Dynamics of Structures

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Lecture – 03
Introduction to Dynamics of Structures
Components of Dynamic System

Hello everyone. So, in the last class, we basically discussed how to idealize a real structure into a more simplified model that can be used to analyze our structures subject to dynamic loading. And we set up a single degree of representation of those systems. So, we are going to extend that discussion today.

And we are going to see different examples and how to represent each and every system through a spring mass and damper representation. And we will be going to learn about the inertial forces, the damping forces, and the stiffness forces.

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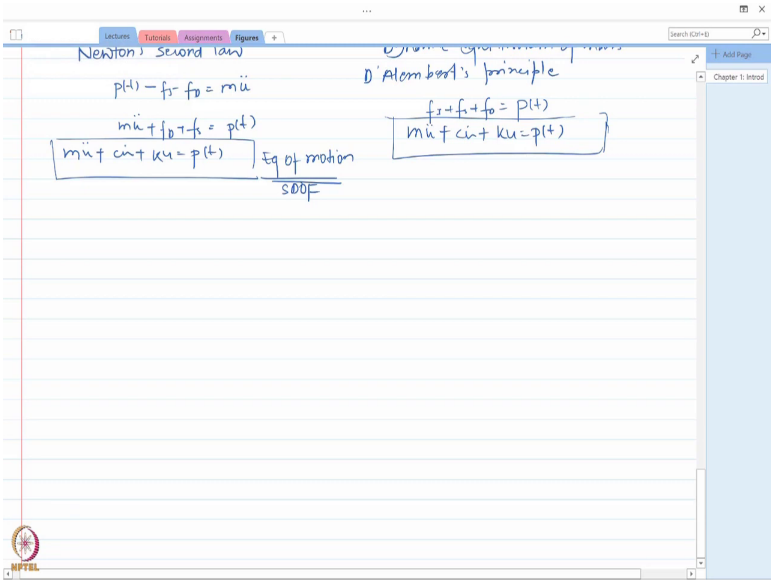

And first we are going to look into the frame type of structure and then we will also going to consider different type of the structures that we might encounter and solve some problems. So, let us get started. So, as we discussed we first consider the frame representation.

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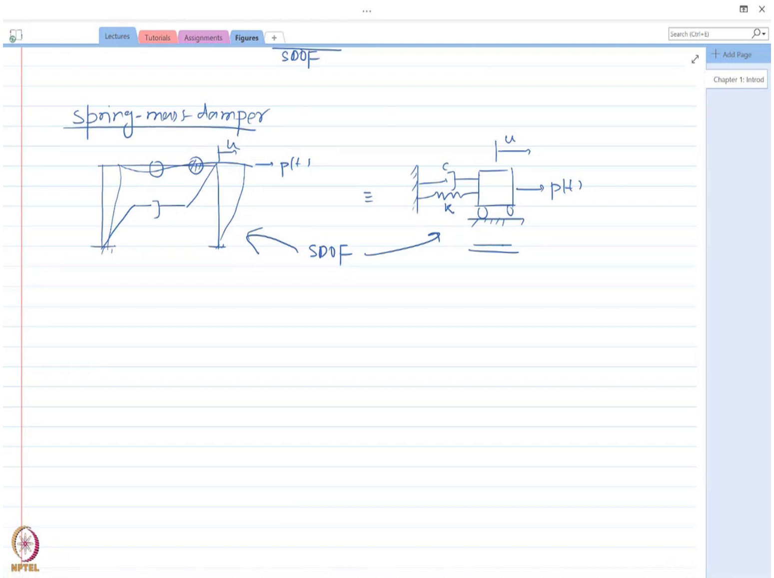

Now, let us see how we can simplify the same thing to a spring-mass damper system. So, the next thing that we are going to do is the spring-mass damper representation which is nothing but you know just another way of looking at a dynamic system.

So, for example, if I had this frame in which I had three components in which I said I had a damper here. I had a frame here which provided me the stiffness, and then I also had a mass that was not here of course initially it was here, and then under the action of load, it displaced by u.

