Dynamics of Structures
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Seismic isolation
Lecture - 29
Dynamics of Base-isolation Systems

Welcome back everyone. In today's lecture, we are going to discuss the dynamics of a seismically isolated structure under the action of earthquake excitation. So, we are first going to learn how to represent a seismically isolated structure in terms of a multi degree of freedom system.

Then look at its frequencies and the important parameters that actually define a seismically isolated system and then, further discuss how those parameters actually represent the behaviour in terms of its dynamic characterization and what are the important things that, need to be considered in the analysis of a seismically isolated building.

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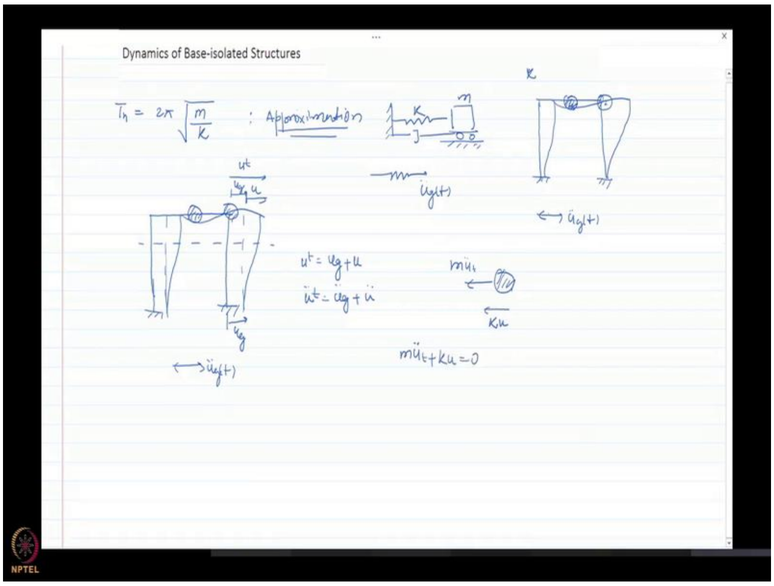

So, let us start with a very simple formula that we have been using and utilizing again and again. You know that the time period of any structure is $2\pi\sqrt{m/k}$. We have been utilizing this to understand the concept of base isolated structure as well.

Now, what is the inherent assumption here in terms of the degrees of freedom? If I am writing this formula, what can I say about the degrees of freedom for the structure that I am writing this formula? Can I say it is a single degree of freedom system, only valid for single degree of freedom system?

A simple representation could be, this mass stiffness and if you want to put damping that is fine as well or so this is the spring mass damper representation or if you want to write in terms of frame representation, then you may be you may consider this interpretation.

This is your u. So, the horizontal stiffness is represented by the overall stiffness of this frame and you can have a ground motion here. Now, what happens during the earthquake, you would have a ground motion like this. So, in both directions, now we said that this is the formula we are going to use, but as you might imagine in reality, no base isolated structure would be a single degree of freedom system.

So, in the end, this is an approximation and like any approximation or assumption, it is valid for certain cases and this works out to be good for base isolated structure. Because of the nature or the relative flexibility of the isolation layer would not be valid for any other type of a structure, unless you have. That is what we are going to looking today that when would it be valid the single degree of freedom representation.

If it is not valid, can we do a simple analysis of 2 degree of freedom system so that we understand how the superstructure and the isolator behave; how and what kind of participation do they have in the overall response.

So, those are the things that we are going to look into today and before we do that, I will just quickly go through some concept and brush up some of your concept you might already be aware of. But we will do that anyway of earthquake excitation of a single degree of freedom system and a multi-degree of freedom system.

So if let us say I have this frame here and I have this mass here, I am assuming that the whole floor mass is actually concentrated at this level. Now, due to earthquake, there would be a ground movement. So, the whole structure would move with the ground and then there would be relative deformation in the structure.

So, first there would be a rigid body motion of the structure due to ground movement. Let us say this would be u_g, which is basically the displacement of the ground and we are representing the ground motion as ground motion acceleration history $\ddot{u}_g(t)$. Now, then we would have this deformation here and let us say this mass moved here.

So, this relative deformation, we are going to denote as the deformation of the structure. This is the deformation of the ground and this is the total deformation u^t. So, $u^t = u_g + u$, relative deformation of the mass. This mass, we are considering.

So, remember the degree of freedom that we are representing, we are saying that this whole frame can be represented as a single degree of freedom system, if members are axially rigid. So that whatever this node is going through, axially the same is would be going through the right node.

The overall, if we considered the lateral deformation is what denotes as the degree of freedom. So, the total displacement of the mass is $u_g + u$ and the total acceleration would be can differentiate it twice, it could be this. Now if you want to write down that, the equation of motion for this one, we know that you can perhaps cut this structure here and write it.

So, if you consider the free body diagram of this, acceleration is always total acceleration. But deformation or the force in the frame is due to relative deformation. Because of rigid body motion of the structure, it does not induce any internal forces.

So, the internal forces would be whatever the relative or the lateral stiffness of the structure is, let us say, this is $k \times u$, the relative deformation and there is no other force. Remember, whatever force that is being applied, if you are thinking from the ground that is coming into this total acceleration. So, the equation of motion that I will get is this here $m\ddot{u}^t + ku = 0$.

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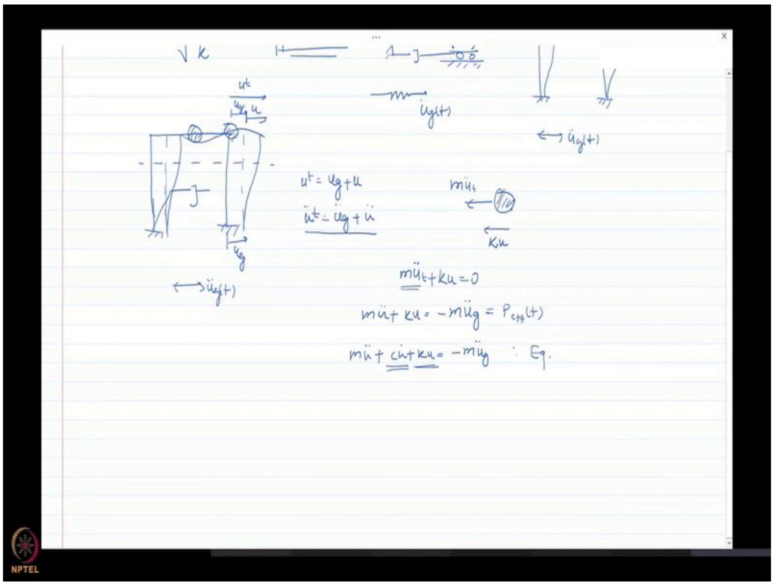

If you substitute $u^t = u_g + u$, you can further write it as $m\ddot{u} + ku = -m\ddot{u}_g$. So, I understand, many of you already aware of this, but I am just repeating it for the sake of completeness.

So, this is for a single degree of freedom system and if you had a damper, you could perhaps include that as well. So, this is the undamped system. But let us say you had a damper like this. So, remember damper also depends on the relative velocity in the structure between the points, between the points of connection of the damper.

