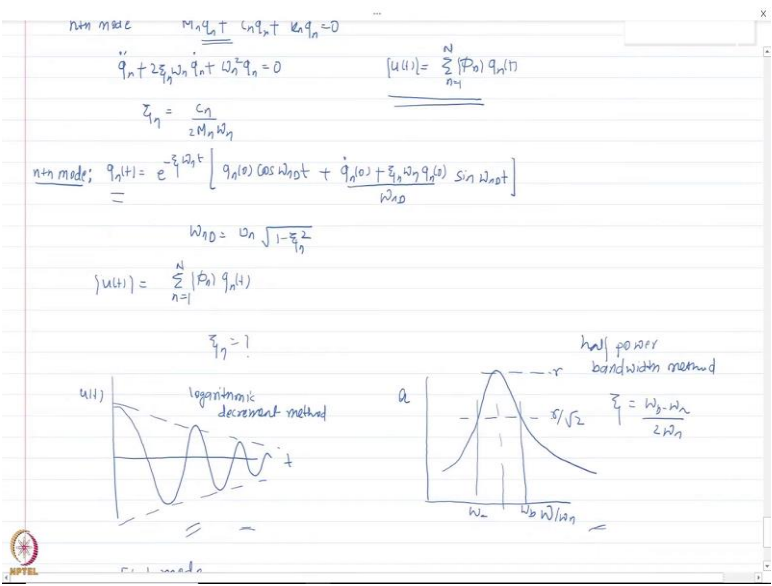
Dynamics of Structures
Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Module - 01
Forced Vibration of MDOF Systems
Lecture - 24
Damping in MDOF Systems

Welcome back everyone. In the last few lectures, we saw that how to set up the equation of motion and saw how to find out the free vibration response of a multi-degree of freedom system. Now, we discussed about damped system as well as undamped system. We learnt few methods when we were discussing single degree of freedom system such as how to estimate damping in a system.

Now, we are going to adopt that procedure for a multi-degree of freedom system and see how we can utilize different type of damping models to represent viscous damping in a multi-degree of freedom system. So, in addition to talking about the significance of damping in the multi-degree of freedom system, we are also going to look at few ways in which we assign damping to a multi-degree of freedom system which includes Rayleigh damping as well.

(Refer Slide Time: 01:33)



Let us start the lecture. If you remember typically for single degree of freedom system to find out damping what we had done, experimentally we had considered first free vibration response. I am giving initial displacement and then taking the ratio of the successive peaks amplitude and find out the damping using logarithmic decrement method.

In the other method for find out damping we had found out the peak response of a single degree of freedom system as we change the excitation frequency. So, let us say this is ω/ω_n and this is the peak acceleration *a* here. So, we had obtained a frequency response curve like this. Whatever peak response we have such as *r*. we basically have to draw a horizontal line which was at a value $r/\sqrt{2}$ and wherever it cuts the frequency response curve we needed to find out ω_a and ω_b.

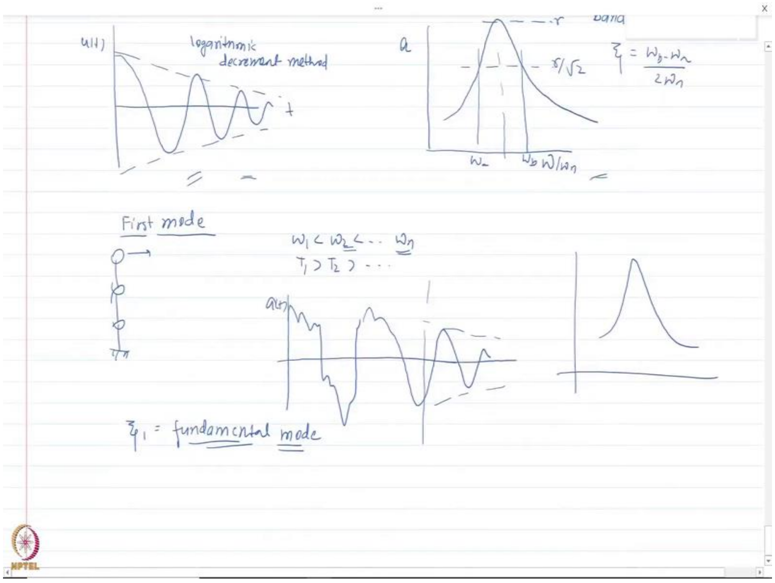
Utilizing these ω_a and ω_b, damping could be determining as-

$$\xi = \frac{\omega_b - \omega_a}{2\omega_n}$$

This was called as Half-power Bandwidth method. Now the question is when we have response of a multi-degree of freedom system, the response is not due to a single mode. Either the first mode or second mode, the response is always due to contribution of all the modes.