The same thing can also be represented to a spring-mass-damper system which is nothing but a spring here, a damper here, and then I have the mass here on a frictionless surface. I am applying a force *P(t)*, this let us say K, this is c. So, under the action of this force, it goes through a displacement u. So, this is just another way of representation of the same system.

So, just keep that in mind. The single degree of freedom system can be represented with either of these two cases. For further analysis and derivation, we would be just using this representation while keeping in mind that this representation is some sort of SDOF system. It might not be a frame type of structure, it might be some other type of structure, but in the end, this is a single degree of freedom representation of some sort of system.

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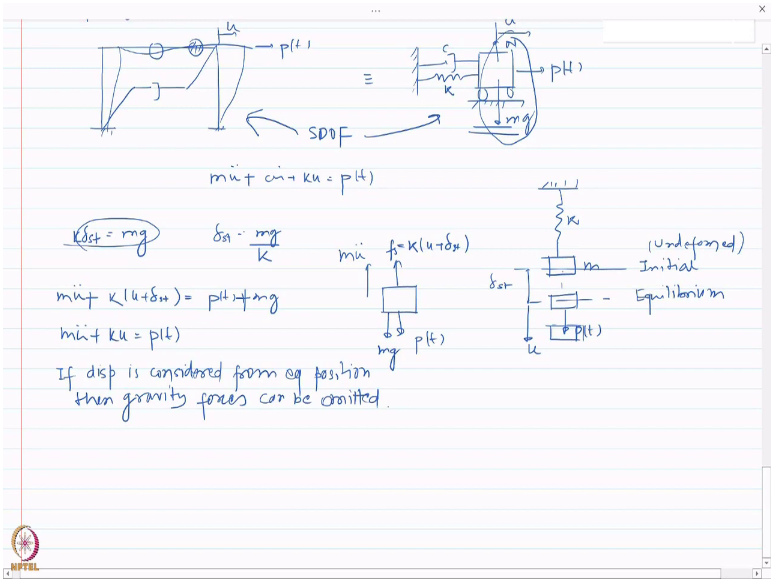

Now, let us again write down our equation of motion. So, in this case, this is our equation of motion for both systems. Now, the same thing you know I mean many places you would also see instead of this, you are going to see a vertical spring mass damper.

So, both are like you know represent the same type of system, but just one thing that you need to keep in mind I am just going to discuss that the effect of gravity. Now in this case, if it is horizontal you have gravity acting *mg* and there is a normal reaction. However, if you are considering the motion in the horizontal direction then these two quantities do not come into play.

So, you do not need to worry about those. However, if you consider only spring here, damper we can similarly have just a spring-mass system here.

So, I have mass *m* now this is a spring *K*. Now, as you can imagine if you suspend mass m from a spring, then the spring would not be undeformed. As soon as you hang a mass *m* from a spring, it will let us say this was initial undeformed position.

As soon as you hang a mass m, it will come to some initial equilibrium position. And let us say let us call this δ_{st} and this is your equilibrium position. So, the body is in equilibrium now. And if you just apply again principle of statics, I can just write $K\delta_{st}$ which is the force in the spring that should be equal to *mg*. So, initial deformation is nothing but $\delta_{st} = mg / K$, not very difficult.

Now, what will happen? If you apply a further displacement *u*, the system will start to oscillate because there is a restoring force or there is a force in the spring which will pull this mass back towards its equilibrium position, and there is an *mg* which is acting down. Let us say force *P(t)* is again acting here.

So, now let us say I am applying a force *P(t)* due to which I have displacement *u*. So, if you consider the free body diagram of this mass. In this let us call this is a new position. Which are the forces that are acting? its P(t) downwards, and because the spring is deformed. What is the total deformation in the spring? Is a $\delta_{st} + u$. So, the total deformation would be $K(\delta_{st} + u)$ alright. So, just keep that in mind while writing the equation.

So, let me write it again here, this force here is $f_s = K(\delta_{st} + u)$. And if there would have been a damping force, there would have like a damper here, you could have written similarly the damping force upward. But now for this case let us just write down the equation of motion.

