Dynamics of Structures
Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Free Vibration of MDOF Systems
Lecture - 23
Free vibration-Undamped and Damped

Welcome back everyone. So far, we have learned how to set up the equation of motion of a multi degree of freedom system and we also saw that a multi degree of freedom system is represented by different modes. So, the total response is actually contribution of the response of each mode.

In today's class, what we are going to learn is basically, how to get the response of a multi degree of freedom system in terms of response at each degree of freedom. So, we are talking about basically displacement response. And we will extend the same concept to find out other type of response quantities, like shear forces and moment. And we are going to see for both type of system; undamped and damped system. How to actually get the expression for the displacement response.

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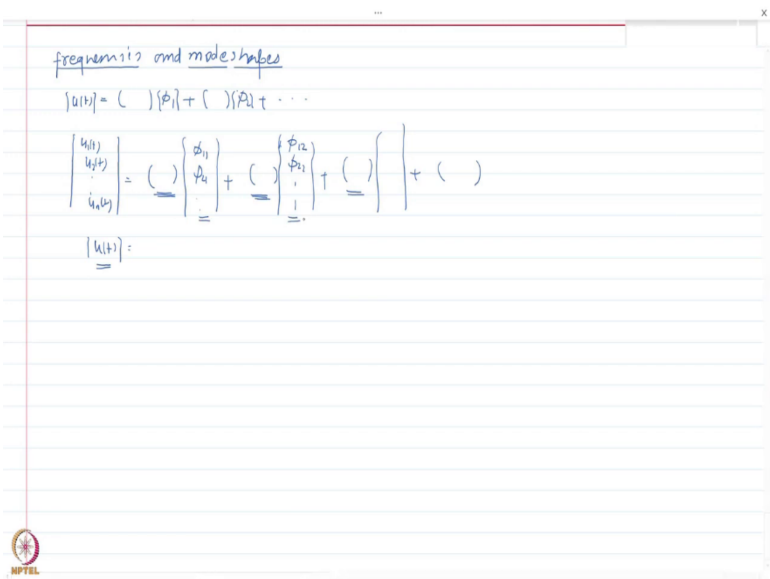

So, let us get started. In the last class we discussed basically the frequency and the mode shapes of a multi degree of freedom system. So, we have frequencies and mode shapes. And we said that, if multi degree of freedom system is excited through some excitation in general, the response of each degree of freedom would not be harmonic.

But there exist few characteristic shapes in which, if they are provided displacement according to those shapes then the multi degree of freedom system would have harmonic response at each degree of freedom and it is going to maintain its shape throughout the vibration, and those characteristic shapes are actually called mode shapes.

And, because the response is harmonic each of these modes would have their own frequencies which is basically the time taken to complete 1 cycle of motion, that is the time period and we can get frequency using the relationship as $\frac{2\pi}{T}$. So, we discussed that and we said that the total response of a system would be contributed by each of these mode.

We said that the total response would be combination of each the contribution due to each of these mode shapes. And depending upon how much of those contribution are some modes are going to govern the response, and we are going to learn about that later, but right now, let us consider that the total response which is nothing but the response at each degree of freedom let us say and so on.

It can be written as, a response due to all mode shapes which are represented through this shape vectors and so on. Now, we already know the frequency, we already know the mode shape, so the problem statement here is given the vector u, is it possible to find out the factors that need to be multiplied to these mode shapes so that we can get the modal decomposition or modal expansion we call it of this displacement vector here.

So, basically our objective here is to find out these factors here, because once we derive expression to find out these factors then we can find out the total response as a linear combination of contribution due to different mode shapes. So, let us see how do we do that.

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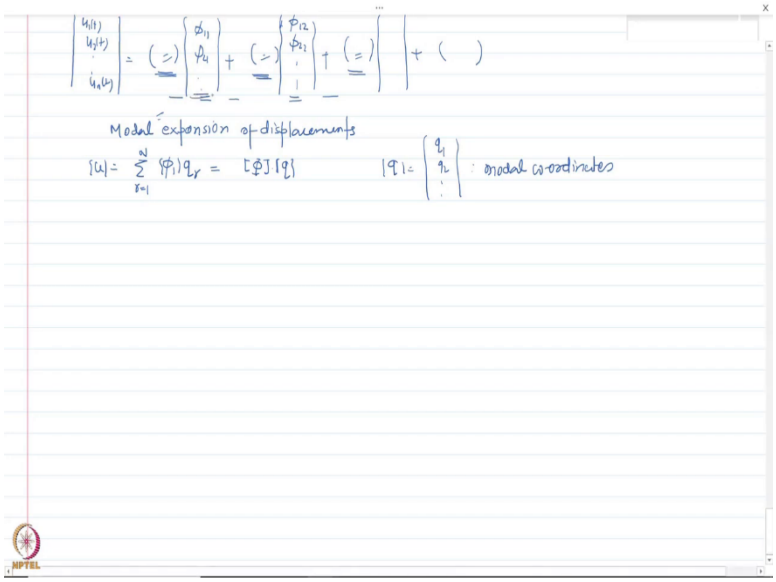

So, this is basically called modal expansion of displacements.

$$\begin{bmatrix} u_1(t) \\ . \\ . \\ u_n(t) \end{bmatrix} = () \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ . \\ . \end{Bmatrix} + () \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ . \\ . \end{Bmatrix} + ...$$

And let us look at that how to get those factors. So, let us assume that the vector u is basically, so I am just going to write this expression that we had written here in terms of summation. So, let us say we have N modes here, N degrees of freedom, so we will have

$$\{u\} = \sum_{r=1}^{N} \{\phi_1\} q_r = [\phi]\{q\}$$

So, these are called modal coordinates. The vector q is basically q_1 q_2 and so on; and these are called modal coordinates. And these are the factor that are required to find out the total response by combining different modes here. So, let us see how we can find out this individual q_r here. Now, we are going to utilize again the condition of orthogonality of the modes which basically says that.