So, can we utilize these methods to find out the damping of individual modes? So, we need to modify these methods a little bit or we need to adapt these methods little bit to find out the damping of a multi-degree of freedom system for a particular mode and the limitations of these methods are that typically we cannot find out damping of all the modes but may be only for first mode.

(Refer Slide Time: 04:10)



Why is that? When we consider response of a multi-degree of freedom system or the acceleration response at some place. Typically, what happens? as I have mentioned the fundamental mode is the mode for which the frequency is lowest, or the time period is highest.

So, let me write it like this $\omega_1 < \omega_2 < ....\omega_n$. So, $T_1 > T_2 > ....T_n$. So, through these methods if I measure acceleration response histories at some point due to contribution of all the modes.

So, it would be something like this, but as the time progresses, because of the damping what happens? In the previous chapter where we have discussed that the damping actually damps out the higher frequencies first and the lower frequencies later.

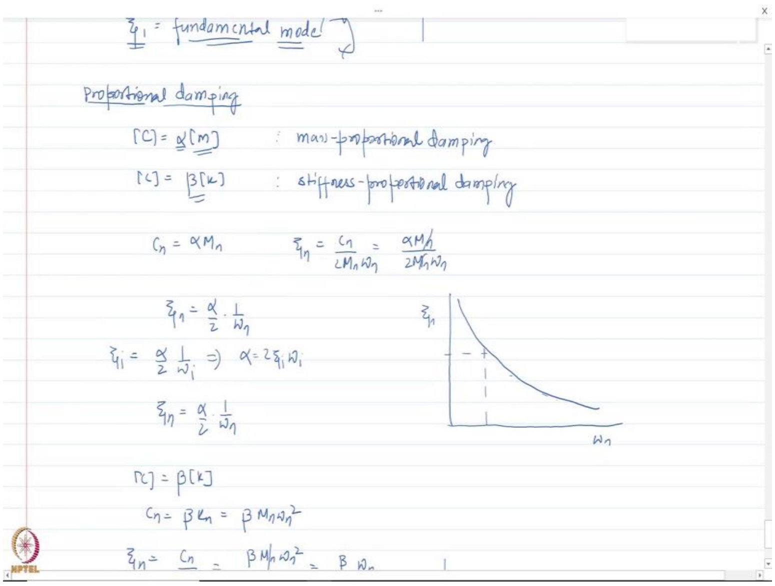
So, after some time the response that is remaining is primarily because of the first mode or the fundamental mode for which the frequency is the smallest. So, utilizing that may be let us say after this point we can find out the damping using logarithmic decrement method here or we can also find out from the frequency response curves, but in this case again considering the steady state response where the contribution of all the higher modes have actually damped out.

So, these two methods we can only find out damping for possibly the fundamental mode, or the primary mode or the first mode. If we need to find out the total response, then we need these damping value for each and every mode.

And how do we get that? We cannot simply assume it to be 0, and the only information is that we have the damping may be from the experiment for the first mode or may be what do we do, we can also assume that the second mode might be like somewhere close to the first mode.

And the damping might be similar, but the thing is that we only know that damping for at max one or two mode. So, are there any methods through which knowing the damping for one or two modes can allow us to find out damping for all other modes? So, that is the problem statement here.

(Refer Slide Time: 07:21)



So, to overcome this limitation what do we do? We consider proportional damping. Now, let us see what is proportional damping? It is a mathematical way to find out the damping for several modes by using the damping for one or two modes. Now, we need this damping matrix, or we need the damping ratio. The basic requirement is that the damping matrix should be classical or we should be able to diagonalize the damping matrix.

Now, we know that mass and the stiffness matrix are symmetric, and they can be diagonalized. So, if we can write down our damping matrix as a linear multiplication of let us say mass matrix.

$$[C] = \alpha [m]$$

$$[C] = \beta [k]$$

The resulting damping matrix would be diagonal matrix if we can find out this vector alpha (α) here. Similarly, I can write down this as some multiplication of the stiffness matrix again my damping matrix would be diagonalized because my stiffness matrix is a symmetric matrix.