So, then, you could perhaps write this as $m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$ and this quantity here, the $c\dot{u}$, relative damping force and there is stiffness force, they these are all depend on the relative deformation and relative velocity. But acceleration, we included as total acceleration because it is always measured with respect to inertial frame. So, this is our equation of motion for a single degree of freedom system.

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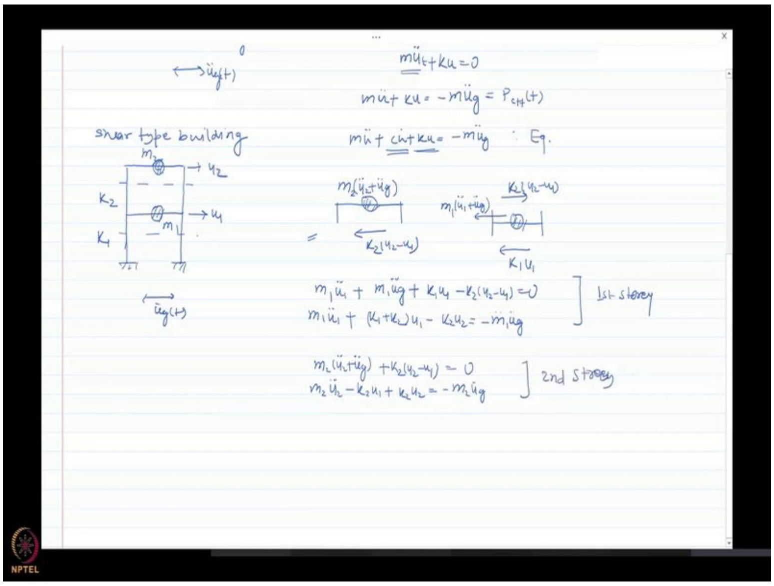

If you understand this, then we can perhaps write it for a multi-degree of freedom system as well. So, let us say I have a multi storey shear type building. The assumption in a shear type building is that the flexural deformations are small, the axially members are rigid. So, overall we can represent the degrees of freedom by deformation at each floor.

Now, let us say masses are again concentrated here. This is m_1 and this is m_2 and you can have again forces u_g; ground motion acceleration $\ddot{u}_g$ applied like this. So, like what we had done for this, we can also write down the equation of motion for this by considering equilibrium of the masses one at a time. So, if you consider the equilibrium of mass m_2, which is at this position. Let us say storey stiffnesses, we are considering as k_1 and k_2.

So, the force here would be in opposite direction of deformation which would be u_2 - u_1, because storey stiffness is k_2. So, then that force would be $k_2 \times (u_2 - u_1)$, the storey drift. Again, this would be the total acceleration. So, it would be $m_2(\ddot{u}_2 + \ddot{u}_g)$.

Similarly, for the first storey, if I consider like this, the force from the top would be equal and opposite to the force that you are applying here. So, this would be $k_2 \times (u_2 - u_1)$ in the opposite direction. And below, it would be k_1 due to storey first storey and then, u_1 minus this which would be 0 and the inertial force would be $m_1(\ddot{u}_1 + \ddot{u}_g)$.

So, we can write down separately the equation of motion for each of these. It would be $m_1\ddot{u}_1 + m_1\ddot{u}_g + k_1u_1 - k_2(u_2 - u_1) = 0$ .So, what you will get $m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = -m_1\ddot{u}_g$. So, this is for first storey.

Similarly, for the second storey, if we consider the equation of motion for this, we can write it is and remember there is another term here that I have brought from this side.

$$m_2\ddot{u}_2 + m_2\ddot{u}_g + k_2(u_2 - u_1) = 0$$

$$m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = -m_2\ddot{u}_g$$

So, this is second storey. Now, you know that this both equations can be combined as a single matrix equation.

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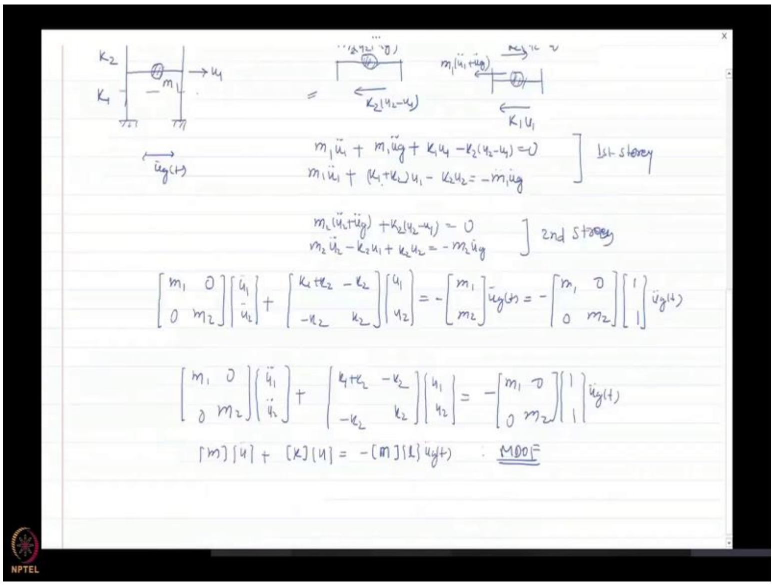

If you combine this, this equation here and this equation here, these can be combined and written in a matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{u}_g(t)$$

So, like we did for a single degree of freedom system, we can generalize the same equation for a multi-degree of freedom system as well, where this is mass vector $[m]$ times the acceleration vectors $\{\ddot{u}\}$, then this stiffness vector $[k]$ times the displacement vector $\{u\}$ and this is equal to mass matrix $[m]$ times the influence vector $\{l\}$ and then $\ddot{u}_g(t)$. This is the equation of motion for multi-degree of freedom system. Now, for a single degree of freedom system, we did not have any concept of contribution factor or anything because there was only one mode.

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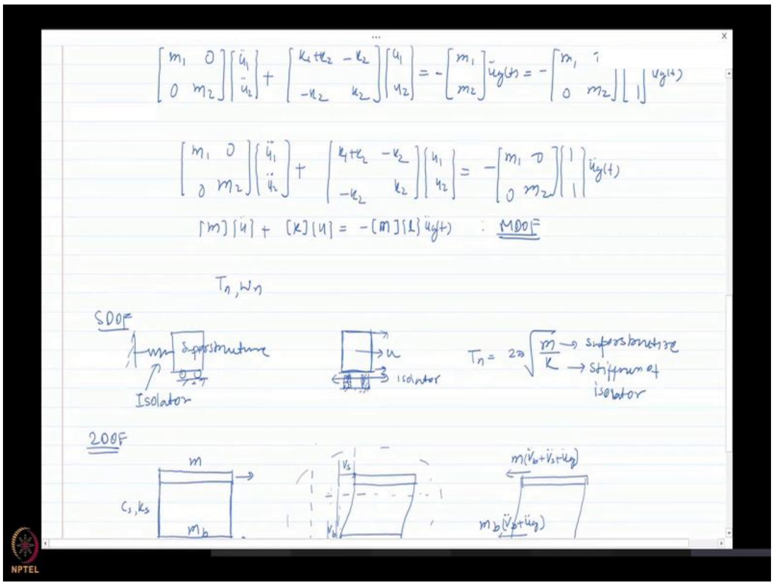

But as you know for a multi-degree of freedom system, it is represented by different modes and the overall response is some of the contribution of response due to each mode. That becomes important because your structure has multiple modes, multiple modes also means that multiple time period or frequencies.

