Dynamics of Structures
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Module - 02
Multi-Degree-of-Freedom Systems
Lecture - 20
Equation of motion

Hello everyone. Till now, in this course when we discussed about Idealization of a Real System, we only talked about single degree of freedom system.

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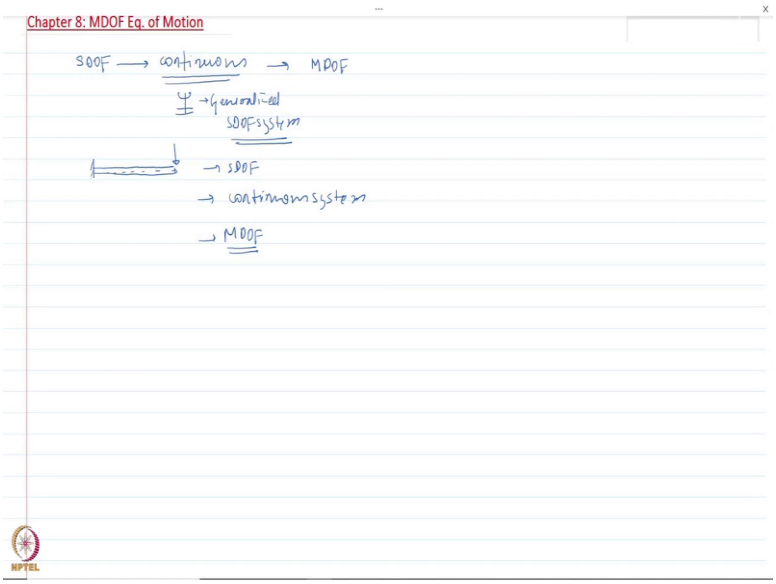

Now, single degree of freedom system, although it is the most simplified representation of a dynamic system, it may not always be appropriate. So, we may have to consider Multi Degree of Freedom System (MDOF), which may be able to represent the behavior of the structure more accurately. So, to get into that, first we look at different type of multi degree freedom system, and then like we did for a single degree of freedom system we will first set up the equation of motion for a multi degree of freedom system and then see how the equation of motion can be solved.

So, let us get started with this lecture. Till now, we have studied about single degree of freedom system and then we also studied about continuous basic systems, in which this continuous system was reduced to a single degree of freedom system or a more specifically generalized SDOF system by using a shape function.

Now, we said that this continuous function which was reduced using ∅ to a generalized SDOF system. We said that this was an approximate method because, we did not exactly know what was the deflected shape ∅. However, now what we are going to do, we are going to simplify our system to a multi-degree of freedom system, and we are going to find out the exact deflected shape using the methods of analysis in the subsequent chapters.

And, as we have previously discussed any system in reality; if you consider a cantilever beam, so in reality it would have infinite degrees of freedom. Now, it might be appropriate sometimes to reduce it to a single degree of freedom system, which would of course provide us limited result in terms of deformation. If we want know the variation of internal forces then we analyze it as a continuous system using this method, that we have discussed in previous chapter.

However, if we want more accurate representation, then we can approximate the same system as multi degree of freedom system and depending upon the number of degrees of freedom we considered we can get results that are quite accurate.

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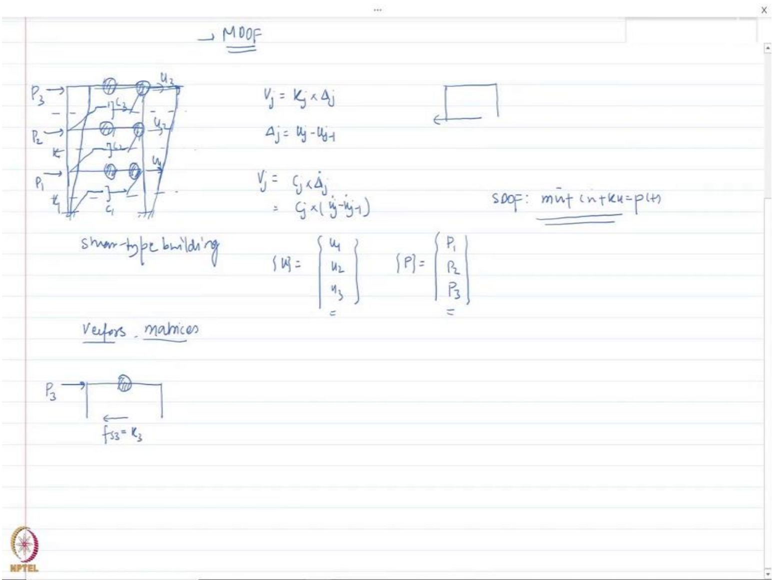

To demonstrate the application of multi degree of freedom system, we will start with a shear type building. We discussed in detail, what does a shear type building mean in terms of its behavior. So, we said that for a shear type building, the masses are lumped at each floor and we represent the deflected shape using horizontal degrees of freedom at each floor level.

So, if it is a three degree of three-story building, it would have three degree of freedom system, and let us say the deflected shape would look something like this. I do not know exactly what that deflected shape is, but I would be able to find that out, exactly.

However, before that what I need to do is set up the equation of motion. So, first let me draw the deflected shape, it would look something like this here or there is no flexure here. Remember, in a shear type building there is no flexure deformation in the beam. So, it looks like something like this, and this one also looks like something like this, so the masses are here, and we are going to represent the degrees of freedom as u_1, u_2 and u_3.

Now, like we have previously done, we can also assign a damper to it, which would something like this between each ends of a story. So, between the two floors of a story, the floor in the roof of the story. Let us say, so this damper has like c_1, c_2, c_3.