This the first approach is called Mass Proportional Damping. The second approach is called Stiffness Proportional Damping. But the question might come what is the physical significance? I mean how can we simply assume it to be like you know proportional to mass or a stiffness matrix.

Well, it may be stiffness proportional matrix can be justified saying that my damping matrix in the end is basically represents the relative velocity between two points and the stiffness matrix is represents the relative stiffness between two points like a storage stiffness.

However, how do we justify the mass proportional matrix because it is basically means that whatever your masses are, this damping is basically proportional to the mass matrix or if a heavier mass is there it, would provide you some resistance. So, this represents the aerodynamic damping.

So, directly we cannot say any physical significant, but we have mathematically. We can write down our damping matrix as a linear multiplication of mass or stiffness matrix and we can find out those vectors to see if that indeed would give us a diagonal matrix that can somehow represent the damping.

And we can equate the damping in a particular mode using this approach to the damping for the mode that is available, and the rest of the damping can be found out. So, let us look at one at a time. Let us first look at mass proportional damping. So, for mass proportional damping $[C]=\alpha[m]$.

So, if I diagonalize this, I can write down the diagonal *C* matrix as alpha times diagonal *M* matrix or if I write it in terms of element. All you need to do is to multiply premultiply with the transpose of the model matrix and then post multiply with the modal matrix. So, this is the relationship you will get as

$$C_n = \alpha M_n$$

Now that is the case then the damping ratio for the n^th^ mode can be found out as-

$$\xi_n = \frac{C_n}{2M_n\omega_n} = \frac{\alpha M_n}{2M_n\omega_n} = \frac{\alpha}{2}.\frac{1}{\omega_n}$$

The only unknown here is alpha (α) and how do we get alpha (α)? Well, let us say from experiment we know the damping for n^th^ mode which would be in most cases the primary mode but let us for the numerical aspect we specify for any other mode.

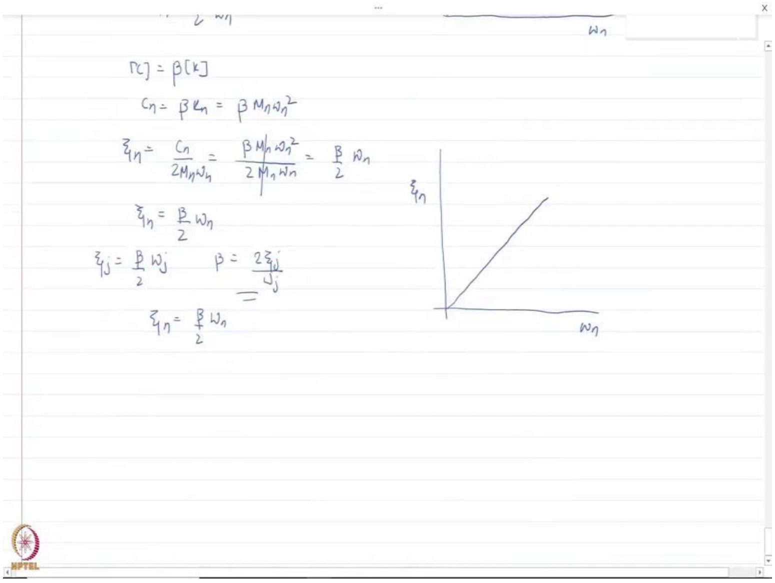
So, let us say we specify the damping for the i^th^ mode and that is equal to $\xi_i = \frac{\alpha}{2}.\frac{1}{\omega_i}$. ω_i is known to us from modal analysis. So, $\alpha = 2\xi_i\omega_i$.

And once alpha (α) is known from the calculation for one of the mode, for the rest of the modes the damping can be found out as $\xi_n = \frac{\alpha}{2}.\frac{1}{\omega_n}$ and if you look at the variation of the damping here. Let us say this is ξ_n versus ω_n.

So, this is like a hyperbolic distribution where basically ξ_n is equal to $\frac{\alpha}{2}.\frac{1}{\omega_n}$. So, if we specify for first mode here, the damping in the rest of the modes would be smaller than the first mode. So, this is the mass proportional damping. Now, let us look at the stiffness proportional damping in which basically the damping matrix is beta a constant time the stiffness matrix $[C]=\beta[k]$.

So, that constant is beta (β) here. I can write for the n^th^ element of the diagonalized damping matrix now as $C_n = \beta K_n$. Where, $K_n = M_n \omega_n^2$.