Dynamic behaviour when you try to predict, you very well understand what happens to a single degree of freedom system, when it is applied to different type of load. For example, harmonic lode, earthquake load. It is much easier to visualize, but it is not so easy for a multi degree of freedom system.

Because you have multiple modes with multiple frequencies and if you say that the overall response is governed by a typical mode, then you need to know how much is that mode is contributing and what would be the effect of higher mode.

So, all these things are topic of interest, if you start to go much into the dynamics of it. But we can rely on that to explain the behaviour of the multi-degree of freedom representation behaviour of a base isolated structure as well and we are going to use the same concept extended to base isolated structure, the equations that we have derived.

Let us see what happens to a base isolated structure. We said that till now we have considered base isolated structure to be a rigid mass on top of any spring or anything or if you want to draw like this, perhaps this would be a good representation, this is the superstructure mass, this is the isolator, stiffness in the horizontal direction.

Although, I am drawing it horizontally, this is spring mass system, in reality, you have a base isolated building and you have multiple isolator. We assume it to be almost rigid. This is a single mass and through because it is rigid, whatever displacement you take here, the same is the displacement here.

So, we say that assume the displacement as u and that to be the degree of freedom for this base isolated structure and the horizontal stiffness is the stiffness of the isolation layer. Then, we say that this is a SDOF representation and then, we say that T_n would be $2\pi\sqrt{m/k}$; where, m is the mass of the superstructure, k is stiffness of isolation layer, not the superstructure.

But as you can imagine in reality, superstructure would not be rigid right; it would be I mean it would have some finite flexibility and it is important that we take that flexibility into consideration to completely understand the behaviour. So, let us see how we do that.

Let us see I had a base isolated structure in which superstructure is being represented by this frame. Then, there are isolators which I am representing through these two isolators and this mass here let us call it m_b, which is the mass of the base mat and m, which is the mass of the superstructure. So, mass of the basement and the mass of the superstructure and we are assuming that all the column, mass everything has been lumped with this superstructure mass.

Now, the stiffness of the isolation layer, I will denote it as k_b, that is the stiffness of the isolation layer and the damping coefficient of the isolation layer, let us say c_b. Similarly, for the superstructure, for superstructure is a damping is c_s and the stiffness is k_s.

This isolation is fixed at the ground and it is connected to this base mat and there is a ground excitation. So, now, this is a 2 degree of freedom representation, where we are going to represent deformation at this level and deformation at the superstructure level to represent the displacements at the isolation level and the superstructure to represent the overall behaviour.

Now, again of course it will not be a 2 degree of freedom system; but a 2 degree of freedom system is a much better representation in terms of understanding the dynamics or the contribution of superstructure and isolation. So, the overall structure, I am dividing between the isolation layer and the superstructure layer and I am trying to study the dynamic behaviour of that.

So, let us see under the action of this ground motion, what will happen? First, if the structure was here, there would be some ground movement. So, this is the initial position of the structure; the ground movement, let us say it is u_g, that would be the rigid body motion of the overall structure. Then, there would deformation at the isolator level.

Let us represent it through this deformed shape and we will say that this deformation is v_b. Let me write it here; this is v_b deformation here. Now, the superstructure would also have some relative deformation. So, let me just write it and draw it here and this relative deformation, let us say v_s.

Do not go by the scale because v_b would typically be much higher than v_s; but just for the sake of drawing, I have drawn it like this. So, v_b is the relative deformation at the isolation layer with respect to u_g or with respect to ground and v_s is with respect to the isolation layer. Now, we wish to write down the equation of motion again similarly for this one and it is up to us, how do we write down the equation of motion.

So, what I am going to do here, in this case, first I am going to write down the equation of motion of this whole system and in the second case, I am going to just write down the equation of motion of the upper slab. If you wish you can do isolation layer at one time and then, the overall or the top layer at other time, like we did in the previous cases.

But for these cases, from the mathematical manipulation point of view, it is much easier for me if I first write down the equation of motion for the whole structure and then, the top slab. But in the end, because there are 2 degree of freedoms, you would only get 2 independent equations; not more than 2. So, it does not matter how do you choose to write down the equation of motion, you would still get only 2 independent equation of motion.

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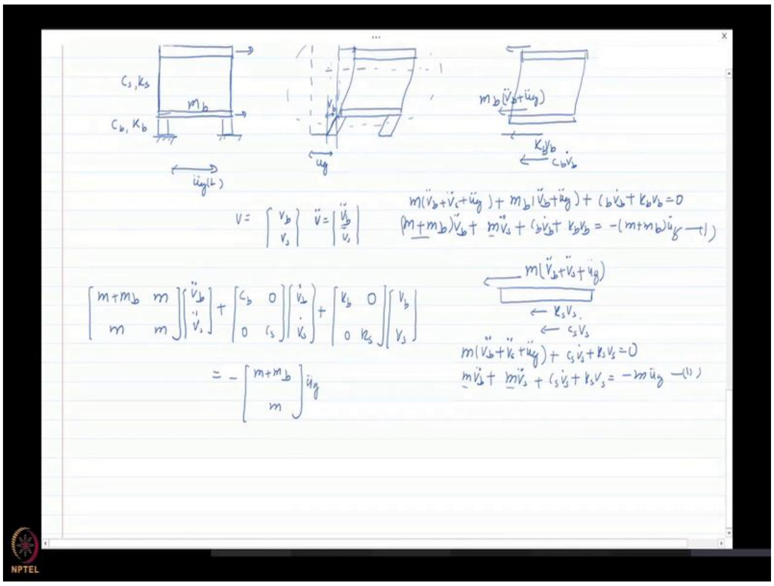

So, let us first consider the whole structure and the consider the free body diagram of the whole structure. This is the mass m; this is the mass m_b. Now, if I consider a free body diagram of this and I am cutting the structure here, the elastic forces would be due to the isolation layer only and that would be k_b, which is the base isolation stiffness base isolated stiffness times the relative deformation of the isolator which is here v_b.

Now, my structure is moving right. So, I have to apply inertial forces and if I apply the pseudo forces, it would be opposite to the direction of motion. So, for the top mass, it would be mass times total acceleration. Now, the total acceleration would be $u_g + v_b + v_s$ due to each of these three quantities.

So, it would be $\ddot{u}_g + \ddot{v}_b + \ddot{v}_s$ and inertial forces on the mass below m_b would be m_b $\ddot{u}_g + \ddot{v}_b$. Are there any other external forces that are acting on it? No, I mean because we are considering damping now, we will be also need to consider the damping force here in the isolation layer and it would be simply c_b times the relative velocity, which let me denote by $\dot{v}_b$.

So, if I write down the equation of motion for this one, let me just write it down;

$$m(\ddot{u}_g + \ddot{v}_b + \ddot{v}_s) + m_b(\ddot{u}_g + \ddot{v}_b) + c_b\dot{v}_b + k_b v_b = 0$$

Now, I need you to do one thing; I have written down for the whole structure, now what do you do? You just cut this structure at this point, only the top, the superstructure and then, similarly draw the free body diagram and write down the similar equation of motion for the top layer.