Now, we said that, at any story if we consider in multistory building, so let us say: at any story j, if there is a story deformation, there would be story shear. So that is why it is called shear type building, because the deformed shape is only represented through shear deformation in the building.

So, if there is a relative deformation between two floors, a floor and the roof of a story, then there would be story shear that is generated. One, would be due to the elastic deformation of the columns that connects the floor and the roof of that story and the second would be if we are considering damping. So, the first is a stiffness contribution to the story shear which is written as k_j, which is the story stiffness, k_j times the story drift.

The story drift is nothing but deformation at level j minus deformation at the story below that, so, $u_j - u_{j-1}$. So, this is the stiffness contribution. Similarly, we can also find out the damping contribution to the story shear as c_j times story drift velocity let us call it, so c_j times $\dot{u}_j - \dot{u}_{j-1}$.

Now, we have already obtained the solution for single degree of freedom system which was of this form. If you remember, this is $m\ddot{u}+c\dot{u}+ku=P(t)$. Now for this case, only one degree of freedom u was considered, but now if you look at here, you do not have a single degree of freedom.

So, we do not have a single deformation, we have u_1, u_2 and u_3. Similarly, forces might be at different level. So, let us say this is P_1, P_2 and P_3. So, we have forces and deformation that are not a single valued function, but it is a vector of multiple quantities.

So, here instead of a single quantity, the deformation u actually is a vector. Let us represent it as a vector, if it is a 3-story building, its u_1, u_2 and u_3 and if its n-story building, it could be u_1, u_2 upto u_n. Similarly, the force P, the applied force can be represented as another vector P_1, P_2 and P_3 and how do we write it depends on where the forces are being applied or whether the degrees of freedoms are being defined. So, u and P are written like that.

If somehow the equation of motion for this type of multi degree freedom system, it can be reduced in this form here, then the job would become much simpler, because I have already the solutions that we have derived for different type of loading and conditions. Now, there would be a key difference as I said between SDOF and MDOF system and that would be you now have multiple quantities here.

So now, we are not dealing with the single quantity, we would be dealing in vectors to represent the force vector or the displacement vector. The matrices which you would see is used to represent the stiffness matrix, mass matrix and the damping matrix. So, let us see how to derive the equation of motion for a multi degree freedom system starting with is three story building that we have.

So, to do that I am going to cut this building at each floor as I have shown here and then draw the free body diagram and then the write down the equation of motion for each of these three stories. So, let us do that.

So, let us first draw the free body diagram of the topmost floor. Remember we have a mass m here, there is a force P_3 which is being applied, now it is moving in the rightward direction as u_3 and the floor below it is moving in the rightward direction with the u_2. So, differential displacement or the story drift is u_3 - u_2.

So, there would be stiffness contribution of the force when I cut this story and it would be f_{S3}, which could be equal to k_3. So, I am assuming the story stiffness to be represented as k_1, k_2, k_3. Then I have another force, if I have defined the dampers between the story as f_{D3}.

So, this is f_{S3} equal to k_3 times u_3 - u_2 and f_{D3} is c_3 times $\dot{u}_3 - \dot{u}_2$. Because it is moving on the rightward direction it would have inertial force which is equal to mass of that flow times $\ddot{u}_3$ which is the acceleration of that mass.

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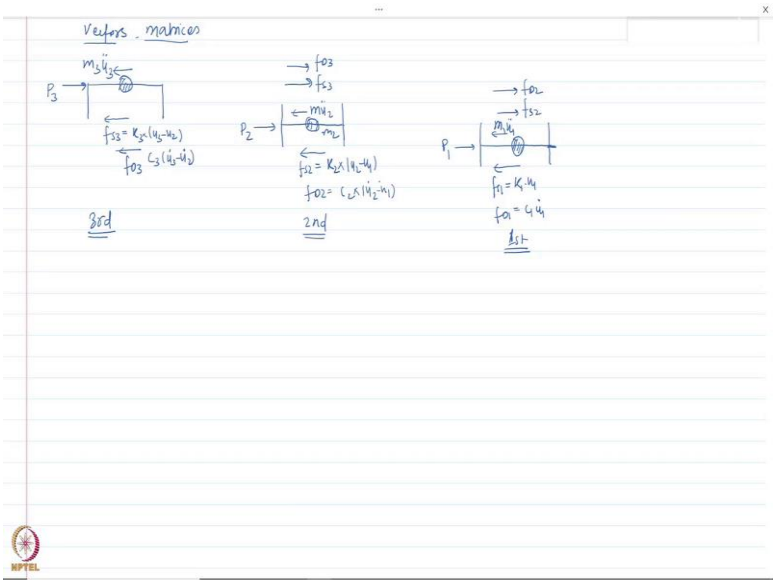

So, this is the free body diagram of the third story. Now, if we consider for the second story, I have forces form above and below both. Now, from the above, the forces could be equal and opposite to f_{S3} and f_{D3}. So, I have f_{S3} in this direction and f_{D3} in this direction, and I have this applied force which is equal to P_2. This is m_2. So, I have the inertial force which is acting opposite to the direction of motion as mass, times, the acceleration.

And then from the story below, again I would have a force f_{S2} which is equal to the story stiffness times the story drift at that floor remember, that deformation here is u_2 and below it is u_1 and that and would be opposite to the direction of the deformation. So, f_{S2} is this much, and then f_{D2} would be $c_2 \times \dot{u}_2 - \dot{u}_1$. Then, now let us draw the free body diagram for the story that we have.