(Refer Slide Time: 14:08)



So, I can write that. So, the damping in the n^th^ mode can be written as-

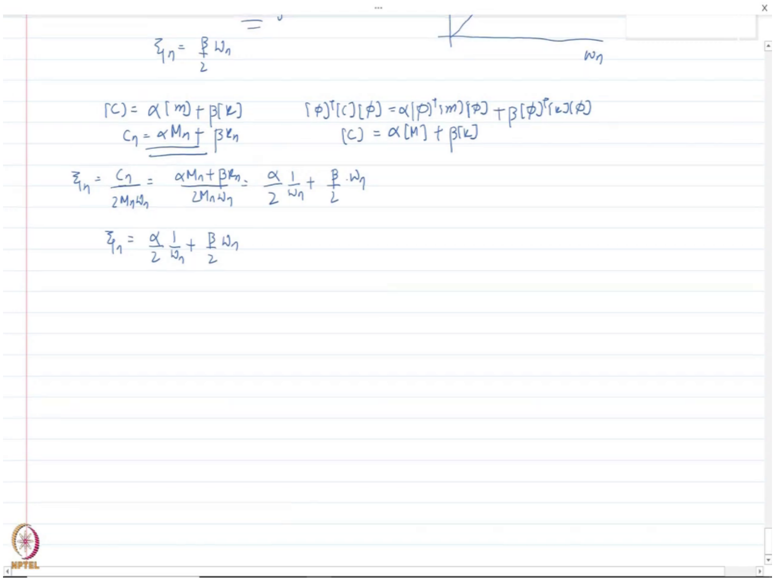
$$\xi_n = \frac{C_n}{2M_n\omega_n} = \frac{\beta M_n \omega_n^2}{2M_n\omega_n} = \frac{\beta}{2}\omega_n$$

So, my damping in the n^th^ mode can be written as $\frac{\beta}{2}\omega_n$. So, again like we did for the mass proportional damping if we assume damping for the j^th^ mode or if we know the damping for the j^th^ mode, then $\xi_j = \frac{\beta}{2}\omega_j$.

So, $\beta = \frac{2\xi_j}{\omega_j}$ and once beta (β) is known from the damping specification for one of the modes for the rest of the modes, we can find out $\xi_n = \frac{\beta}{2}\omega_n$. And if we look at the variation for this if this is ξ_n here and this is ω_n here, this is actually a linear variation between ξ_n versus ω_n where the slope is actually $\frac{\beta}{2}$. This is stiffness proportional damping.

Now as I said we cannot directly justify in some sense that these actually represents some damping mechanism, but in the end, we have to specify the damping to the system to using some mechanism through which we can fix the damping value for the modes that we know the values from experiments and for the other modes, we have to specify using some mathematical formulation. In reality the damping is neither mass proportional and neither stiffness proportional.

(Refer Slide Time: 16:04)



But if we write it the overall damping matrix, then the linear combination of both mass plus stiffness proportional maybe then we can say that the damping is being contributed by the mass and the stiffness solvers. So, this would be closer to the reality than the previous two damping mechanisms that we have consider.

So, in this case my damping matrix is written as a linear combination of mass and the stiffness matrix $\left([C]=\alpha[m]+\beta[k]\right)$ and again because mass is a symmetric matrix, and the stiffness is a symmetric matrix. For typical structures damping matrix can also be diagonalized here and it can be written as-

$$C_n = \alpha M_n + \beta K_n$$

Or

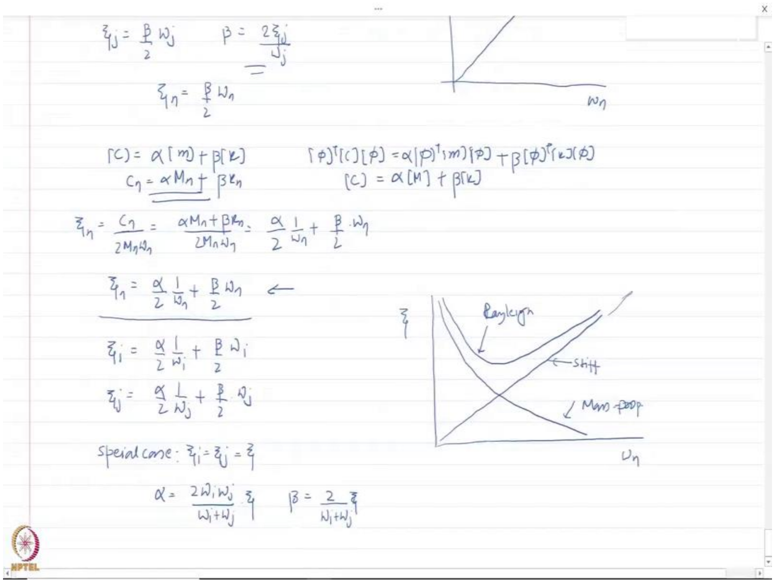
$$[\phi]^T [C][\phi] = \alpha [\phi]^T [m][\phi] + \beta [\phi]^T [k][\phi]$$
$$C_n = \alpha M_n + \beta K_n$$