Once you are done with that, like we did here, combine the whole thing or write it with whole two simultaneous equation like this in terms of matrices. Remember on the left-hand side, the acceleration vector or the displacement vector that we are looking for is v equal to v_b and v_s and the acceleration vector would be $\ddot{v}$ is equal to $\ddot{v}_b$ and $\ddot{v}_s$.

So, accordingly, you must arrange equations like we did for this and write it like that. Let us discuss the solution now and we will compare. So, we have already written this for the whole structure, now we can let us say just isolate or draw the free body diagram of the superstructure.

So, for that, if you consider the superstructure, let us say this is my superstructure, it would have inertial force opposite to the direction of excitation or the pseudo inertial force suppose into the direction of excitation which would be $v_b + v_s$, because this is the total displacement and we are differentiating it twice and then, it would have because of the superstructure, so we are considering only relative you know deformations, it would have k_s times v_s because v_s is the drift of the superstructure.

Similarly, the damping force as well c_s times $\dot{v}_s$. So, we can write down our equation of motion for the superstructure as

$$m(\ddot{u}_g + \ddot{v}_b + \ddot{v}_s) + c_s \dot{v}_s + k_s v_s = 0$$

Let us now further write down these two equations so that are they can be written in the matrix forms. Combine all the terms which are multiplied by $\dot{v}_b$, acceleration term $\ddot{v}_b$ and $\ddot{v}_s$, together and same for the displacement and then, see what do we get.

So, here, I would get as

$$(m + m_b)\ddot{v}_b + m\ddot{v}_s + c_b \dot{v}_b + k_b v_b = -(m + m_b)\ddot{u}_g$$

$$m\ddot{v}_b + m\ddot{v}_s + c_s \dot{v}_s + k_s v_s = -m\ddot{u}_g$$

So, these are the two equation that I want to combine in the matrix form and if I do that, what would I get? If I write it in the matrix form

$$\begin{bmatrix} m+m_b & m \\ m & m \end{bmatrix} \begin{Bmatrix} \ddot{v}_b \\ \ddot{v}_s \end{Bmatrix} + \begin{bmatrix} c_b & 0 \\ 0 & c_s \end{bmatrix} \begin{Bmatrix} \dot{v}_b \\ \dot{v}_s \end{Bmatrix} + \begin{bmatrix} k_b & 0 \\ 0 & k_s \end{bmatrix} \begin{Bmatrix} v_b \\ v_s \end{Bmatrix} = -\begin{Bmatrix} m+m_b \\ m \end{Bmatrix} \ddot{u}_g(t)$$

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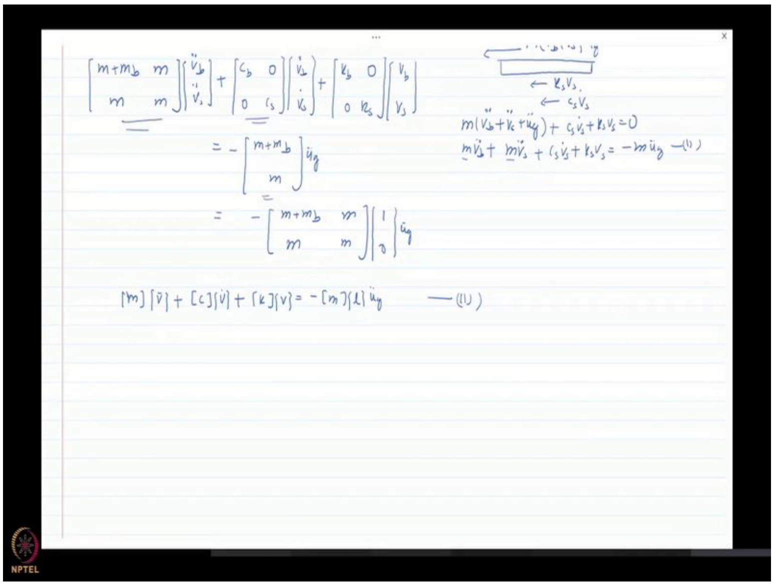

Now, like we did previously, we can further write down this as – {m+m_b, m} and then m which is the same as the mass matrix. So, I am trying to write it same in the mass matrix term and this, I will write it as 1 and 0. Because influence vector, if you multiply this vector with this matrix, then I will get the this vector here.

So, this is my equation of motion with the mass matrix as let me write it here mass times acceleration plus damping times the velocity vector plus the stiffness matrix times the displacement vector and this is equal to mass influence vector and u_g, where the mass vector and everything is again and damping matrix all are shown here.

So, this is the multi-degree of freedom equation for my base isolated system and this is what we need to solve to find out the frequencies of the base isolated structure and the superstructure or participation factor in every sense. So, this is what we are going to utilize. But before we get into that, let us define some useful quantities.

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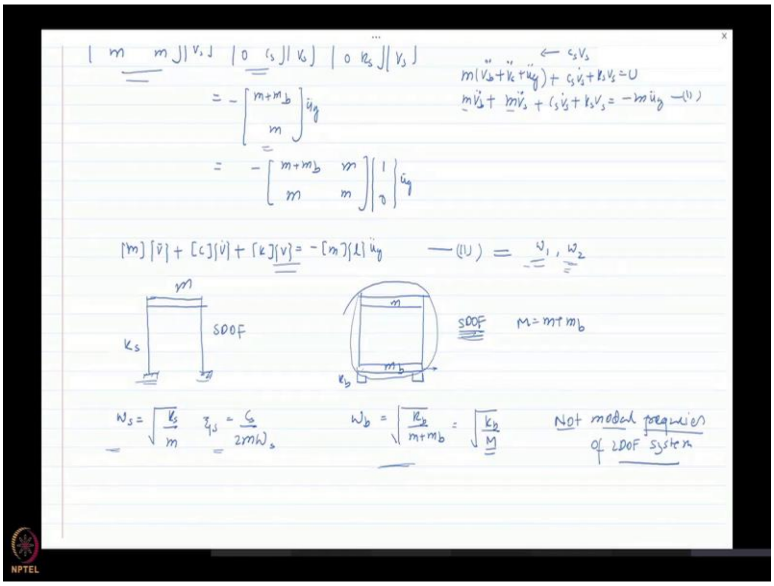

So, whenever we talk about a fixed base building and base isolated building, we always compare the frequency of the fixed based superstructure or the building to the base isolated structure; is not it?

We say that if the fixed based frequency is 1 hertz and the base isolated frequency is 0.1 hertz, how they contribute to the overall response. So, that is why we always talk in terms of the fixed base building and the base isolated building.

So, what I am going to do here? I am going to consider the superstructure as a fixed base with mass m, stiffness k_s and this is a single degree of freedom representation of the superstructure. So, for this, I know that for the fixed base superstructure, I can find out the frequency as $\sqrt{k_s / m}$ and damping as $c_s / 2m\omega_s$.

Similarly, for the base isolated structure, if I consider not a 2 degree of representation that I have considered above; but a single degree of representation, just for the argument. Why I am doing that, I will come back to that later.