I am going to follow the same procedure that I have done for the previous two the stories. So, I would have f_{S2} here equal and opposite to the forces that I have applied on the story above and f_{D2}; there is force P_1 here and then I have the inertial force which is m_1. So, $m_1 \times \ddot{u}_1$ and then I have force which is equal to $f_{S1} = k_1 \times u_1$; however, at the second the floor is basically the ground, so it is that has 0 velocity.

So, I will add this as one $k_1 \times u_1$ and similarly, f_{D1} would be $c_1 \times \dot{u}_1$. So, I have now three free body diagram. I am starting with the third story, because it just makes the job easier for me to explain in terms of forces from the top to bottom.

Now, we can go ahead and write down the equilibrium equation for each floor. So, the dynamic equilibrium equation of motion for each floor and then combine them as a single equation in terms of stiffnesses and matrices. In terms of writing down the equation, let me first write down the equation of motion for the first story.

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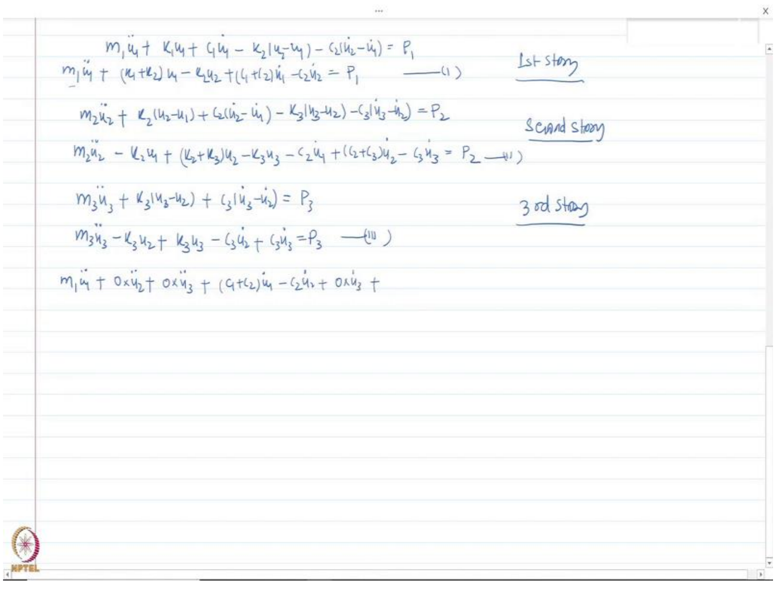

So, I am going to write down here; if you look at here,

$$m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1u_1 - f_{S1} - f_{D1} = P_1$$

I have already given you f_{S2} and f_{D2}. So, let me write it here this is $k_2(u_2 - u_1)$ and then minus $c_2 \times \dot{u}_2 - \dot{u}_1$ and this is equal to the applied force which is P_1.

So, we can write down in this form and then let us combine the similar terms together. So, this is,

$$m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + (k_1 + k_2)u_1 - k_2u_2 = P_1$$

This is for the first story. Now, let us go ahead and write down the similar equation of motion for the second story. So, the for the second story again I can write similarly as,

$$m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) - f_{S3} - f_{D3} = P_2$$

The expression for f_{D3} and f_{S3} are actually given here. So, let me just write that,

$$m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) - k_3(u_3 - u_2) - c_3(\dot{u}_3 - \dot{u}_2) = P_2$$

So again, I can combine similar terms together as a coefficients of the deformation, acceleration and velocity and write this one as.

$$m_2\ddot{u}_2 - c_2\dot{u}_1 + (c_2 + c_3)\dot{u}_2 - c_3\dot{u}_3 - k_2u_1 + (k_2 + k_3)u_2 - k_3u_3 = P_2$$

So, this was for the second story. Let us write down the equation of motion for third story, which in fact is quite simple, because there are no forces from above. So, I would write it as

$$m_3\ddot{u}_3 + f_{S3} + f_{D3} = P_3$$

Which can be written as

$$m_3\ddot{u}_3 + c_3(\dot{u}_3 - \dot{u}_2) + k_3(u_3 - u_2) = P_3$$

So, again let we write it as

$$m_3\ddot{u}_3 - c_3\dot{u}_2 + c_3\dot{u}_3 - k_3u_2 + k_3u_3 = P_3$$

So now, what we see, we have three equations here: (I), (II), (III). So, these are three simultaneous equation and you know you might have studied this in vector algebra, how to write this simultaneous equation in terms of a vector and matrices.

So, we combine them together and simplify it even further and write down only for once. So, I know that for acceleration there are coefficients, m_1, m_2, m_3 and then vectors that we are considering would be acceleration vector, velocity vector and the deformation vectors.

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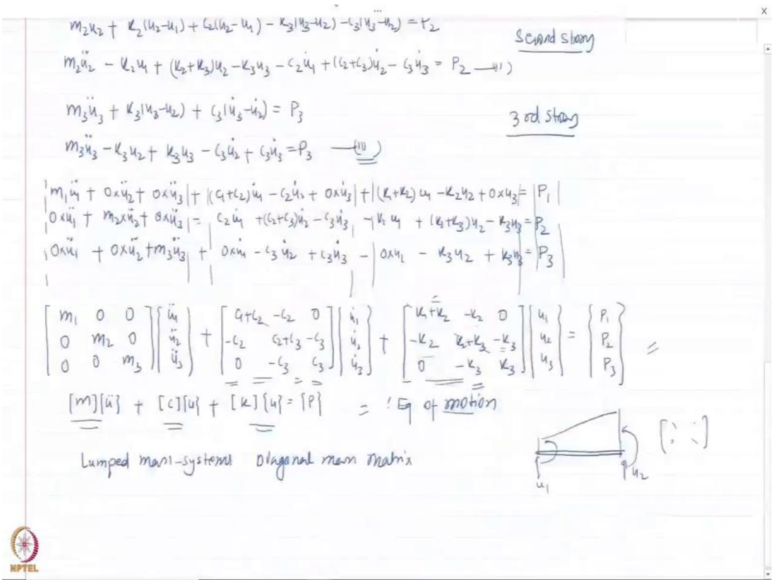