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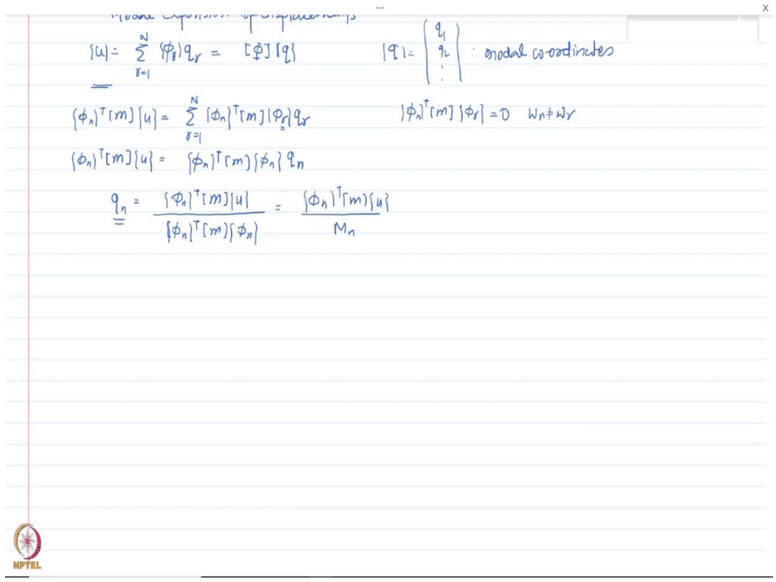

If I have two different modes then, if I take the product with respect to the mass matrix ok the product of the transpose of one mode times the mass matrix times the product of another mode that would be equal to 0, as long as those are different mode shapes.

$$\{\phi_n\}^T[m]\{\phi_r\} = 0$$

So, what I am going to do here, I am going to pre multiply this expression that we have written with $\{\phi_n\}^T$.

$$\{\phi_n\}^T[m]\{u\} = \sum_{r=1}^{N}[\phi_n]^T[m][\phi_r]q_r$$

Now, as we know due to modal orthogonality, when I expand this summation, when I execute this summation what will happen most of the term will vanish except when r is equal to n. That is the only term that is going to survive. So, knowing that the expression that I will get is actually

$$q_n(t) = e^{-\xi w_n t}\left[q_n(0)\cos w_{nD}t + \frac{\dot{q}_n(0)+\xi_n w_n q_n(0)}{w_{nD}}\sin w_{nD}t\right]$$

Now, if you look at the denominator, this is nothing but the diagonal element of the diagonalized mass matrix. So, basically what I am saying here, this is nothing but, in the denominator, I can write this as M_n which is the n^{th} diagonal element of the diagonalized mass matrix.

So, we have found out q_n now. So, we can do that for all the mode shapes and find out these factors, multiplicative factors for all the mode shapes. Hence, and once we know that then we can combine modes using those factors. This would be clearer after we do one example. So, let us take an example that we have been doing till now.

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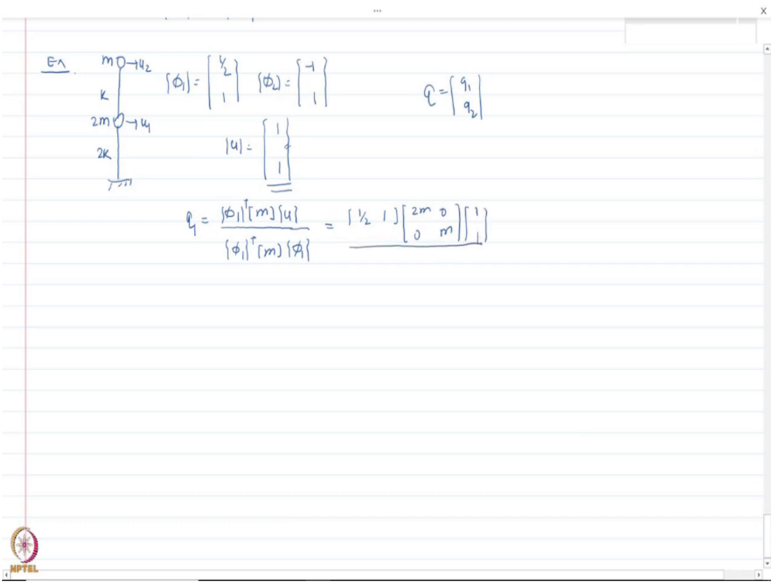

So, basically, I have the same 2 story representation of a building in which this here is 2k this is k this is 2m and this is m. Now, we know that the mode shapes for we had derived the mode shapes, and we had got that, but this is degree of freedom 1 and this is 2, so this was half and 1. We had normalized with respect to the top story or u_2.

$$\{\phi_1\} = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Now the question is, that at any time instant let us say u is given as

$$\{u\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

and what I have to find out basically, the modal expansion of the displacement vector. So, I have to find out the expansion of this in terms of some factor times the first mode shape plus some factor times the second mode shape. So, basically, I have to find out those factors

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

So, let us see how do we do that. So, my q_1 will be nothing but

$$q_1 = \frac{\{\phi_1\}^T [m] \{u\}}{\{\phi_1\}^T [m] \{\phi_1\}} = \frac{\begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}} = \frac{4}{3}$$

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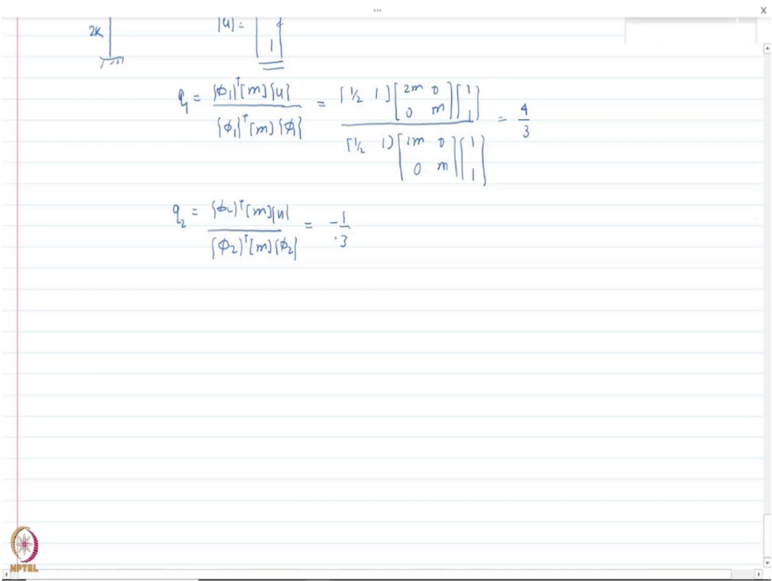

Similarly, q_2 I can find out as

$$q_1 = \frac{\{\phi_2\}^T[m]\{u\}}{\{\phi_2\}^T[m]\{\phi_2\}} = -\frac{1}{3}$$

So, you got this factors, let us see whether we get the same expression not.

