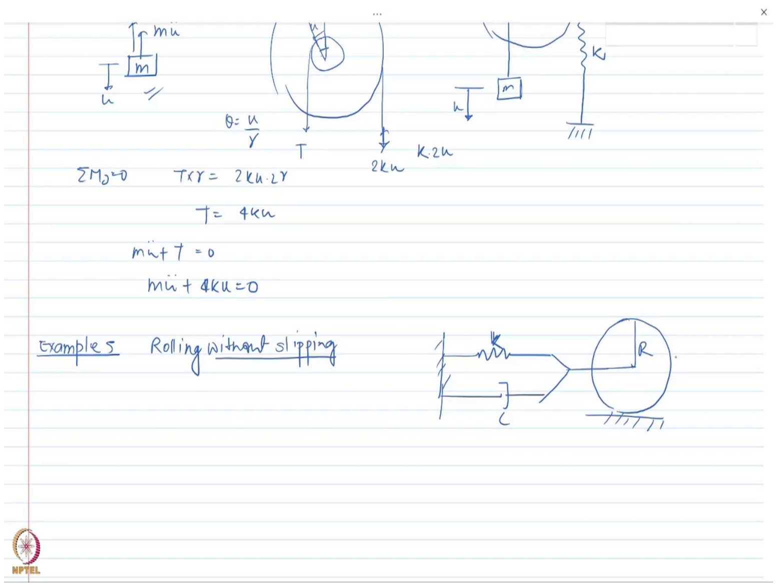

Now, let us see our example 5. In example 5, I again have a spring mass system except in this case I do not have a block but I have a rotational disc here. I need to find out what is the equation of motion, if there is rolling without slipping.

So, if that is the condition and these constants are given to you. The radius is basically R here, you need to find the equation of motion of this system here. So, take some time and then, try to attempt this problem so, that we are going to come back to again this later and then, solve it alright.