With the stiffness k_b here and the overall mass of the superstructure would be now what it would be a capital M which is $m + m_b$, mass of the superstructure at this one. So, when I consider a single degree of representation of the same building that I have been doing at 2 degree representation, I can write that the frequency as $\omega_b = \sqrt{k_b / m + m_b} = \sqrt{k_b / M}$. Remember, initially when the structure is not isolated, you only have m.

But when you need to isolate the structure, we need to provide an additional base mat here like we discussed, that for the base isolated structure you need to have either a rigid arrangement of plinth beams or you can provide a very rigid base mat.

Now, these are the two frequencies that we already discussed that what would be the frequency of the superstructure and the base isolated structure, but when we consider this equation and 2 degree of freedom representation, now we know that from this, we are going to get two frequencies; ω_1 and ω_2 because it is a 2 degree of freedom system.

Can you tell me, would this ω_1 and ω_2 be same as ωs and ω_b? Remember, these are not the modal frequencies, these are just the frequencies of these two structures that we have defined. These are not the modal frequency of 2 degree of freedom system. It will not be same.

Although, it might be close depending upon some situation, some parameters; but this is a 2 degree representation with frequency ω_1 and ω_2 and these are two separate structure, a superstructure that been considered as a single degree of freedom system. So, you got this frequency and then, the whole base isolated structure that I have again got the frequency as.

When we consider 2 degree of representation of the same building, we know that we will get frequencies ω_1 and ω_2, I do not know what this frequency would be. What I have done below here? I have considered 2 single degree of freedom system. Remember this is the same system; the second one, this one here is the same system as above; but now being considered as a signal degree of freedom system.

So, in general, this single degree of representation frequency would it be same as the multi-degree of freedom system frequency? If you consider a frame and then, you consider a single degree of freedom system or if you consider a multi-degree of freedom system; would the corresponding frequency of both these two would be same? What do you think? Now, we have considered in that 2 degree of freedom system, the flexibility of the superstructure. Like in this case, when I am considering single degree representation of the same structure, I am considering this whole thing to be rigid. I am saying that there is no relative deformation. So, there are no forces generated because of this one and the overall response of the system, can represented through the deformation at the isolation layer. But above I am not doing that. So, in general, the frequencies of this and 2 degree of freedom representation and this two single degree of representation system will not be same. But we will see that for base isolated structure, it might be close because of some parameters or because of the inherent property of the isolators that we use. We will see that how we do that.

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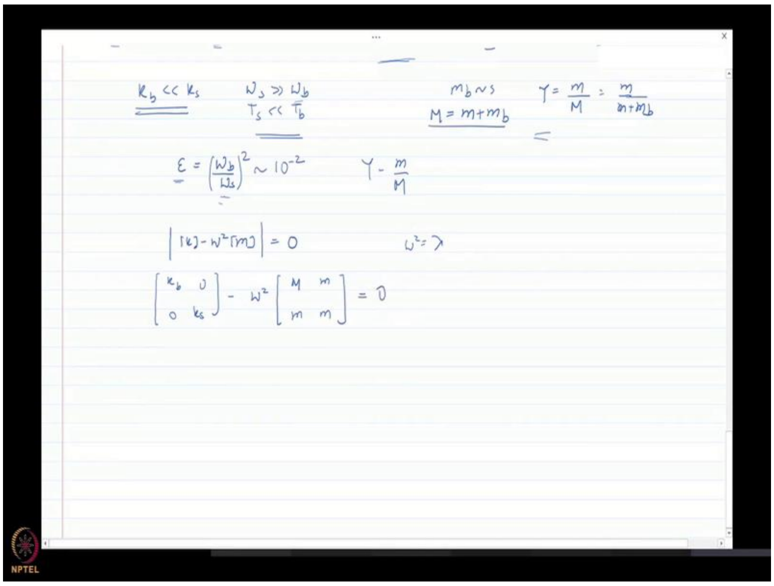

If ω_b is closer to ω_1, then we can say that the very first approximation that we had considered it is a good approximation. In this case, M here is $m + m_b$ and k here is k_b and why that is a good representation? It depends on the frequency ratio, we will come back to that.

So, in general, can I say stiffness of the isolation layer k_b would be much smaller than k_s, the superstructure stiffness? This, these are the assumptions for base isolated structure. We say that the base isolation layer is much flexible compared to the superstructure in a base isolated structure.

So, this assumption is ok for a base isolated structure, the base isolated stiffness is much smaller than the superstructure stiffness. So, that is why if you consider the equation for ω_s and ω_b, I can say that superstructure frequency would be much larger than the overall base isolated structure frequency or vice versa. The time period of the superstructure would be much smaller than time period of the base isolated structure.

This m is not negligible. It is comparable to m_b or mass of the base mat is not negligible it is. So, maybe I should write it like m_b is of the same order of the superstructure. So, the total mass, which is $m + m_b$. This is not negligible or there is not magnitude of order difference between this and any other quantity.

So, we define a term γ, which is the ratio of the mass of the superstructure by total mass, where total mass is $m + m_b$. However, I can say that $m + m_b$ are comparable, but typically k_b is much smaller than k_s.

So, that is why I can make this assertion, assuming that masses are almost of the same order; but stiffnesses are much smaller. Because imagine superstructure, unless it is like a very flexible, let us say very tall steel building and the time period is above 1 second or something like that. If you have any structures, typical time period is less than 0.5 second or let us say 0.6 second and the base isolated time period is typically above 2 second.

So, if I can make this claim, let me define a parameter ε which is the ratio of base isolated frequency to the superstructure frequency and the square of that. Why I am doing that, I will come back to that, because what happens when I start solving this equation, I tend to get these terms ε and γ.

Then I need to have some idea of the estimate that what are the order of these terms. So, that if I have to neglect some terms, I can make any inform decision based on this. Now, if ω_b, which is the base isolated frequency, if it is much smaller. The ω_s, which is superstructure frequency, it is much larger than the base isolated frequency. What we observe that this is of the order of 10^{-2} or smaller.

So, because now see, I am comparing the square of the quantity. So, if initially it was 0.1 when you consider the square that ratio becomes 10^{-2}. I am considering a square because when I solve this multi-degree of freedom equation, then I actually get terms like this square terms.

I am just defining these terms because I am going to encounter them later when I solve this equation and then, it becomes helpful to know these equations. So, I have two term $\varepsilon = (\omega_b / \omega_s)^2$ and then, I also have $\gamma = m / M$.

So, how do we solve a multi-degree of freedom system, how do we get the frequencies? Eigenvalue problem. We take the determinant of this quantity here $[k] - \omega^2[m]$ and the determinant of this, we set it equal to 0.