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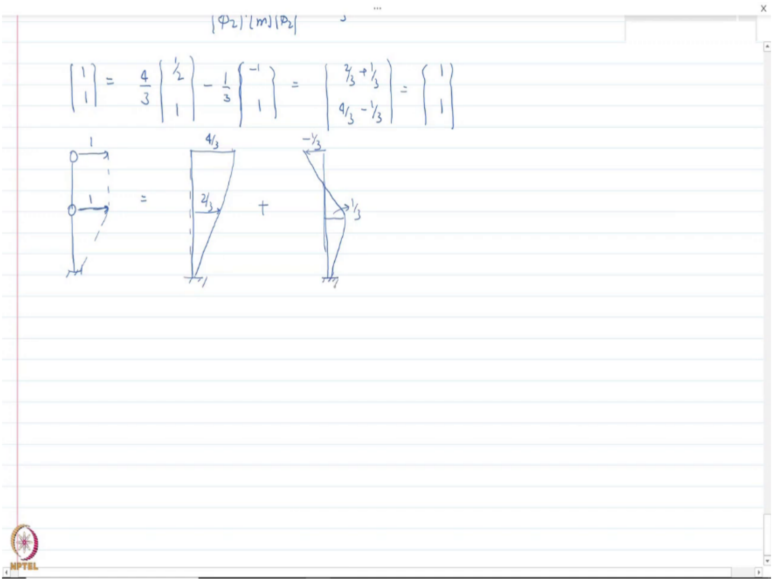

And, if we want to represent this in terms of or like you know using some schematic in terms of the mode shape, problem is that or the statement the problem statement is initially it is given displacement which is 1 at each degree of freedoms unit displacement at both positions.

So, this is 1 here and this is 1 here. Now, this is equal to ok if you look at the first mode ok, the mode shape is so this is basically 2/3 and 4/3. So, I can just write it like this is 4/3 and this is 2/3 here, plus sum of the second terms which is +1/3 and minus -1/3. So, let me again draw this here. So, this is what we get.

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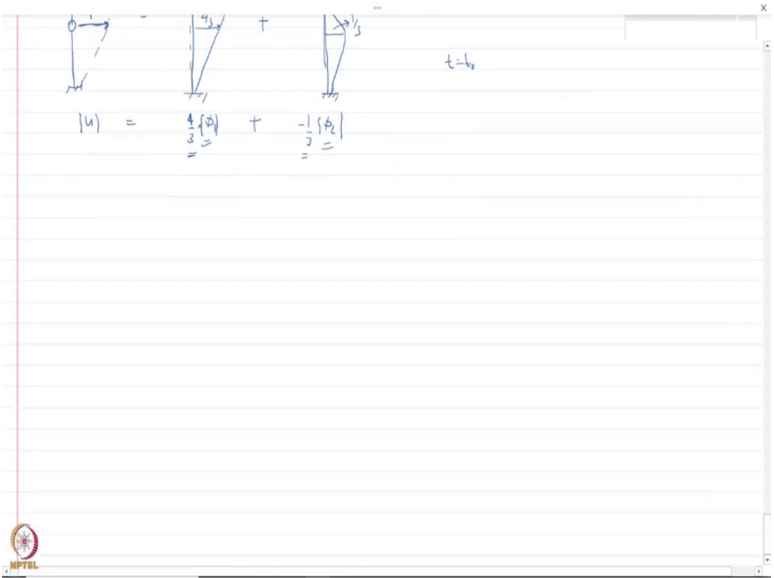

So, the total response which is basically u is represented like this.

$$\{u\} = \frac{4}{3}\{\phi_1\} - \frac{1}{3}\{\phi_2\}$$

So, you can see that through the modal expansion at any time instead whatever the displacement is given can be expanded in terms of contribution of the first mode and the second mode and so on.

If it has n modes then there would be n such mode shapes, and vice versa it can be also true. So, the idea is that, if we have n number of modes they can be combined using these factors q_1 q_2 and so on, to be find out the total displacement at all degrees of freedom.

And, we are going to see how we are going to get these vectors, now remember, this q that we have obtained here it is at any time instant, but in general q is a function of time. So, this is for any time fixed time instead of $t=t_0$, but in general q is basically the function of time which is the time variation. So, at any time it would be distributed or contributed by different nodes. So, let us get into that. So now, what we are going to do here.

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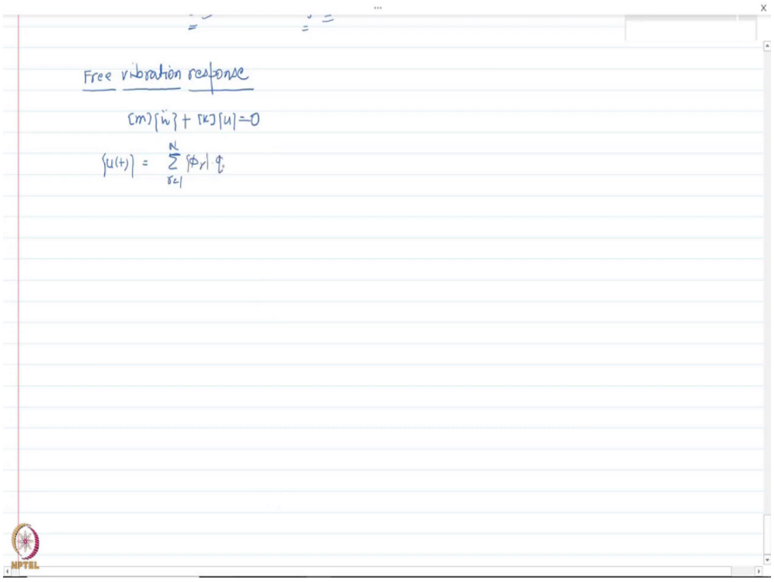

Actually, going to solve the equation for free vibration response. So, remember that our equation of motion is

$$[m]\{\ddot{u}\}+[k]\{u\}=0$$

We have said that my, now we are going to write in terms of as a function of time t. So, this is nothing

$$\{u(t)\}=\sum_{n=1}^{N}\{\phi_n\}q_n(t)$$

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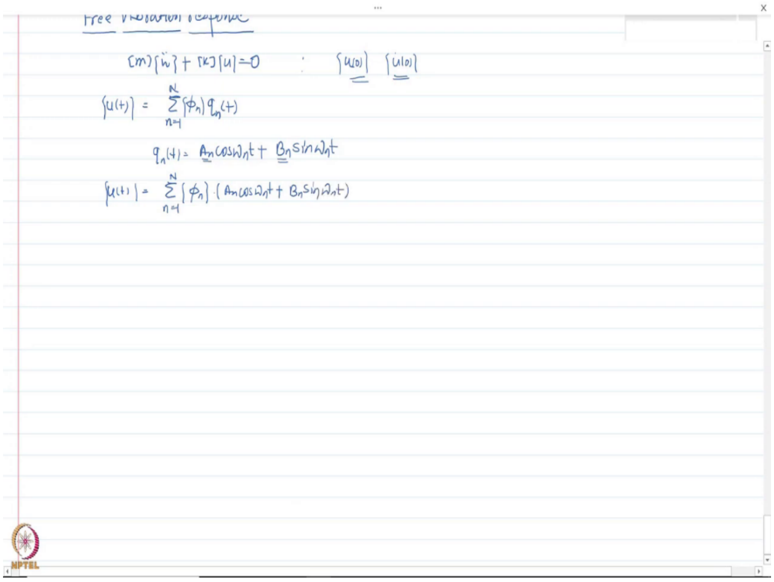

So, we need to find out this $q_n(t)$ and we have previously said that, because each mode shapes respond as they respond in the harmonic fashion, so the displacement q and t which is the generalized displacement or the time variation of that mode shape can be written as

$$q_n(t) = A_n \cos w_n t + B_n \sin w_n t$$

where this A_n and B_n are basically constants that need to be found out you using initial condition.