So, Now you have *fs*, then you have P(t) downward. Now, there is also gravity that is acting. So, I can write *mg* that is acting downwards. Is there any other force? No, I do not think so.

So, now, write down the equation of motion. So, what do you have here and sorry there is another force I completely missed that the most important force. If it is moving down either you can put a pseudo force through upward. So, let us say $m\ddot{u}$ or you could have used Newton's second law, and then the resultant of all these forces would have been equal to $m\ddot{u}$.

So, let us write $m\ddot{u} + K(\delta_{st} + u)$ this should be equal to $P(t) + mg$. So, now, if you see I write it like this $K\delta_{st} = mg$. So, this cancels off and I can just simply write it as *P(t)*.

So, this gives me an important conclusion that if the displacement of the body of course for a single degree of freedom representation, if displacement is considered from equilibrium position then gravity forces can be omitted.

So, in this case, you do not need to consider gravitational force if you are considering the deformation from the position of equilibrium because initially that gravity force was being balanced by the initial stored energy or the initial internal energy which is $K\delta_{st}$ in the system. So, the gravity force can be omitted.

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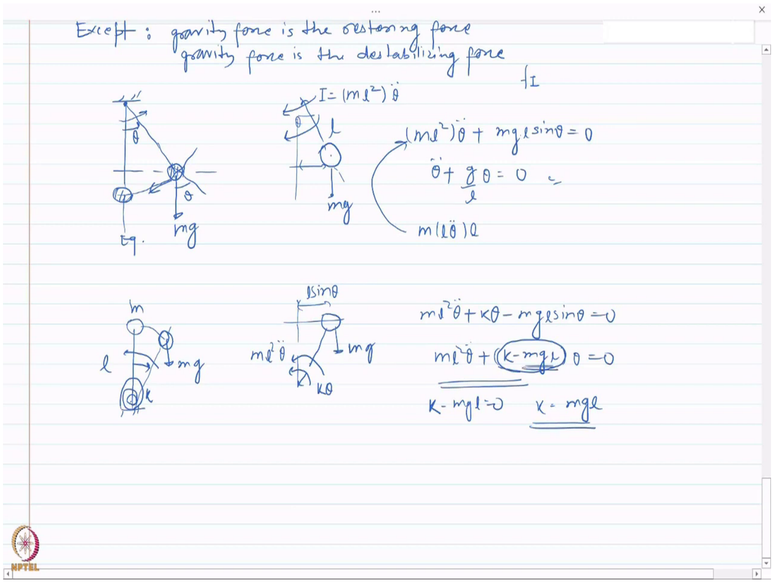

But there are two exceptions to this rule, so let us say except when gravity force is the restoring force, we will see an example of that the gravity force is the destabilizing force. So, considering the first example where the gravity force is steering is the restoring force. The most and the one of the simplest example is a pendulum mass system in which you have a pendulum-like this, which is pin connected so constraint to rotate about this point.

So, this rigid rod is here and then it is rotating about this point. So, let us say this is θ. So, let us derive the equation of motion of this body. So, what do we have here, let us say this is the trajectory of the motion I have a force which is acting *mg* here, and as it moves in the anticlockwise direction by the rotation θ, what will happen? I have a force acting *mg*.

This force will have a component, which is tangent to the direction of motion or towards this direction. So, that will act as a restoring force as it oscillates about the position of equilibrium remember this is the position of equilibrium. This gravitational force acts as a restoring force.

So, if I have to write down the equation of motion, let us draw the free body diagram of this system. This is θ here. So, there would be a certain force here in this direction, and either you can write down the equation considering the rotational equilibrium or you can consider the translational equilibrium of this mass.

Now, if you consider rotational equilibrium about the point of rotation, you have this *mg* here and this distance here is if this is θ and the length of the pendulum is *l*, this would be $l\sin\theta$. It is creating an anticlockwise movement. Now, I have assumed the direction of θ to be anticlockwise.

So, if I have to apply a pseudo force, it would be opposite to the direction of θ which would be again clockwise. So, this would be *I* equal to whatever the moment of inertia of this about this point which is nothing but $ml^2\ddot{\theta}$. So, instead of remember f_I, here I have I_α .