So, that is where I get this from just considering the nth diagonal element.

So, again

$$\xi_n = \frac{C_n}{2M_n\omega_n} = \frac{\alpha M_n + \beta K_n}{2M_n\omega_n} = \frac{\alpha}{2}.\frac{1}{\omega_n} + \frac{\beta}{2}.\omega_n$$

(Refer Slide Time: 18:45)



So, let me rewrite it again.

$$\xi_n = \frac{\alpha}{2}.\frac{1}{\omega_n} + \frac{\beta}{2}.\omega_n$$

So, this is the expression. Now, we have two unknown constants α and β. So, we need two equations. So, now we need specification of damping or assumption of damping for two of the modes.

So, either we can find it based on some experimental evidence or we can just assign it some values which represent some representative value of the actual damping in the system. So, what we are going to do? We are going to consider specified damping in two modes. Let us say those are ith and jth mode.

$$\xi_i = \frac{\alpha}{2}\frac{1}{\omega_i} + \frac{\beta}{2}\omega_i$$
$$\xi_j = \frac{\alpha}{2}\frac{1}{\omega_j} + \frac{\beta}{2}\omega_j$$

So, now what you have two simultaneous equations in terms of α and β and it can be solved and once the α and β are known. Then utilizing this expression here $\left(\xi_n = \frac{\alpha}{2}.\frac{1}{\omega_n} + \frac{\beta}{2}.\omega_n\right)$, I can find out damping for any other mode. In this case for a special case let us say when $\xi_i = \xi_j$, the solution for α and β can be written as-

$$\alpha = \frac{2\omega_i\omega_j}{\omega_i + \omega_j}.\xi \quad \beta = \frac{2}{\omega_i + \omega_j}.\xi$$

Now if you look at the variation for the Rayleigh damping, remember for mass proportional damping it was varying like this, and for the stiffness proportional damping it was varying like this. For Rayleigh damping it is basically sum of both damping mechanisms. It actually varies like this.

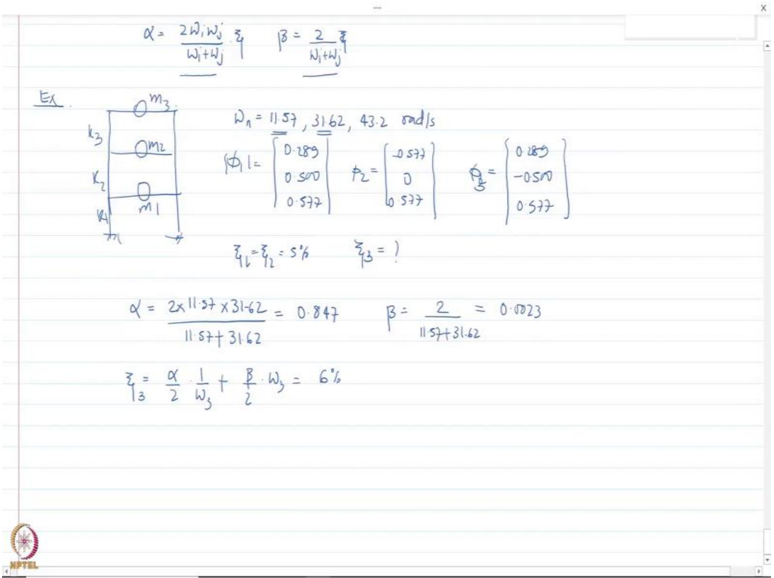
So, this is mass proportional, and this is stiffness proportional, and this is Rayleigh damping. So, the Rayleigh damping overcomes few limitations of the mass proportion or the stiffness proportional damping which are if we assign certain value, it is stiffness proportional damping. It would mean that if multiple degree of freedom system has large number of modes, the damping would increase with the mode frequency, and it would become unbounded after certain point.