If this is our equation of motion the initial conditions are typically given at displacement vector at time t=0 and the velocity vector at t=0, and utilizing this we find out A_n B_n and w_n is basically the frequency, the circular frequency of the n^{th} mode.

So, let us see how do we get the constant A_n B_n, because, once we get that then we can find out the overall displacement vector. If you utilize the initial condition we have the expression for u here, so let me just rewrite it. This is nothing but

$$\{u_n(t)\} = \sum_{n=1}^{N} \{\phi_n\}(A_n \cos w_n t + B_n \sin w_n t)$$

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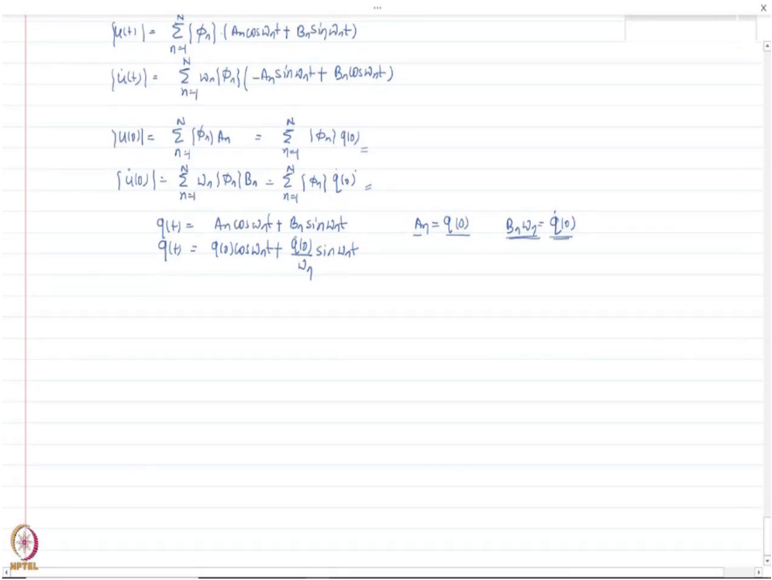

And, we can similarly get the velocity by differentiating with respect to time. The same expression here,

$$\{u_n(t)\} = \sum_{n=1}^{N} w_n \{\phi_n\}(-A_n \sin w_n t + B_n \cos w_n t)$$

So, let us substitute t=0 to get these expressions here. So, that will give that would give us

$$\{u(0)\} = \sum_{n=1}^{N} \{\phi_n\} A_n$$

$$\{u(t)\} = \sum_{n=1}^{N} \{\phi_n\} q_n(t)$$

So, this is the expression to that we would utilize to find out the constants A_n and B_n. Now, if you look at A_n and B_n and let us say my q(t) is written as this expression here,

$$q_n(t) = A_n \cos w_n t + B_n \sin w_n t$$

Now, if we say that let us assume

$$A_n = q_n(0)$$
$$B_n w_n = \dot{q}_n(0)$$

Then I can write this expression as,

$$q(t) = q(0)\cos w_n t + \frac{\dot{q}(0)}{w_n}\sin w_n t$$

Remember, these are also constants here, but why I am writing it like this, so that my q(t) is similar to the expression that we had obtained for single degree of freedom system in terms of u(0) and $\dot{u}(0)$, but now we obtain here in terms of the modal coordinate.

So, I am just making this substitution here, so that it is we can correlate it to single degree of freedom system. So, if that is there then this expression can be written as

$$\{u(0)\} = \sum_{n=1}^{N} \{\phi_n\} A_n = \sum_{n=1}^{N} \{\phi_n\} q(0)$$

$$\{u(0)\} = \sum_{n=1}^{N} w_n \{\phi_n\} B_n = \sum_{n=1}^{N} w_n \{\phi_n\} \dot{q}(0)$$

Now, we can look at these two expressions and q(0)and $\dot{q}(0)$ are nothing but you know the expressions similar to what we had derived here. So, we can utilize the same expression to find out the value of q(0)and $\dot{q}(0)$.

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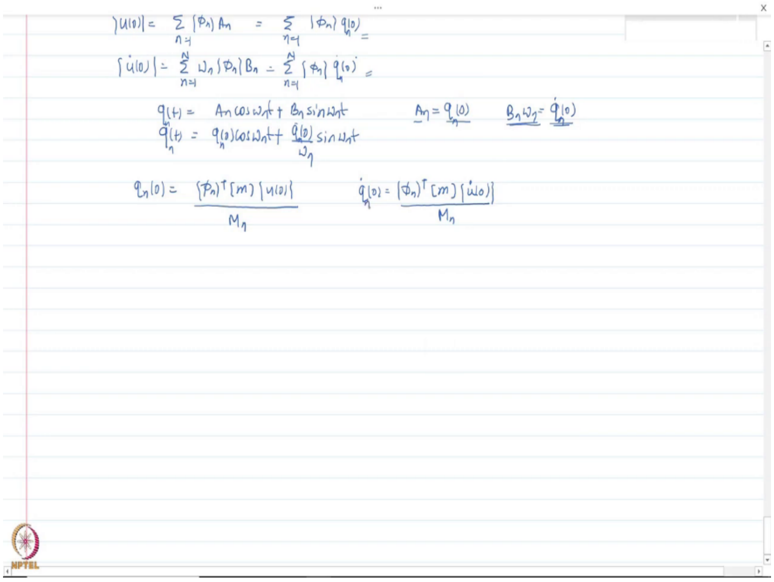

So, basically, so then remember this is $q_n(0)$ for n^{th} mode. So, just be careful with that all things we are doing for the n^{th} mode. So

$$q_n(0) = \frac{\{\phi_n\}^T [m] \{u(0)\}}{M_n}$$

$$\dot{q}_n(0) = \frac{\{\phi_n\}^T [m] \{u(0)\}}{M_n}$$

So, utilizing these two expressions we can get, q(0)and $\dot{q}(0)$.