Now, in this case, let us write down the equation (I) like this;

$$m_1\ddot{u}_1 + 0\times\ddot{u}_2 + 0\times\ddot{u}_3 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + 0\times\dot{u}_3 + (k_1 + k_2)u_1 - k_2u_2 + 0\times u_3 = P_1$$

There are no coefficients to $\ddot{u}_2$ and $\ddot{u}_3$. So, I am going to write this one as 0 coefficient. Plus, now route first the damping term, because conventionally we write down the damping term before the stiffness terms.

So, in terms of damping, if we look at it, we have $(c_1 + c_2) \times \dot{u}_1$ and then the coefficient of $\dot{u}_2$ which is $-c_2$ and there is no coefficient of $\dot{u}_3$, I am just going to write it 0. Similarly, the stiffness terms, it would be $(k_1 + k_2) \times u_1$, $-k_2u_2$, $0 \times u_3$ and this is equal to P_1.

I am going to do exactly the same thing for the equation (II) and equation (III) as well.

$$0\times\ddot{u}_1 + m_2\ddot{u}_2 + 0\times\ddot{u}_3 - c_2\dot{u}_1 + (c_2 + c_3)\dot{u}_2 - c_3\dot{u}_3 - k_2u_1 + (k_2 + k_3)u_2 - k_3u_3 = P_2$$

Now, let us write down the third equation;

$$0\times\ddot{u}_1+0\times\ddot{u}_2+m_3\ddot{u}_3+0\times\dot{u}_1-c_3\dot{u}_2+c_3\dot{u}_3+0\times u_1-k_3u_2+k_3u_3=P_3$$

So now, if you look at it all of these are multiplication of some matrices and vectors. For example, if you consider this part here, and then this part here, and then this part here, let us compartmentalize these three sets of equation like that. So, this one is nothing but

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}+\begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} P_1 \\ P_3 \\ P_3 \end{Bmatrix}$$

So, if you look at carefully, this three sets of simultaneous equation can be written it like this. And, if you look at this carefully, this is the mass matrix for a multi degree of freedom system and here it is a 3 degree of freedom system for the three-story building.

This is the acceleration vector plus the damping matrix times the velocity vector, and then the stiffness matrix times the deformation vector, and this is the applied load vector. So, we have written down our equation of motion, again in a similar form that we have been using for a single degree of freedom system.

How to solve that, we will discuss later. But the first step is to actually find out the equation of motion. So, this is the equation of motion which is the first steps towards the solution of a multi degree of freedom system and throughout this chapter our goal is to develop or determine the equation of motion for a multi degree of freedom systems.

Now, let us come back, look at this equation that we have here and then make out some observation here. So, if you look at here, what do you see about the mass matrix here? The mass matrix is actually a diagonal matrix. So, there are only elements m_1, m_2, m_3 which are along the diagonal and the off-diagonal terms are 0.

So, mass matrix is typically obtained as a diagonal matrix as long as it is a lumped mass systems. So, for lumped mass systems where the degree of freedom corresponds to the lumped masses, then you are going to obtain diagonal mass matrix.

You might obtain off diagonal terms for a continuous or distributed mass system. Let us say you have a beam in which the mass is distributed along length. In that case, if you define degree of freedom let us say u_1 here and u_2 here to represent let us say translational motion like this, then you might see that your mass matrix would not be diagonal and you would have term off diagonal term as well.

But typically for a multi-story buildings shear building if you have a lumped mass system where the degrees of freedom have been defined at each mass then you would get a diagonal mass matrix.

Now, let us look at the stiffness matrix and the damping matrix. The first observation is that a stiffness and the damping matrix are symmetric matrix. If you look at it, we have the diagonal elements $k_1 + k_2$, $k_2 + k_3$ and k_3, but the off diagonal terms are actually mirror image of each other along that diagonals; so - k_2 here, - k_2 here, - k_3 here, - k_3 here and same, is the case for the damping matrix, it is also symmetric.

Now, as long as our system is linear, what we would see that the mass, the stiffness matrix that we get could always be symmetric. Third observation that we see that, if we define our damping matrix or the dampers in such a way that it represents the same degrees of freedom to which the stiffnesses have been defined, then the form of the damping matrix would be same as the stiffness matrix. This is specially true for multi degree or multi story building in which dampers are defined at each story.

Although, we will discuss later that we do not actually get our damping matrix like this, because in reality there is no damper between different story, except in the cases where you actually install dampers. But you get the damping matrix we utilize some other methods to get the damping matrix, utilizing the mass and the stiffness matrices, but that is for the later discussion. What is important to note here is that, if I have a multistory building and I have defined the dampers between different stories, then the form of the damping matrix would be similar to as the stiffness matrix.

Now, once I have found out the equation of motion for multistory building. Let us consider some different representation. Remember, for a shear type building, there is no flexure deformation, all the mass at concentrated at the floor level and the degrees of freedom are basically horizontal deformation at each level.