So, let me just write it as

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

The moment of *mg* about this point is nothing but $mgl\sin\theta$. And for a small oscillation, I can approximate $\sin\theta = \theta$. And I can write it as

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This is my equation of motion. I could have obtained the same thing if I had considered the translational equilibrium of the mass. So, in that case, you could have considered. Let us say this is translating about this point.

So, you can write it as basically what would be the tangential velocity at this point? It would be l times a tangential acceleration which is $l\ddot{\theta}$. So, the $m\ddot{u}$ is $ml\ddot{\theta}$. And if you consider the moment about this point, it would be nothing again a $ml^2\ddot{\theta}$ which is the same thing here.

Now, this is the case where gravity is acting as a restoring force and it is coming in the equation of motion. So, I could not neglect gravity here right because initially, gravity was not producing any internal energy in the system at is at this position (Refer Slide Time: 15:42).

Now, in the second case, let me take an example again where gravity is the destabilizing force. And I am just going to invert this pendulum, and then what I have here is a rotational spring with rotational stiffness constant *K*. I have *m* here and deformed position it would look like something like this, where the weight of this one would always be acting downward. So, let me just write it as *mg* here, and this is the direction of motion.

So, again if I consider the rotational equilibrium, I can have *I* that is the moment of inertia of this body about this point. And let us consider just.

So, there would be *ml*, again I am assuming *l* to be the length of this. So, $ml^2\ddot{\theta}$ and there would also be restoring force in this rotational spring here which would be $K\theta$, but then I have *mg* acting here. And this distance here is, if this is θ, this is *l*, this is $l\sin\theta$.

So, $ml^2\ddot{\theta}$ and $K\theta$ both of them are producing anticlockwise, and then a clockwise moment due to *mg* is $mgl\sin\theta$. And for a small angle, of course, $\sin\theta$ becomes θ, and so I can write it as -

$$ml^2\ddot{\theta} + K\theta - mgl\sin\theta = 0$$

Or

$$ml^2\ddot{\theta} + \left(K - mgl\right)\theta = 0$$

So, this is the equation of motion.

Remember, the critical value of the weight or the critical value of stiffness at which the oscillation would not happen or this would destabilize would be what?

$$(K - mgl) = 0 \text{ or } K = mgl$$

So, in this case, the gravity was reducing the stiffness of the system, its stiffness would have been $K\theta$. However, gravity is acting against it and reducing the stiffness of the system. So, it is acting as a destabilizing force.

So, in these types of typical cases, you could not possibly neglect gravity. However, if gravity is balanced by spring forces, in the equilibrium position, then it can be neglected in setting up the equation of motion if the displacement is considered from the equilibrium position. So, keep that in mind.

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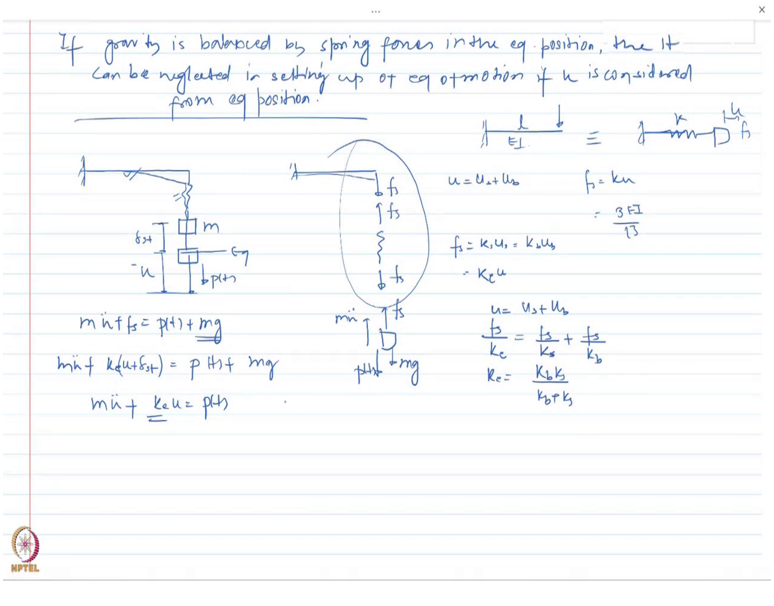

Now, let us come down to another example in which I will show you, how to find out for a combined system in which you have multiple springs, not a single spring, but a single degree of freedom system. How to find out or set up the equation of motion? So, let me take a very simple example. So, I have a cantilever beam here alright.