Then we get equation in terms of ω and the frequencies provide ω^2. You can think it as eigenvalue, whatever the values of frequencies, we get provide the modal frequencies. So, we can do that here.

$$\begin{bmatrix} k_b & 0 \\ 0 & k_s \end{bmatrix} - \omega^2 \begin{bmatrix} M & m \\ m & m \end{bmatrix} = 0$$

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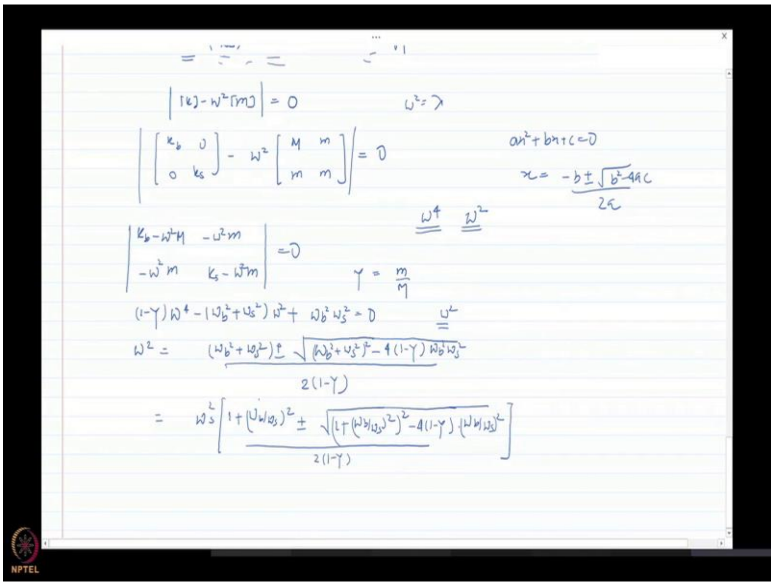

So the characteristic equation that I will get in fact, let me do this. Why do not you write it the characteristic equation and then, solve? Remember if you have the equation like this, your $x = (-b \pm \sqrt{b^2 - 4ac})/2a$.

So, take the determinant of this, the equation that you will get, it would be in terms of ω^4 and ω^2. So, it would be a quadratic equation in terms of ω^2. Please get that and by making this approximation of ε and γ, try to get ω_1 and ω_2.

You will be getting the values of ω_1 and ω_2. You will see that how these frequencies are relating to the frequencies that we have defined here for two single degrees of freedom systems. So, go ahead and try to solve get first the characteristic equation and then, solve it and then utilize these approximations to see what do you get as ω_1 and ω_2 and how do they relate to ω_b and ω_s. It is a very interesting exercise for base isolated structure, let us see, what do we get? I hope at least, you got the characteristic equation.

Remember, if I substitute this and then, I solve this, what I would get as my determinant as this would be

$$\begin{vmatrix} k_b - \omega^2 M & -\omega^2 m \\ -\omega^2 m & k_s - \omega^2 m \end{vmatrix} = 0$$

I can multiply this and rearrange the term, I can write this as

$$(1-\gamma)\omega^4 - (\omega_b^2 + \omega_s^2)\omega^2 + \omega_b^2\omega_s^2 = 0$$

So, remember, this is a quadratic equation in terms of ω^2. Now, further what we can do or let me just do this first write down. So, it is quadratic in terms of ω^2. So, my ω^2 I can write it as

$$\omega^2 = \frac{(\omega_b^2 + \omega_s^2) \pm \sqrt{(\omega_b^2 + \omega_s^2)^2 - 4(1-\gamma)\omega_b^2\omega_s^2}}{2(1-\gamma)}$$

Now, in this case, if you take ω_s^2 outside, I will have

$$\omega^2 = \omega_s^2 \left[\frac{1 + (\omega_b / \omega_s)^2 \pm \sqrt{(1 + (\omega_b / \omega_s)^2)^2 - 4(1-\gamma)(\omega_b / \omega_s)^2}}{2(1-\gamma)} \right]$$

Now, what is this term here? This is my ε, that we had is used. First, let me write down all the terms, then we will start to make approximations.

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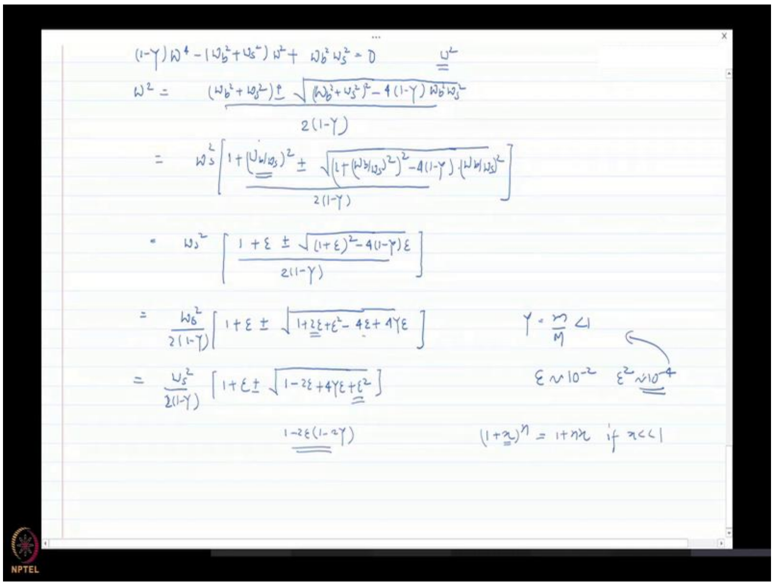

So, I will keep writing this as

$$\omega^2 = \omega_s^{\,2}\left[\frac{1+\varepsilon \pm \sqrt{(1+\varepsilon)^2 - 4(1-\gamma)\varepsilon}}{2(1-\gamma)}\right]$$

$$\omega^2 = \frac{\omega_s^{\,2}}{2(1-\gamma)}\left[1+\varepsilon \pm \sqrt{1+2\varepsilon+\varepsilon^2-4\varepsilon+4\gamma\varepsilon}\right]$$

$$\omega^2 = \frac{\omega_s^{\,2}}{2(1-\gamma)}\left[1+\varepsilon \pm \sqrt{1-2\varepsilon+4\gamma\varepsilon+\varepsilon^2}\right]$$

Now, remember, we said that this quantity γ is m / M, which is less than 1, but it is not a very small quantity. It is still comparable right, we told that above. However, ε is of order of 10^{-2}. So, ε^2 would be order of 10^{-4}. So, very small compared to this. So, utilizing that, I can perhaps neglect this and then, I have the term $1 - 2\varepsilon(1 - 2\gamma)$.

Now, what we can do in this case? We have to now simplify this. So, that now, we know that if I have something like this $(1 + x)^n$, a polynomial expansion of this, if x is a very small quantity can be written as $1 + nx$. If x is very small, so we are going to utilize that as well and then, see what happens.