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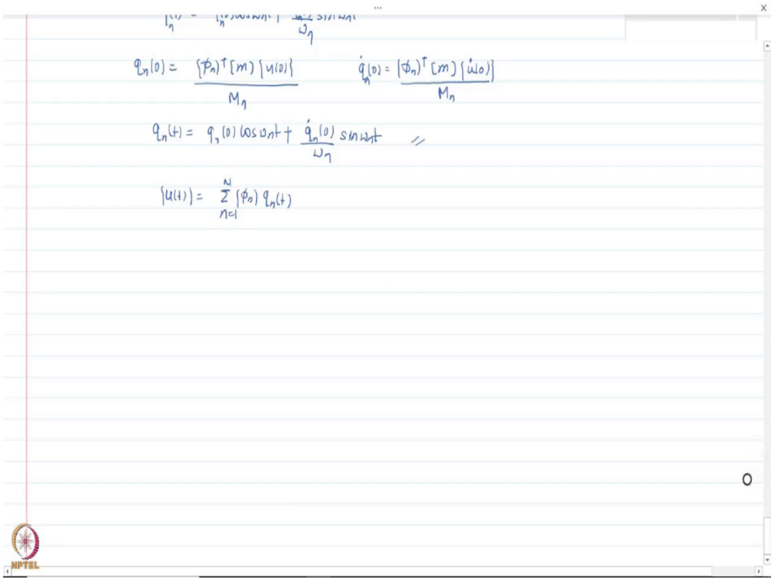

And, once we have that then the expression for $q_n(t)$ is known.

$$q_n(t) = q_n(0)\cos w_n t + \frac{\dot{q}_n(0)}{w_n}\sin w_n t$$

Once $q_n(t)$ is known then we can do that for each and every mode and we can find out the total response using this summation that we had derived. So, this is basically

$$\{u(t)\} = \sum_{n=1}^{N} \{\phi_n\} q_n(t)$$

So, basically, we saw that how to find out the response, the free vibration response of an undamped system.

And, basically what we do? We find out the time variation of the generalized coordinate for each mode, we find out the constants unknown constant using the initial conditions, and finally we combine all those modes using the modal coordinate at any time t. And, we can find out the total response is a function of linear combination of all those mode shapes. So, we can do an example of this and then see, demonstrate the free vibration response.

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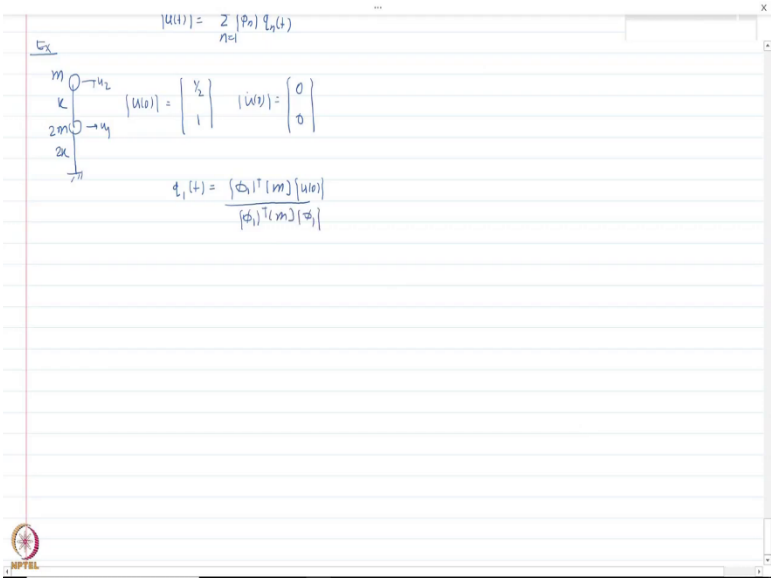

So, let us again take the same example. So, this is here 2k and k here, this is 2m and m here, again ϕ_1 and ϕ_2 have been given to it. Now, what is given has initial conditions have been given to us.

$$\{u(0)\} = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

$$\{u(0)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

So, let us see what do we get as the overall response, the u(t) value. To do that, first I need to find out these factors $q_1(t)$ and then $q_2(t)$. Let us see how do we get that. So basically,

$$q_1(0) = \frac{\{\phi_1\}^T [m] \{u(0)\}}{\{\phi_1\}^T [m] \{\phi_1\}} = \frac{\begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}} = 1$$

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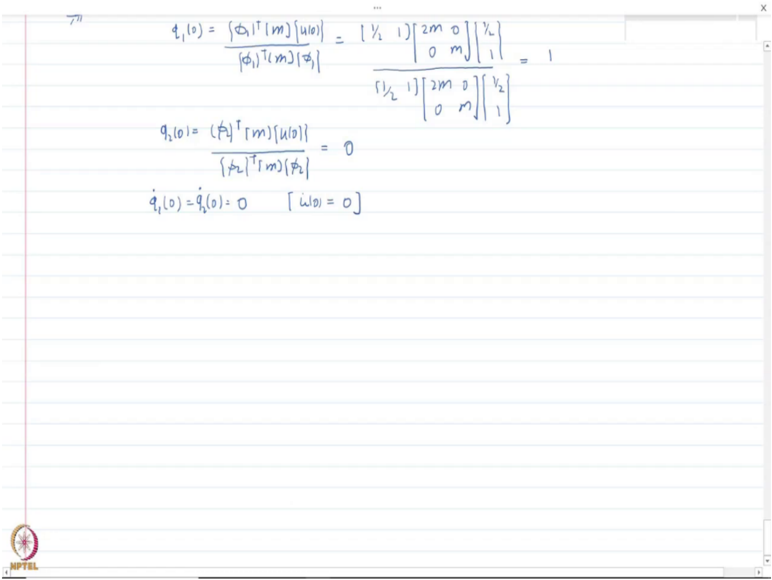

$$q_2(t) = \frac{\{\phi_2\}^T [m] \{u(0)\}}{\{\phi_2\}^T [m] \{\phi_2\}} = 0$$

$$\dot{q}_1(0) = \dot{q}_2(0) = 0$$

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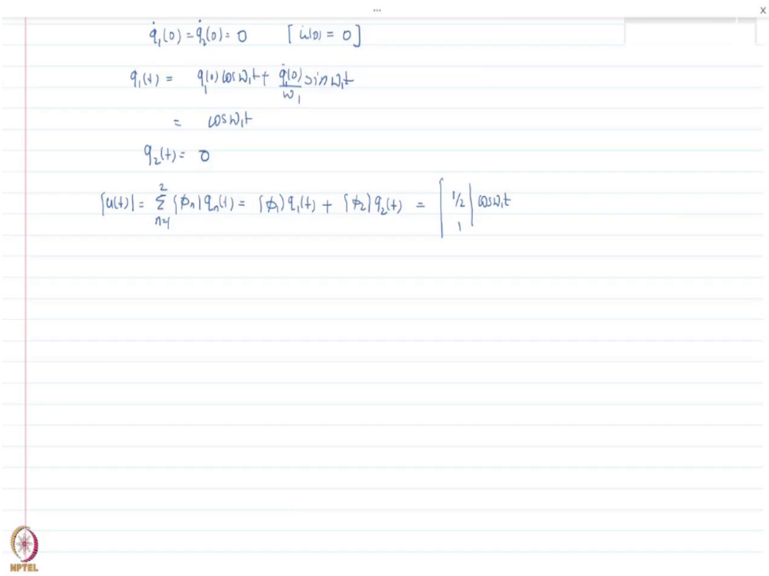

So, we can go ahead and, in our expression, we can substitute

$$q_1(t) = q_1(0)\cos w_1 t + \frac{q_1(0)}{w_1}\sin w_1 t = \cos w_1 t$$

So, these two; $q_1(t)$ and $q_2(t)$, we have obtained as these expressions.