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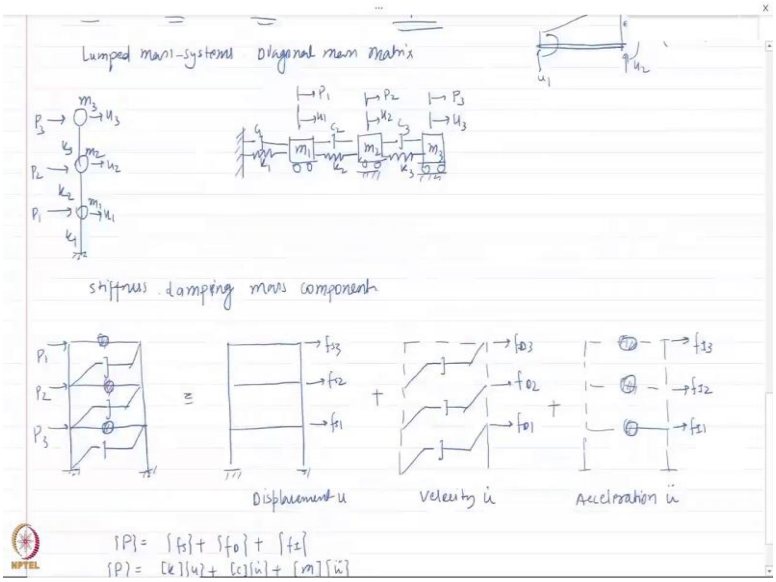

So, many a times you would also see, this representation is lollipop representation being used to define a shear type building. And basically u_1, u_2 and u_3 are used to define the degrees of freedom here, and the story stiffness is k_1, k_2 and k_3. So, you should be aware of these representation in terms of dynamic behaviour. This is used to represent the same multi story building. So, this is P_1, P_2 and P_3 with the masses m_1, m_2 and m_3. So, this is also used to represent the same shear type building.

Another representation that is we have been considering is the spring mass damper representation. Again, we can utilize the same method to draw the spring mass representation, remember the first story is fixed to the ground. So, to draw the spring mass representation, let us start with the first story here. I have the mass m_1 which represent the first story mass. Let us draw the damper as well. So, this m_1 which is the first story is now connected to the second story. Let us first write down k_1 and c_1 representing the stiffness and damping of the story the first story here. Then this is connected to the second story through the spring k_2 and the damper c_2.

This is m_2 is connected to third story, c_3 and k_3 here, and this is m_3 and the degrees of freedom are u_1, u_2 and u_3 with respect to initial undeformed position or in this case that would also be the equilibrium position. The forces that are being applied are P_1, P_2 and P_3.

So, this is another spring mass damper representation of the same system and you know we can again consider the equilibrium motion and get the similar kind of equation the equation that we have got here. So, this is the second type of representation that we typically consider.

So now, we have seen that different type of representation. Now, let us consider different components of a multi degree of freedom system and we are going to utilize, again the same shear type building. We will see the utility of that representation later.

Basically, we know that for single degree of freedom system, any structure would have three components under dynamic load: the stiffness component, then the damping component, and then the mass component.

So, let us say this is the system here, same three-story building. I have applied forces P_1 here, P_2 here and P_3 here. So, this is the combined system.

This overall system is combined representation of the three components which is: the first one is the stiffness component in which we consider the bare frame without mass or damper. So, we have bare frame in which we apply forces f_{S1}, f_{S2}, f_{S3} which represents the stiffness forces in the system.

In the second one, we are going to consider the damping component that would only have the damper and no frame or mass. So, only the damper would be there. So, in this case again I have f_{D1}, f_{D2}, f_{D3}. So, this is for displacement u, this is for velocity $\dot{u}$. The third components is the mass component where we are not going to consider the frame or the damper, but we are only going to consider the masses at different levels.

So, I only have these masses at different level and the forces that are that are being applied are the inertial forces. So, I am going to write it here f_{I1}, f_{I2} and f_{I3}. In general, the total applied force is the sum of the individual components. So, let me just write down the acceleration here.

So, the vector P here is sum of vector f_S, vector f_D and vector f_I. Now, we already know that the same expression we had been used for single degree of freedom system. We can write this as P as equal to: remember f_S, we have found out as the stiffness matrix times the displacement vector, the damping forces f_D we have found out as damping matrix times the velocity vector and f_I as the mass matrix times the acceleration vector.

We are going to see in the next section itself that this representation is quite useful in other type of method to find out the equation of motion. So, the method that we are going to discuss it is called Influence Coefficient Method. Now, you might have come across this influence coefficient method in your structural mechanics courses or advance structural mechanics courses. We will see that how to utilize that to get the equation of motion. Let me give you background why we need to do that.

If we look at this equation of motion, so this can be either obtained using free body diagram and then writing down three simultaneous equation. Now, this sometimes becomes little bit complicated for complex systems.

You will see that this method, that influence coefficient method that we are describing, many times is useful to obtain the same equation of method motion using an alternative method. So, this is just an alternative of finding out this equation of motion.

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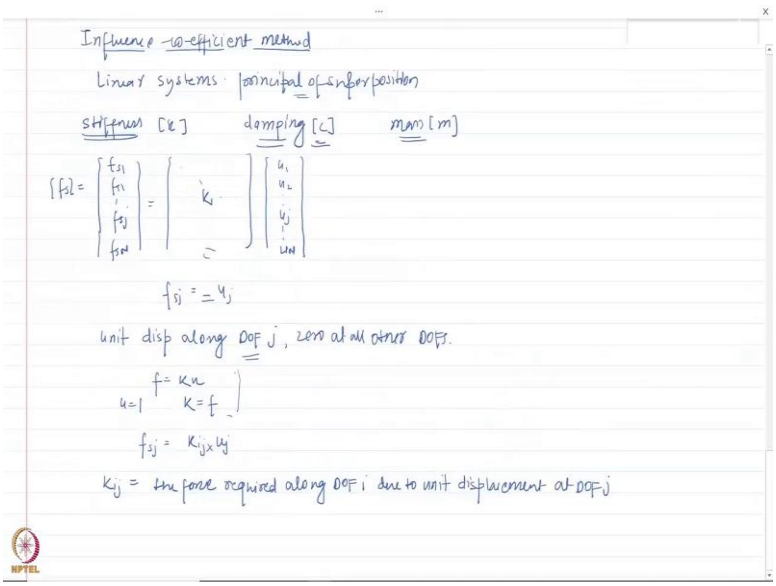

So, utilizing this influence coefficient method, we are again going to find out the equation of motion, that is what we plan to discuss. Remember that, we are only considering linear systems. So, for linear systems principle of superposition is also valid.