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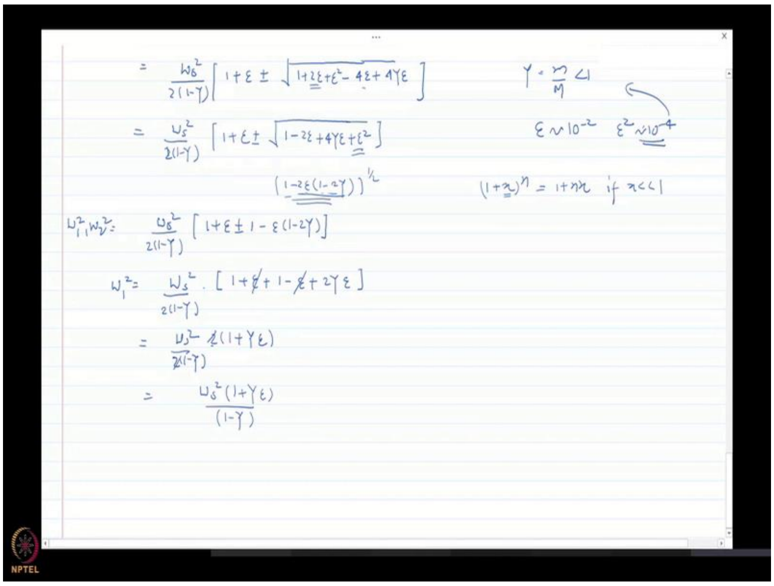

So, I have

$$\omega_1^{\ 2}, \omega_2^{\ 2} = \frac{\omega_s^{\ 2}}{2(1-\gamma)}\left[1+\varepsilon \pm 1-\varepsilon(1-2\gamma)\right]$$

Now, remember, my ω_1^2 and ω_2^2 would to when I consider first the positive sign here and then the negative sign here and I do not know yet which one correspond to what. So, let us first take the positive sign and then, see what we get.

$$\omega_2^{\ 2} = \frac{\omega_s^{\ 2}}{2(1-\gamma)}\left[1+\varepsilon+1-\varepsilon+2\varepsilon\gamma\right]$$

$$\omega_2^{\ 2} = \frac{\omega_s^{\ 2}}{(1-\gamma)}\left[1+\varepsilon\gamma\right]$$

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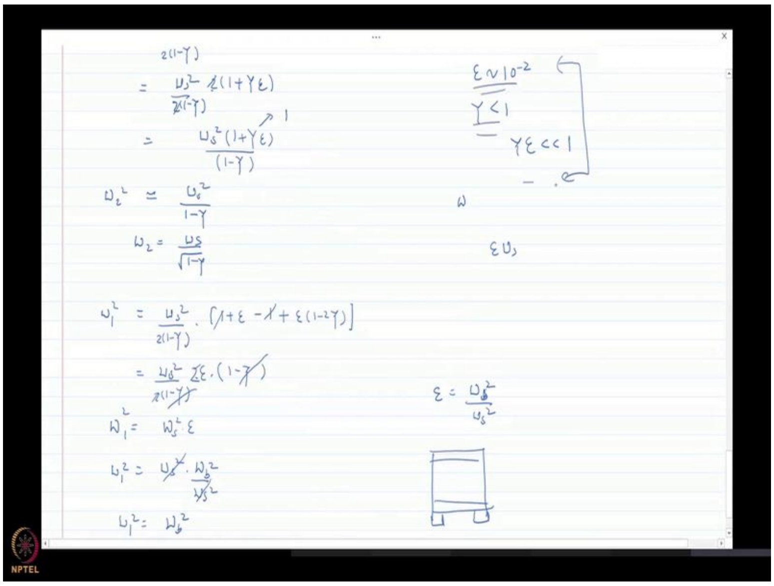

Now, for this one, I said that ε is order of 10^{-2}; γ is less than 1, but not very negligible. So, can I say γε term can be neglected with respect to 1? However, the denominator term γ cannot be neglected with respect to 1, but γε can be. So, this term again would be omega s square divided by this term would become equal to or not equal to approximately equal to 1 and I have this here.

So, what I am getting here is actually the higher frequency. So, maybe this is ω2 square. We will see that what do we get as the other term, when I take the minus sign here. So, $\omega_s^2/(1-\gamma)$. Now, let us see when we take the negative term, then what do we get?

$$\omega_1^2 = \frac{\omega_s^2}{2(1-\gamma)}\left[1+\varepsilon-1+\varepsilon-2\gamma\varepsilon\right]$$

$$\omega_1^2 = \frac{\omega_s^2\varepsilon}{(1-\gamma)}\left[1-\gamma\right]$$

$$\omega_1^2 = \omega_s^2\varepsilon = \omega_s^2\frac{\omega_b^2}{\omega_s^2} = \omega_b^2$$

So, ω_1^2 as ω_b^2 and ω_1 as you just take this square root could be ω_b. What is ω_b? The ω_b was frequency of the base isolated structure when considered as a single degree of freedom representation. Even when I solve a 2 degree of freedom representation of the same building, I am still getting my ω_1 as ω_b and ω_2 as $\omega_s / (1-\gamma)$.

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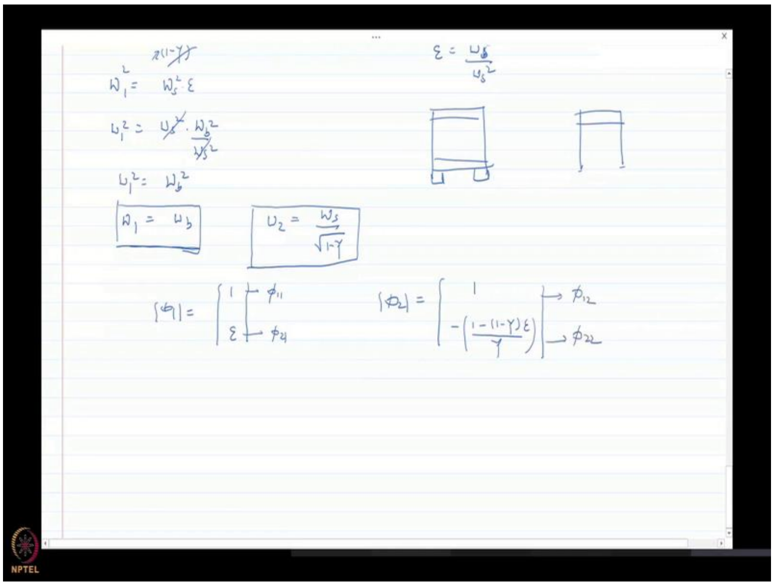

So, for a base isolated structure, even when you consider 2 degree of freedom representation if these assumptions are valid that your frequency of the superstructure is much higher than the base isolated structure. The first mode is the base isolated mode; the second mode is the deformation in the superstructure, but the frequency is not exactly equal to ω_s, it might be close to that, but not exactly equal to ω_s.

Now, is this important conclusion or not? That even when you solved a 2 degree freedom representation, your first mode was coming that is coming out to be equal to the base isolated mode and the second mode is the deformation in the superstructure. Because ω_s is what? It is the frequency of the fixed base superstructure. Now, you can go ahead, once you have the frequency, you can find out the mode shapes as well right.

That is not difficult to do. But we are not going to do that. I am just going to write down the final answer, you can go and verify if you like. Your Ø1 would be mode shape and this is the base isolated mode shape 1, ε and Ø2 would be 1, -(1-(1-γ)ε/γ). Now, as you know this is what? This is Ø11 and this is Ø21; this is Ø12 and this is Ø22.