So, remember,

$$[u(t)] = \sum_{n=1}^{2}\{\phi_n\}q_n(t) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\cos w_1 t$$

So, this was the first part, let us again find out the response for another case in which.

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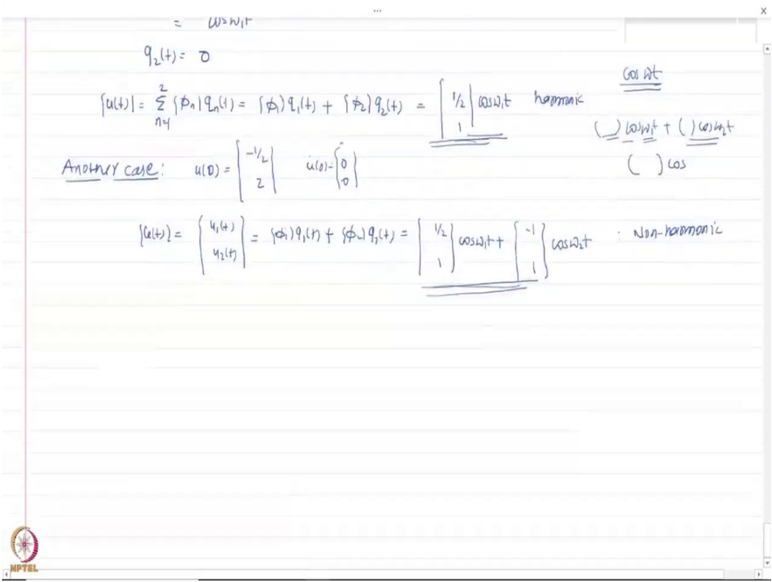

In this case the initial displacement has been given as

$$u(0) = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}$$

$$\dot{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, we can again follow the same procedure and we can write down the displacement as

$$\{u(t)\} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \{\phi_1\}q_1(t) + \{\phi_2\}q_2(t) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cos w_1 t + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos w_2 t$$

So, this is the response for the second case. Now, the reason that we did like in the 2 cases, I wanted to show you something.

You have not you have not look at the response here and the response here, what can you tell me about these responses. If you look at it, and if you the response for the first one, this is actually a harmonic motion at each degree of freedom right, because it is a coswt type, as a cosine function. However, in the second one, I have some function times some vector times $\cos w_1 t$ plus another vector times $\cos w_2 t$.

So, this is not a harmonic variation because I have two frequency and it cannot be written as some constant times some cos or sine variation. Had it been the same angle omega 1 and omega 2 equal to omega, then I would I could have done that, but not in this case.

So, in the first case, the response is harmonic; whereas, in the second case the response is non harmonic. One more thing to notice here, in the first case I can see only the contribution due to the first mode, there is no contribution of the second mode, there is no $w_2 t$ term here.

However; here you can see that there is contribution of both modes. So, only the contribution of first mode in the first case, while in the second case there is contribution of both mode. And that can be explained directly if you look at the initial displacement.

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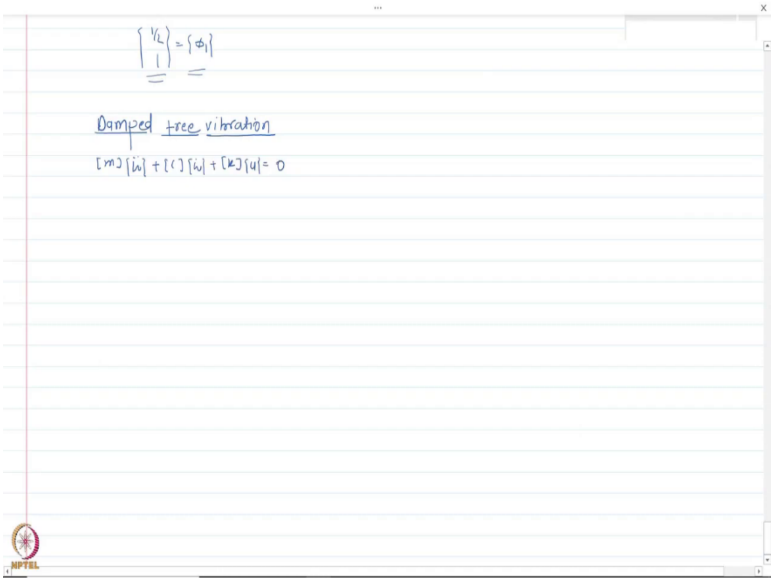

For the first case, remember the initial displacement was half and 1 which is also the first mode shape of the given structure, and by definition as we have previously discussed, if you assign initial displacement to the structure which is according to the one of the characteristic shapes, then it is going to respond harmonically while maintaining it is shape, and it is going to vibrate in a particular mode of vibration.

So, there would be no contribution of any other mode. So, that is why we do not see any contribution of other modes, because it is being vibrated with initial displacement which is one of it is mode shapes. Compare that to second one, the initial displacement here is not one of its mode shapes. So, in general, when we provide initial displacement we will have contribution of both modes and the response would not be harmonic at each degree of freedom ok.

And we have just seen example of that, whatever we had discussed. So, I hope this example is clear. Now, let us get into damped free vibration ok. So, we have discussed undamped free vibration, now let us discuss damped free vibration. Now, for the damped free vibration we will now have an additional damping term here, with the damping matrix C here.

$$[m]\{\ddot{u}\}+[c]\{u\}+k\{u\}=0$$

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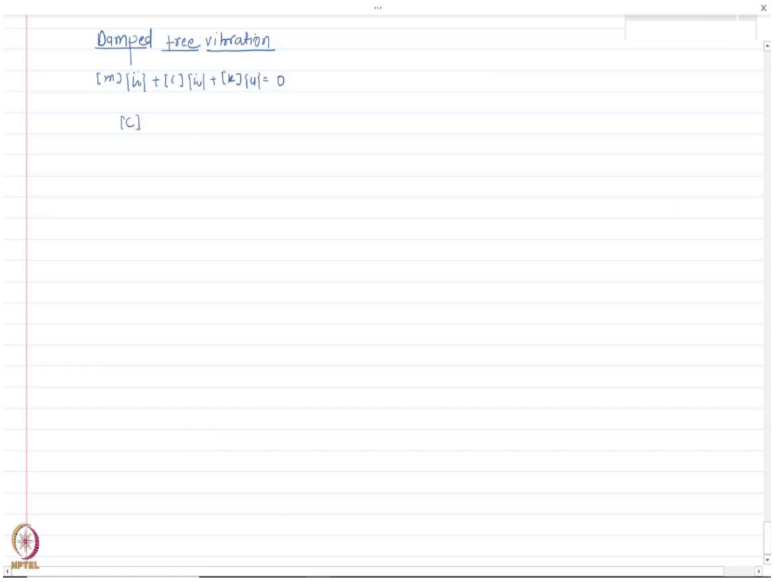

Now, in this case if you look at it here, I have this damping matrix. Now, the reason I was able to solve the undamped free vibration because my mass matrix and my damping matrix could be diagonalized.