In this cantilever beam, what do you see, I have another spring here, then there is a mass that is attached here alright. Now, as you know in the previous lecture, we said that if you have a cantilever beam and there is a point load at the end, it also behaves like a spring right. What did we say if I have a cantilever beam, and then I have a force acting? And if this is length *l* and *E I*, then I said that it could be represented by a spring like how?

How do I represent it? *K* and this force *fs*; $f_s = Ku$, where $K = \frac{3EI}{l^3}$. So, I mean this is not like directly with the two springs, but just like you know you have to represent it like that. So, in this case, I have to set up the equation of motion for this body.

Now, as I discussed in the last problem, we said that if in under the equilibrium position there would be some initial deformation. Let us say that deformation is δ_{st}, and δ_{st} would have some contribution of this beam here as well as some contribution of this spring here, but I do not care. I am going to represent it as a single quantity δ_{st}. So, in equilibrium, let us say it is something like here, this is the equilibrium position, and this is δ_{st}.

Now, if you push it further, then again it will vibrate let us say this displacement is *u* alright. And now I need to find out the equation of motion. So, the best thing for any type of problem is first to draw the free body diagram, it makes your job very simple. So, if I consider this and if I draw the free body diagram of this mass, how would it look like or let us say the whole system, it is coming down.

So at the same level. Because if I draw the free body diagram, I have *fs* that is acting here (at beam end), and then the spring force. What is the value of fs? We will just be going to write it down ok.

So, I have a *fs* here, fs here, and then I have mass *m* here, again the same fs is acting on here (on mass in upward), and that is coming down. So, there would be an inertial force $m\ddot{u}$, the *mg* would be acting here alright. And are there any other forces? I do not think so.

So, if I write down the equation of motion, what would I get?, if there is any external force *P(t)* then I can write

$$m\ddot{u} + f_s = p(t) + mg$$

Now, if I am considering my *fs* to be, *fs* would include the component from the initial deformation and the large one as well.

Let us say in this case whatever the deformation initially had in the displaced position if I am considering *u*, *u* would have some component of the spring (u_s) and beam (u_b) alright.

$$u = u_s + u_b$$

So, *fs* would be

$$f_s = K_s u_s = K_b u_b$$

So, the same *fs* is being applied here which is applying the force in the beam, and the same is through the spring because the same force is transferring here. I could just explain to you through the series and the parallel connection, but I am not going to do that right. Now, I will do that as a consequence of this example instead of using that to derive just simply represent it like that.

Through the *fs*, I can also represent this whole system as all the deformable parts of this system as one system and through a single equivalent stiffness and through the total displacement which is the sum of spring deformation (u_s) and beam deformation (u_b) because now I am considering for the whole system. So, here I can write the equation of motion and remember in this case I also have to consider δ_{st}. Because let us say this is from the position of the deformed, but here if you considering *mg*, I have to consider δ_{st} here as well.

$$m\ddot{u} + K_{eq}\left(u + \delta_{st}\right) = p\left(t\right) + mg$$

But initially, I know that

$$mg = K_{eq}\delta_{st}$$

So, that I can cancel off and I can get

$$m\ddot{u} + K_{eq}u = p(t)$$

So, my only job is to find out K_{eq} which I can easily do, how?

Let us see. I have said that

$$u = u_s + u_b$$

$$\frac{f_s}{K_{eq}} = \frac{f_s}{K_s} + \frac{f_s}{K_b}$$

So, my *K* equivalent would be-

$$K_{eq} = \frac{K_b K_s}{K_b + K_s}$$

And then I can substitute it here the only unknown to get the final equation of motion.