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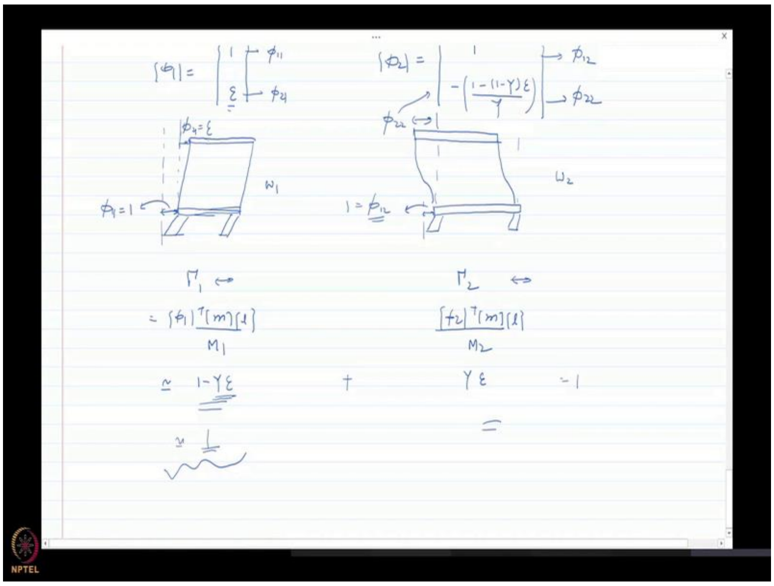

So, if I have to represent these modes, draw these mode shapes, how would that look like Ø11 = 1 and Ø21 = ε, where ε is a very small quantity. So, if I must draw it, I will draw something like this. Small deformation in the superstructure compared to large deformation in the bearing or the isolation layer.

So, remember, we are measuring everything with relative quantity. So, this one is Ø11 which we have obtained as 1; this one is Ø21 which we have obtained as ε, where ε is a very small quantity. So, the in the first mode, what we are seeing here the superstructure contribution is very less.

The deformation in the superstructure corresponding to the first mode is ε very small and we can again go ahead and draw the second mode, what happens in the second mode? So, Ø2 let me draw this. Remember, now Ø2, it is negative; Ø22 is in opposite direction to the Ø12.

So, it would look like something like this. This here is Ø22 and this here with respect to the relative deformation of the base is Ø12, this is equal to 1 and Ø22 is the second quantity here. In the second mode, the superstructure deformation is not negligible compared to Ø12. So, Ø21 is not negligible compared to Ø12. However, if you try and go ahead and find out the modal participation factor which you can do it by writing down the expression for the modal participation factors.

So, you can multiply the mode shape and find out, you would get this approximately equal to 1 - γε and this, you will get as γε. And of course, if you sum this up, it would be equal to 1 because the sum of contribution of each mode should be equal to 1. But what I am saying in the second mode, the contribution of the superstructure is not negligible compared to the contribution of this $\emptyset 12$, which is the base isolation.

However, when you calculate the participation factor of each mode, you will see that first mode is approximately equal to $1 - \gamma\varepsilon$, which you cannot say that this quantity is very small compared to 1. So, this is almost equal to 1; the whole participation is coming from the base isolated mode.

So, what we have derived here is the contribution, the frequency of each mode which we see that it is coming out to be close to ω_1 and ω_2; and the participation of each mode, so the ω_1 was close to the base isolated mode; ω_1 was not close, but the superstructure frequency.

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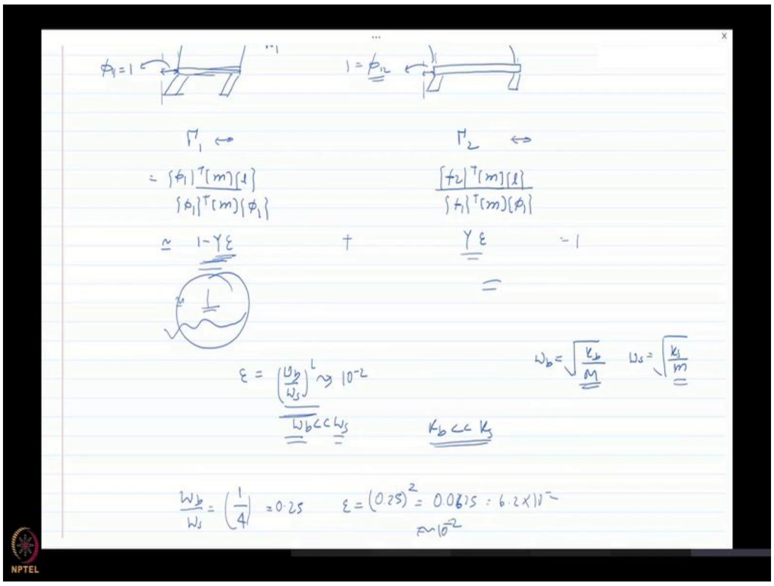

The participation of the base isolated mode is almost very close to 1 and you have very less participation of the superstructure mode. This would only be valid as long as your approximations which we had assumed $\varepsilon = (\omega_b / \omega_s)^2$ is very small which means ω_b / ω_s.

So, as you can see that if you want the whole response to be dominated by the isolated mode, you need to ensure this. The frequency of the base isolated mode is much smaller than the frequency of the superstructure mode and how we can ensure that? If we assume that the masses are comparable, we can ensure this by ensuring k_b is much smaller than k_s so that isolation layer is very flexible.

But remember ω_b is $\sqrt{k_b / m_b}$ and ω_s is $\sqrt{k_s / m}$. So, it is not exactly equal to ω_b or not directly proportional to k_b, it also depends on this mass. But assuming that masses are comparable, not of the order magnitude of difference if we ensure that you can ensure that effectiveness of base isolation structure by ensuring that the contribution of the response mostly come from the isolated mode.

Very important that you understand you might have been discussing all about this that you know I mean contribution of the isolated mode; the first mode is only mostly the isolated mode itself. But having a mathematical appreciation is also important because then you can have an insight into the overall dynamics of a base isolated structure, superstructure should be stiff.

So, typically, this is based on the observation and I mean anybody who has designed or come through lot of structures, they would know that for all the building structures, typically the period is between 0.2 second to 0.8 second or 1 second let us say.

Now, for base isolated structure, we just want to ensure that it is spaced apart from that period ok. So, if the structure period is 0.5 second, we go for at least 4 times that let us say 2 second. So, that my ω/ω_s comes 1 by 4. If I go for 0.5 second fixed and so, I have just reversed that one. Now, this is what? This is ω/ω_s and then this is 0.25. So, what does my ε becomes 0.25^2 which would be 0.0625, I think? So, this is of the order of 10^{-2}; is not it? It is a design assumption you know, as long as you maintain in that period range, you usually get a good spacing between the contribution mode mostly from the isolation mode.

Remember, this is M_1 and M_2 are modal masses. So, maybe instead of writing it like this, I will write it like this one; l is your inference vector.

$$\Gamma_1 = \frac{\{\Phi_1\}[m]\{l\}}{\{\Phi_1\}^T[m]\{\Phi_1\}} \qquad \Gamma_2 = \frac{\{\Phi_2\}[m]\{l\}}{\{\Phi_2\}^T[m]\{\Phi_2\}}$$

So, now, you have a mathematical proof and the understanding of the assumptions that we make, when we talk about the dynamics of base isolated structure.

You should have complete idea now on how the superstructure behaves; what role does it play in the overall response; what do the frequency depend on, in order to ensure that the base isolated structure mostly behaves like a single degree of freedom structure, what are the assumptions that are required. You know all those things, now should be known to you because now you have mathematical background of the same.