Similarly, for the mass proportional damping if we specify based on some modes which is not the primary mode, then what will happen? Greater than intended damping would be assigned for a smaller mode. However, through Rayleigh damping we can consider the modes to which the damping needs to be assigned and all the frequencies that would lie between, then damping would be bounded for those.

So, there are extensive research on the different damping mechanisms, but for the purpose of this course we are only going to study up to this and what we are going to do?

We are going to do one example to see how we implement these damping mechanisms. So, if we have the damping values for few of the modes or assume it, we can find damping for other modes.

(Refer Slide Time: 22:40)



So, let us do an example. In this example, a three-story building is given to us, there are some masses and stiffnesses. This is m_1, m_2 and m_3. So, in general we need to find out the mode shapes and the frequency, but for this case those are given to us.

$$\omega_n = 11.57,\ 31.62,\ \text{and}\ 43.2\ \frac{rad}{\sec}$$

And the mode shapes are also given to us.

$$[\phi_1]=\begin{bmatrix}0.289\\0.5\\0.577\end{bmatrix},\ [\phi_2]=\begin{bmatrix}-0.577\\0\\0.577\end{bmatrix}\ \text{and}\ [\phi_3]=\begin{bmatrix}0.289\\-0.5\\0.577\end{bmatrix}$$

So, the mode shapes are also given, and it even said that for the first two modes, the damping values are given as 5 percent $(\xi_1=\xi_2=5\%)$. We need to find out what will be the damping ratio for the third mode (ξ_3).

Let us discuss the solution to this problem. Remember the damping values are same. So, we can just use these expressions here to find out α and β. When we substitute-

$$\alpha = \frac{2\omega_i\omega_j}{\omega_i + \omega_j}.\xi = \frac{2\times 11.57\times 31.62\times 0.05}{11.57+31.62} = 0.847$$

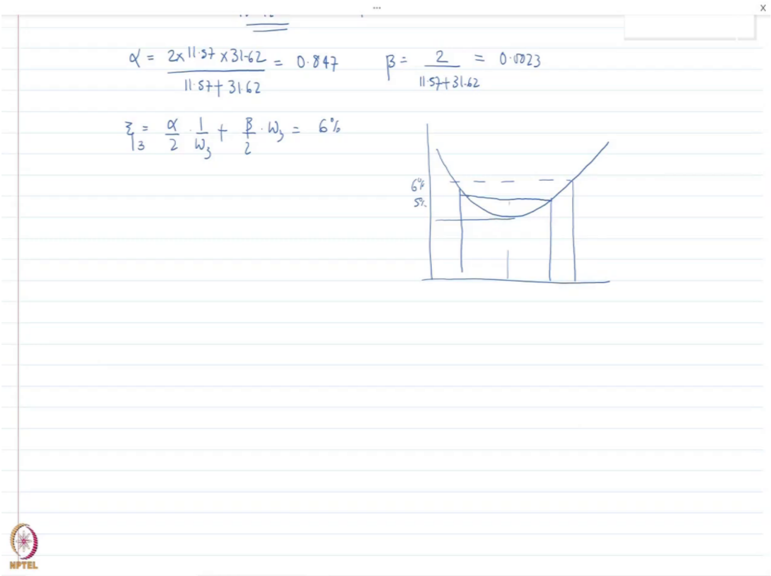
$$\beta = \frac{2}{\omega_i + \omega_j}.\xi = \frac{2}{11.57+31.62}\times 0.05 = 0.0023$$

So, for any other mode damping can be found out as-

$$\xi_3 = \frac{\alpha}{2}\frac{1}{\omega_3} + \frac{\beta}{2}\omega_3 = 6\%$$

So, we saw that utilizing the frequencies and the mode shape if we assume the damping values for the two mode shape, the damping for the third mode can be determined using the Rayleigh damping.

(Refer Slide Time: 25:55)



And one more thing to notice here in this case remember Rayleigh damping is something like this. We had assumed for the first mode. Let us say the second mode is around here and let us say third mode is around here.

So, the third mode which is outside this range, the frequency would be greater than ω_1 and ω_2 the damping is around 6% right compared to the first two mode which is around 5%. If we had considered, for the first mode and some third mode or the fourth mode and wanted to find out the damping in between of the modes, that would be always smaller than the assumed damping.

So, keep that in mind. So, with this discussion we are going to conclude our discussion on damped free vibration. In next class, we are going to start a new chapter on Forced Vibration.

Thank you.