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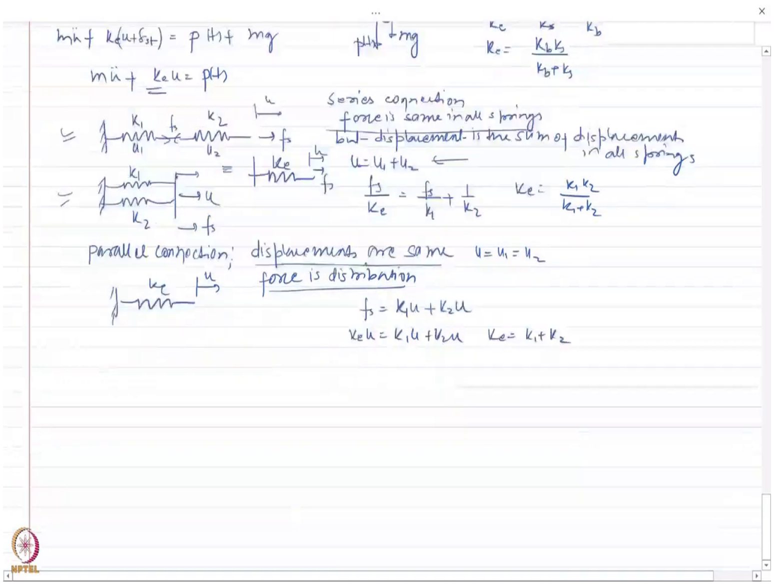

So, this system here where I have two deformed body, it behaves like two springs in some certain sort of combination. We are going to see what typical type of combination I could have?. So, let us consider two situations in which I have a spring like this in which this is K_1, this is K_2, and then, of course, there is a force that is being applied. So, that is one situation.

And in the second situation, I have two springs K_1 and K_2. Now, what can you tell me about this connection (first one) and this connection (second one)? Can I say in this connection in the first one, the force in all the springs is same?. So, if this is *fs*, the same would be the force here *fs* to all of them.

However, the displacement is distributed between these two springs. This is called a series connection of springs in which force is same in all springs, but displacement is the sum of displacement in all the springs. So, basically what I mean to say $u = u_1 + u_2$, and if I am representing this whole system through a single system on which the force *fs* is applied, then there is a displacement of *u* with a spring constant K_e.

Can I write it so this system would have u as what? So, that deformation would be simply force divided by the spring constant. And through this series relationship, I can get it as

$$\frac{f_s}{K_e} = \frac{f_s}{K_1} + \frac{f_s}{K_2}$$

$$K_e = \frac{K_1 K_2}{K_1 + K_2}$$

Now, this is the series connection.

Now, the second type of connection which is called parallel connection displacement is same. So, the displacements are same for all the springs. However, force is distributed between the springs. So, in this case, $u = u_1 = u_2$. And if I need to represent it through a single spring that has *K* equivalent as the spring constant with the same deformation *u*.

Let us see how do I do that. So, as I know my force here basically if I am applying *fs*, *fs* is-

$$f_s = K_1 u + K_2 u$$

However, the equivalent system-

$$f_s = K_e u$$

$$K_e = K_1 + K_2$$

So, the mathematical formulation itself is not very complicated. What I need you to remember when do we say connection is in series or in parallel, this what is important.

For a series connection, force is same in all the springs, but the displacement is distributed among the spring. And for the parallel connection, displacements are same, but the force is distributed. So, using these two, you can determine if the connection is in series or parallel if you ever get confused.

And if you have a combination of series in parallel you can always like you know start taking two springs at a time and reduce it to a very simple system. So, I hope it is clear up to this point alright.