So, utilizing this principle of superposition and the influence coefficient method, we are going to find out the equation of motion.

So, first let us start with the stiffness matrix. Now, using this influence coefficient method, we individually find out the stiffness matrix, then damping matrix, and then mass matrix. Most of the times we only find out the stiffness and the mass matrix, and damping matrix would typically find it as using damping ratio that is obtained experimentally and then combining the mass in the stiffness matrix. So, we do not typically go ahead and do that, but we will be going to discuss that anyway here.

Now, for the stiffness matrix, we have considered forces like f_{S1}, f_{S2} and f_{S3}. So, we say that our goal here is to relate the external forces f_{Sj} or vector f_S is nothing but, f_{S1}, f_{S2}, f_{Sj}, let us say it is goes up to f_{Sn} this equal to k matrix times u_1, u_2, u_j and then u_n.

So, this is how we write down our f_S matrix and the stiffness matrix elements are basically here. So, basically our goal is to relate the external force, that is f_{Sj} to u_j, and whatever the coefficient that we get constitute the stiffness matrix. So, the way we apply this influence coefficient method, we apply a unit displacement along j, while keeping other displacements equal to 0.

So, if I have a 3 degree of freedom system, let us say, I apply $u_1 = 1$ and keep u_2 and u_3 equal to 0, because unit displacement along D(j) and 0 at all other DOFs. Now we know that, typically when we apply any displacement at a degree of freedom, then other degree of freedom would also deform, that I know. However, if we want to maintain that I have unit deformation at a degree of freedom j, but it is 0 at all the other degrees of freedom, I will have to apply forces at each in every degree of freedom, to achieve that state of deformation.

Now, before going into that and further discussing, remember that, if we apply a force f on any spring or a stiffness component, it deforms by a deformation u. So, f equal to $k \times u$. For unit displacement $u = 1$, this stiffness k is force that is being applied on the system. This is the concept that we are going to extend. This is a single degree of freedom we are going to extend to a multi degree of freedom

So, what do we say here, that if I have, let us say f_{Sj} equal to coefficient k_{ij} times u_j. If u_j equal to 1, then k_{ij} is nothing but the force that is being applied on that particular degree of freedom. So, the stiffness coefficient k_{ij} of this stiffness matrix here. The k_{ij} is basically the force that is required, so force required along DOF i due to unit displacement, a unit displacement at DOF j.

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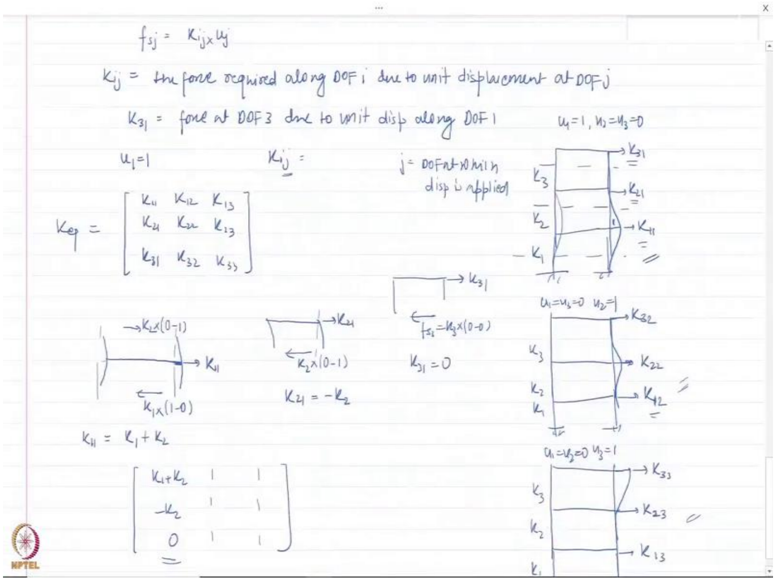

So, if I consider for a three-story building here, if I consider element k_{31} is nothing but the force at DOF 3 due to unit displacement along DOF 1. Now if I apply a displacement u_1 equal to 1.

So, let us say, I have this building and I am applying unit displacement along 1 and keeping others 0. So, the shape would look like something like this. So, what this basically says, if I apply unit displacement 1 here, I need to apply force which is equal to k_{11} here, k_{21} here and k_{31} here and these forces would result in a case where u_1 = 1 and u_2 = u_3 = 0.

Now, if you look at it k_{31}, k_{21}, k_{11} are nothing but the first column of the stiffness matrix. Similarly, if I do like this, in which I apply unit displacement at node 2 and 0 everywhere else, say it would look something like this. Again, I have to apply force which is k_{22} here. Remember, the second subscript of k, if I have a k_{ij}, j is where the displacement is being applied and i is the degree of freedom at which you are considering the force.