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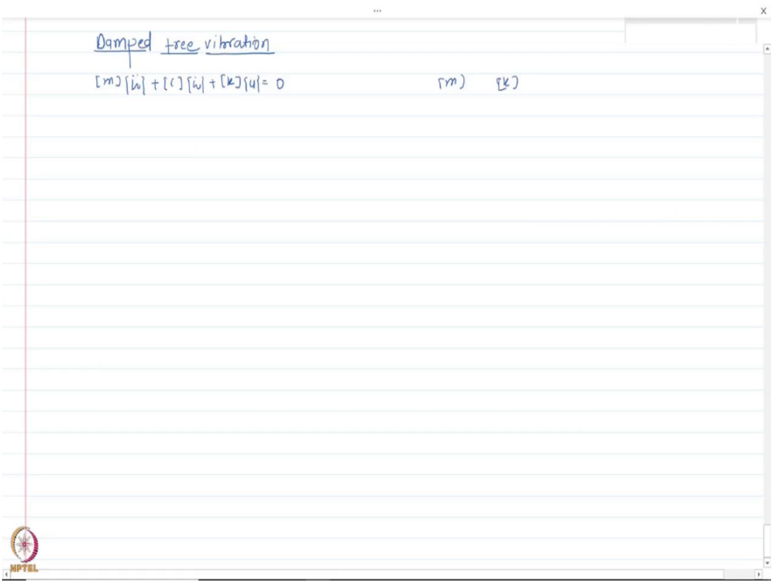

The solution for damp free vibration would depend on the fact that whether my C matrix can be diagonalized or not. Because, if I am able to diagonalize my C matrix or the damping matrix, then I can uncouple all the equation of motion for all the different modes, so I would have n uncoupled differential equations. And then, I could solve those and combine the responses using the modal expansion vectors. So, whether I can solve this analytically or not, it would depend on C.

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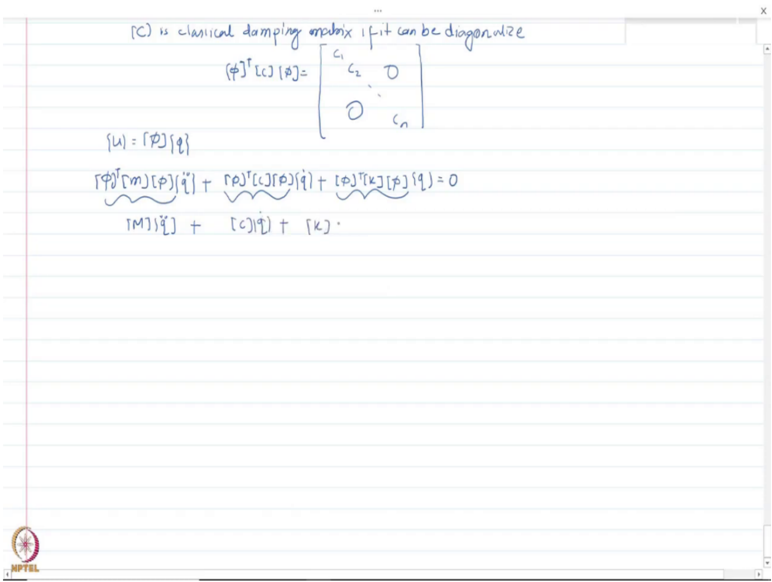

So basically, C plays a big role now. If C is a classical damping matrix, so let me if it can be diagonalized, and how do we diagonalize this? How do we diagonalize any matrix? We multiply with the modal matrix and then see whether it can be diagonalized.

$$[\phi]^T[c][\phi]=\begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & c_n \end{bmatrix}$$

And if it can be diagonalized we call it a classical damping matrix. There is non classical damping matrix as well, and there are some methods of solution for those as well, but that is not within the scope of this course. So, let us consider classical damping matrix where C can be diagonalized and then see if we can get solution to this damped free vibration. So, what we are going to do here, I am going to write my u as

$$[u]=[\phi]\{q\}$$

So, first I am going to substitute this and then I am also going to pre multiply with the modal matrix transpose.

$$[\phi]^T[m][\phi]\{\ddot{q}\}+[\phi]^T[c][\phi]\{\dot{q}\}+[\phi]^T[k][\phi]\{q\}=0$$

Now, as we know this is diagonal mass matrix, this is diagonal damping matrix and this is diagonal stiffness matrix. So, this can be written as

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=0$$

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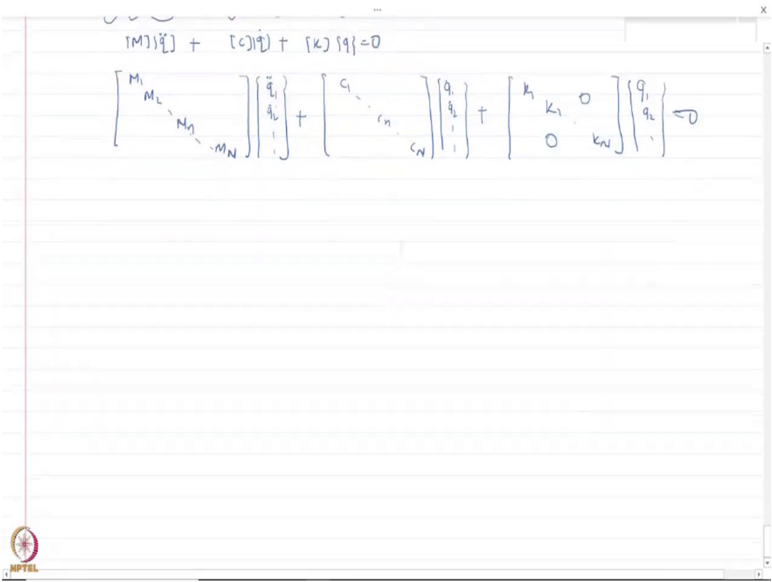

And because, remember all these expressions are in terms of M_1 M_2 and so on. Let us say M_n here like this, and then q_1 q_2 and so on. Similarly, C_1 Cn q_1 q_2 plus K_1 K_2 all these are uncoupled an n uncoupled differential equation.