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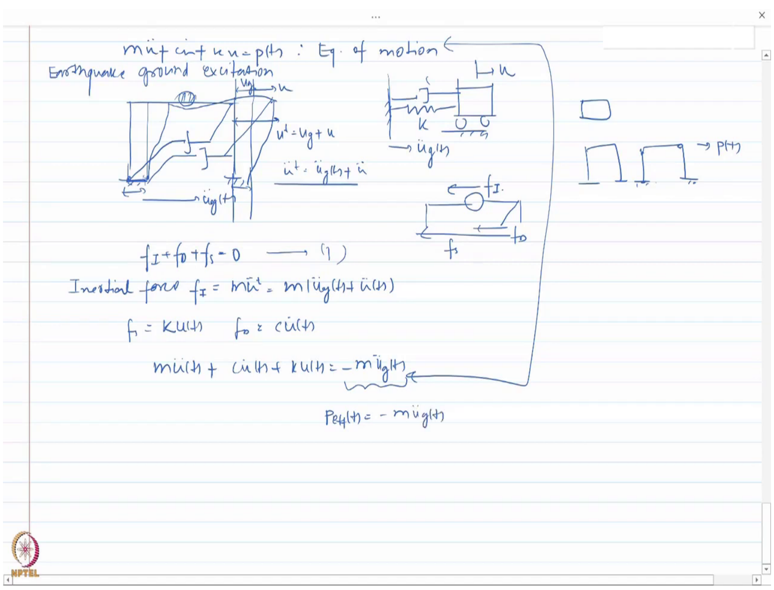

So, the next thing that we are going to do, till now we have said our equation of motion is what?

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

So, this is our equation of motion. Now, one of the very common situation especially for structural and earthquake engineers is a scenario in which the base of the structure is excited by ground acceleration.

So, what I am saying is, I have this frame here or I could do the same thing with the spring-mass representation, I will show you how do I do that. And this is undergoing a ground acceleration. So, this represents an earthquake ground excitation. So, this is an earthquake ground excitation alright.

So, in this case, how do we set up our equation of motion? And similarly, in terms of spring-mass-damper representation it will look like something like this has a damper and then mass, and the support that I have here. This one is not fixed support anymore; it is going through acceleration $\dot{u}_g(t)$. Of course, because of this, there will be some displacement as well. So, let us say it is u and then there is a displacement of the ground due to earthquake.

Similarly, in this case, if I write it like this is happening due to the ground movement, there is some shift due to ground movement, and then of course, due to internal deformation of the structure. So, this much of component u is relative deformation of the structure, u_g is the ground deformation.

And then this total deformation is called u^t which is $u_g + u$. So, this thing you need to know. And then of course, in terms of velocity and displacement, it would be also similar, to the acceleration $\dot{u}^t = \ddot{u}_g(t) + \ddot{u}$.

So, again if I have to set up my equation of motion, let us consider the equilibrium of this body here alright. So, let us say there is no external force being applied here. So, in this case, if it is deforming, then I have the stiffness component f_s, damping component f_D, and then because of the inertia, I have f_I component.

And my equation of motion would be simply –

$$f_I + f_D + f_s = 0$$

Now, let us see how do we write each of these terms here. So, the inertial term, remember my ground itself is moving. So, for the inertial force f_I, I need to have some inertial frame of reference, in this case, the ground is also moving. So, the total acceleration of the body is what? Is $\dot{u}^t$.

So, the inertial force f_I on the body is $m\ddot{u}^t = m(\dot{u}_g(t) + \ddot{u}(t))$. Now, let us see what happens to the damping f_D and stiffness f_s term. Can you imagine in this case whatever deformation happens along the two ends of the structure is what is responsible for the stiffness force? Try to imagine like this. Because of the rigid motion of the structure, no deformation is produced in the structure.

So, you have something like this here, and then it moves here, again there is no deformation produce. So, the stiffness forces because of the relative deformation $u(t)$ in the structure. So, because of that reason $f_s = Ku(t)$, not the total deformation $\dot{u}^t$.

And you know just to explain it, I give an example, consider a spring in which or consider train moving right? and you are standing on the ground, and you are applying a force on the spring correct.

Now, if you apply the force whether you are on the ground or inside the train, put the force in the spring be different? No. However, consider yourself now you are on the ground and running. And the second situation you are inside the train and the running would you think the total inertial force on you would be different?

Yes, because on the ground if you are running, the total inertial force on you would be just your mass times your acceleration. However, in the train, if you are running the total inertial acceleration on you would be or inertial force would be your mass times, your acceleration plus the acceleration of the train.