So, in this case, 2 is the degree of freedom at which the displacement is applied. So, the force applied at this degree of freedom 2 would be k_{22}, k_{21} here it would be force applied at degree of freedom 1 due to unit displacement at degree of freedom 2, k_{23} force applied at degree of freedom 3 due to unit displacement of DOF 2 So, this is the case when, $u_1 = u_3 = 0$ and $u_2 = 1$.

Similarly, so now, these elements if you look at it forms the second column of the stiffness matrix. So, let us just write down the third case here.

In the third case, I have unit displacement at 3 and then 0 displacement everywhere else. So, $u_1 = u_2 = 0$ and $u_3 = 1$. So, I need to apply force at degree of freedom 3 due to unit displacement at degree of freedom 3, force at degree of freedom in this case basically equal to 2 at degree of freedom 3 in this case. Force at degree of freedom 1 due to unit displacement of degree of freedom 3. So, the third one is basically k_{13}, k_{23}, k_{33}. So, using this procedure we can get k equivalent.

Now, if the story stiffnesses are given k_1, k_2, k_3 and similarly for all these cases, we are assuming the same story stiffness. Again, we can consider the equilibrium of each and every floor to find out what is k_{33}, k_{23}, k_{13} or k_{ij} in terms of k_1, k_2, k_3. So, let us take example of this first case and let us cut this at three level and find out what is the value.

So, if I consider the first story which would deform something like this here. So, first story is something like this, where this is unit deformation. Now, to maintain that I am applying a force k_{11} here. Now, because of deformation there would be forces from below which would be story stiffness k_1 times the story drift, which is 1 at this level and 0 at the ground.

So, it would be 1 - 0. Now remember, there would be forces from the above as well, and it would be in this direction. And what would be that force? The story stiffness k_2 times the deformation of the floor above and subtracted by this deformation here. Now, deformation of the floor above is 0, but the deformation of this floor here is this k_2. So, k_{11} would be nothing, but $k_1 + k_2$ here. So, we can write down the equation of motion and get $k_{11} = k_1 + k_2$.

Now, if I consider the story above this, so the second story which is let us say something like this is here. On this I am applying force k_{21} here, there is force from below which in direction k_2 times whatever the deformation at this point which is 0 minus the deformation of the lower below, which is 1.

So, in this case k_{21} comes out to be equal to - k_2 and then I consider the story above which does not have any story drift. So basically, it is k_{31} and because it does not have any story drift, would there be any force here? So, f_{S3} is equal to 0 here. Because f_{S3} is k_3 times u_3 - u_2, both are equal to 0. So, k_{31} = 0.

So, $k_{11} = k_1 + k_2$, $k_{21} = - k_2$, $k_{31} = 0$. And then I still need to find out these two columns which I can do by finding out or repeating the same procedure that I have just described. Now, if you compare the first column, the equation that we have derived here, look at this, $k_1 + k_2$, - k_2, and 0. So, we got exactly the same column and, you will see, if you repeat this procedure here and here you are going to get exactly the same stiffness matrix.

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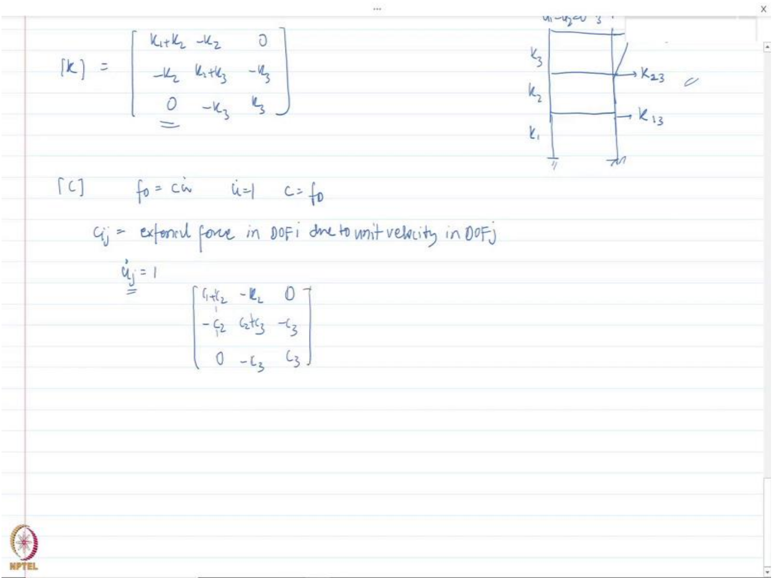

So, this let me write it as - k_2, 0, $k_2 + k_3$, - k_3 and this is k_3. So, this is a k matrix. So, using the influence coefficient method, we have been able to derive the stiffness matrix like this. Now, you might think at this point it does not seem any bit easier from the previous method that we described. But remember, for this case it might not be, but some of the other cases you will see, and we will do some examples.

$$k = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

This method, the influence coefficient methods is much simpler than writing down the simultaneous equation and solving and formulating the equation of motion. This also provides us some insight into the dynamic behaviour of the system. So, this is how we get the stiffness matrix, but remember we still need to find out the damping and the mass matrices.

Now, damping matrix again, we are going to follow the same procedure. We are going to, say remember damping forces is $f_D = c \times \dot{u}$. So, if $\dot{u} = 1$, then the damping coefficient is basically the force applied on that particular degree of freedom.

For multi degree of freedom system again, we are doing to extend that, we are going to say c_{ij}. Remember, for stiffness matrix we said that k_{ij}, for this we are going to say that c_{ij} is nothing but the external force in DOF i due to unit velocity. Now, not the unit deformation, unit velocity in DOF j.