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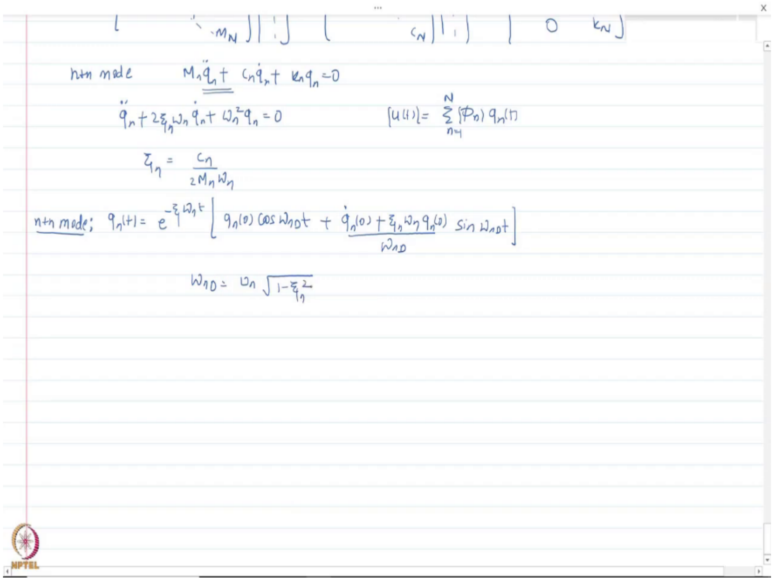

And for the n^th mode I can write it as

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = 0$$

which is of the form of the free vibration response of a single degree of freedom system. So, I have basically decomposed my multiple degree of freedom system through the modal decomposition into n single degree of freedom system, and then I am going to find out the response for each system which would be basically response for each mode in terms of the modal coordinate $q_n(t)$, and once I have the $q_n(t)$ we know that we can combine all the modes using the multiplication of the mode shape times the $q_n(t)$ using this expression here.

Now, like what we did for free vibration response of a damped undamped system, we can do the same thing for the damped system as well ok, and $q_n(t)$ can be similar the obtain. Now, to do that let me first do this, write down the same equation that we have by dividing throughout by M n and writing it in this form

$$\dot{q}_n + 2\xi_n w_n \dot{q}_n + w_n^2 q_n = 0$$

Now, if you look at here, we have damping for the nth mode. So now, for a multi degree of freedom system the way the damping ratio is defined, there is a separate damping ratio for each of the mode. And that damping ratio is nothing but

$$\xi_n = \frac{c_n}{2M_n w_n}$$

And my $q_n(t)$ can be written

$$q_n(t) = e^{-\xi w_n t}\left[q_n(0)\cos w_{nD}t + \frac{\dot{q}_n(0) + \xi_n w_n q_n(0)}{w_{nD}} \sin w_{nD}t \right]$$

This is similar to the expressions that expression that we had obtained for the damped response of a single degree of freedom system. Only now, we are considering everything for a particular mode. So, this is the response for the nth mode. And we have defined the damped frequency for the nth mode as

$$w_{nD} = w_n\sqrt{1-\xi_n^2}$$

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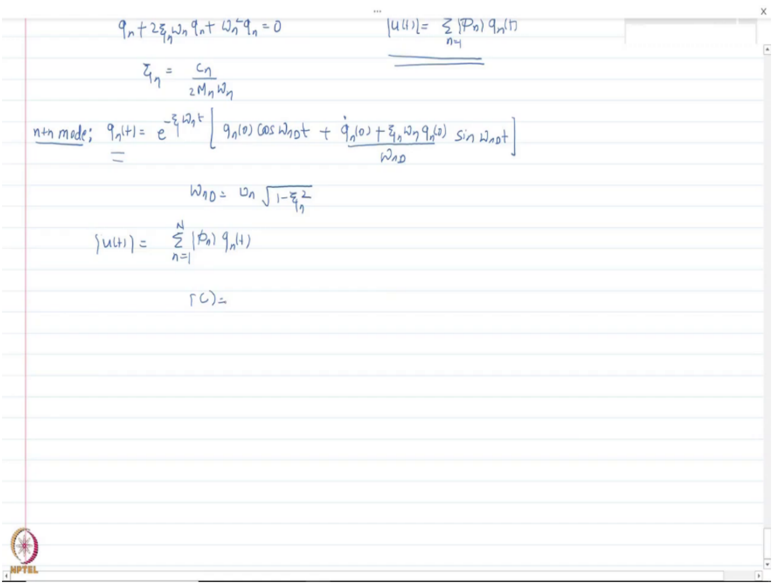

And, the total response I can write using the expression that I had here, has

$$\{u(t)\} = \sum_{n=1}^{N} \{\phi_n\} q_n(t)$$

So, this is how we get the response, the damped response free of a free response of a damped system.

Now, if we notice in this case typically, we do not get the damping matrix I have told you that you know you can get the damping matrix and see if it can be diagonalized or not ok. But, in practice there is no way we could get the damping matrix by just considering damper element in a structure. So, what we typically do for practical analysis, we assume or we determine experimentally the damping ratio for each mode.

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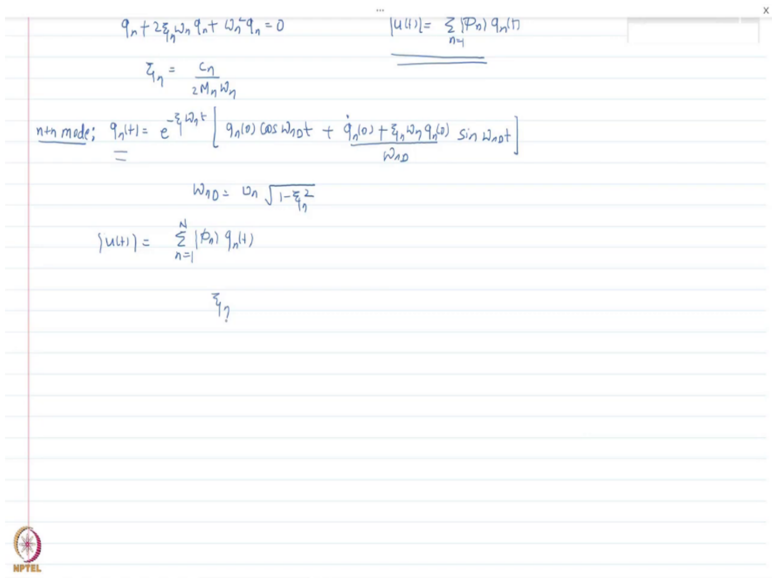

Because, remember now the damping is defined for each of these modes ok. And there are some experimental methods to do that ok, or we can assume some values based on the data that we have observed in the past from the testing or the response of other type of structure, similar structures.