So, there is a difference in terms of what quantities we consider whether it is total or whether it is relative. Acceleration is always total, and displacement is always relative. Similarly, the damping term is also relative because we define damping something like between the two ends of the structure. So, the damping as we previously also discussed is due to the deformation in the structure leading to some energy dissipation.

If you move something from one place to other like you know in simple language rigid motion of a body does not lead to any dissipation in the body unless there is a deformation. So, damping force is also related to deformation. So, in this case, for example, initially my damper was here, and then it moved here. So, the damping term is also relative.

So, it would be $f_D = c\dot{u}(t)$. So, this one is very crucial to understand. Now, if you have substituted all these terms in the equation here, what do I get? I get-

$$m\ddot{u}^t + c\dot{u}(t) + ku(t) = 0$$

$$m\left(\ddot{u}_g(t) + \ddot{u}(t)\right) + c\dot{u}(t) + ku(t) = 0$$

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -m\ddot{u}_g(t)$$

So, although it did not have any external force in here something lies $P(t)$ being applied the top of the structure. We see I am getting a term on the right-hand side.

And if I consider this to the equation of motion here, consider these two here, this is my effective force $P_{eff}(t)$ that is being applied because of the earthquake which is $P_{eff}(t) = -m\ddot{u}_g(t)$. And that is why we say if I have a heavy structure, then it would attract more earthquake forces; if I have a light structure, it would have attract less earthquake forces.

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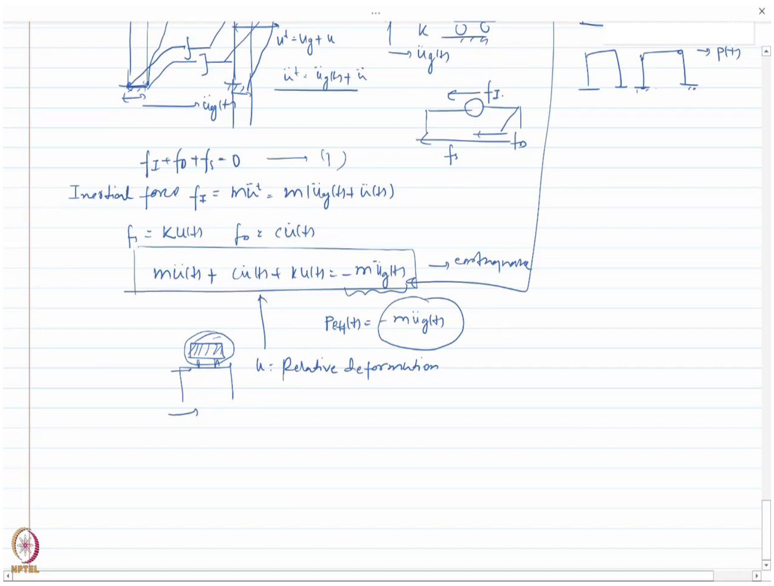

And one of the very good example is what happened during the Bhuj earthquake in Gujarat, see many of the houses, residential houses had this water tank at the top. Now, these water tanks are very heavy masses, supported on the top of the structure where the acceleration is highest. So, you have ground acceleration, then you have a structural acceleration, and then you have this right.

Because these are heavy concentrated masses, the acceleration on these masses or the force due to earthquake on these masses are huge. And this led to failure of many of the overhead tanks and ultimately the failure of the overhead tanks also led to the failure of the roof, and ultimately the failure of the building in many cases. So, a very heavy mass attracts a very large earthquake force alright.

And if that heavy mass is distributed over a larger span or like you know footprint, then the earthquake forces on the individual components are relatively lesser. So, remember that we have considered external force as $P(t)$ and we have also considered for the earthquake. And we saw that for earthquake, I am getting the effective force as minus of m is opposite to the direction of motion.

And this u here is actually relative deformation with respect to the base of the structure or whatever reference point you are using. So, we saw that there are different ways of coming to the equation of motion. And in the next subsequent chapter, we would be dealing with how to solve this equation of motion and some of the physical reinterpretation of the equation of motion. And there are different ways to solve this equation of motion.

So, with this, I would like to conclude this lecture.

Thank you.