So, again we are going to follow the same procedure. We are going to apply $\dot{u}_j$, the velocity at the j^{th} degree of freedom as equal to 1, while keeping velocity at the other degrees of freedom equal to 0. Then finding out the forces that are needed to maintain that velocity profile and that would give me the column for that particular degree of freedom j. And I am going to repeat this for all the degree of freedom, and I will keep on getting this basically the damping matrix.

If it is a multi-degree of freedom system, what you can do? You find out the stiffness matrix and if your dampers are also connected between the same degrees of freedom, you can just replicate that in terms of c_1, c_2, c_3. However, that might not always work, so just be careful. If it is not a shear type building or degrees of freedom corresponds to different a velocity terms, then your damping matrix would be different.

$$c = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

Now the third term is still remaining, that is the getting the mass matrix. So, let us see how do we get the mass matrix.

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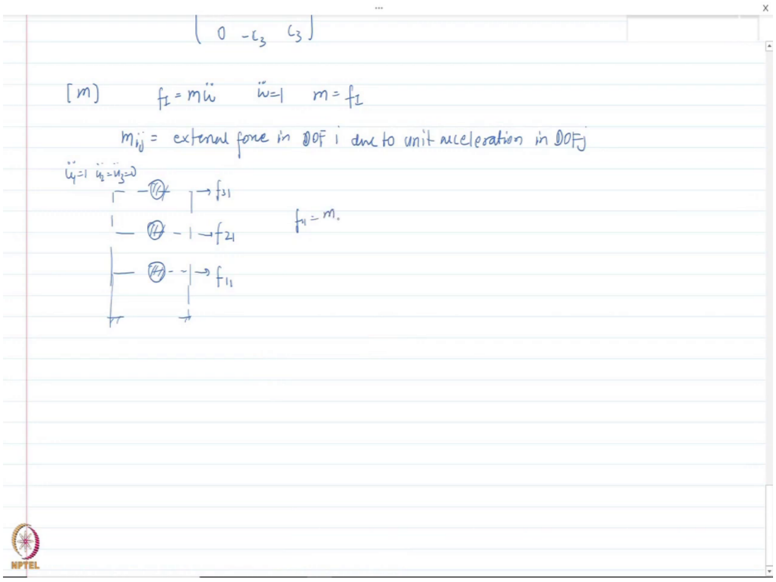

Now, to get the mass matrix, we are again going to employ the same thing. Remember, the inertial forces, mass times acceleration. So, if we apply unit acceleration then mass is nothing but the inertial force that is being applied. The procedure remains same.

In this case we are going to say m_{ij}, which is the element at the i^{th} row and the j^{th} column in the mass matrix and it is nothing but the external force in DOF *i* due to unit acceleration. So, unit acceleration in DOF *j*. Now, for our case, it is very simple, because if you consider the acceleration representation here, I only have mass and if it is a lumped mass system.

Remember, there is no damper here, there is no frame here. Only these masses are here and if I apply a unit acceleration. So let us say $\ddot{u}_1 = 1$ and $\ddot{u}_2 = \ddot{u}_3 = 0$, then this is f_1, due to unit acceleration at 1, this is a degree of freedom 2 due to unit acceleration 1, degree of freedom 3 due to unit acceleration at 1.

So, in this case, if the unit acceleration is only applied at 1, there are no forces in f_{21} or f_{31}, because these are not connected. So, in this case, I can directly say, if the unit acceleration is only applied at mass m_1, m_{11} would be m_1.

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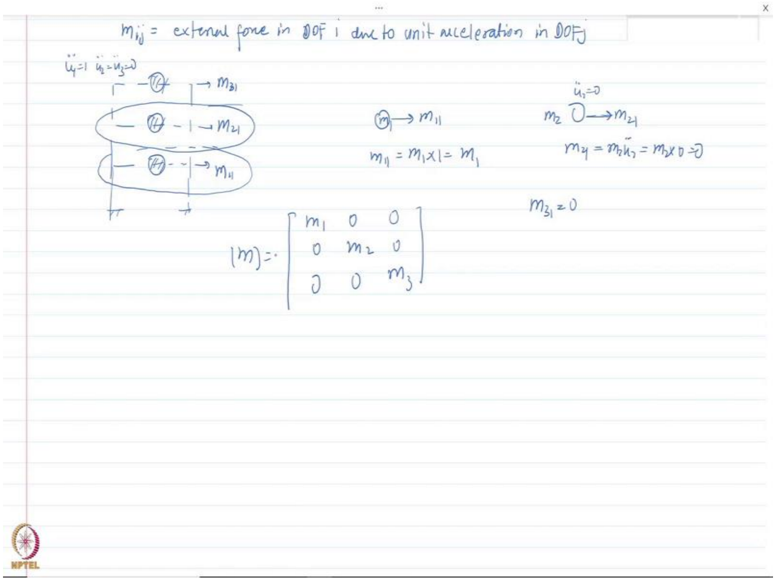

So, let us consider the free body diagram of this, which is a force is being applied is m_{11} and the acceleration is 1 here, and it is not connected to any of the masses. So, what does that mean, m_{11}, the force equal to this is the mass m_1, $m_1 \times 1$ which is equal to m_1.

Now, if we consider this one here, the free body diagram, the acceleration is equal to 0, the force this is applied is m_{21}. So, m_{21} is equal to whatever the mass here is $m_2 \times \ddot{u}_2$ which is equal to $m_2 \times 0$, so 0. Similarly, m_{31} is also equal to 0.

So, this would give us a diagonal matrix, because the first column that I get as m_1 0 0. Similarly, I would get 0 m_2 0. So, for lumped mass system, it would only produce acceleration at that degree of freedom at which you are considering the unit acceleration. This is your mass matrix and this you would always as I previously mentioned get this for a lumped mass system or diagonal mass matrix.

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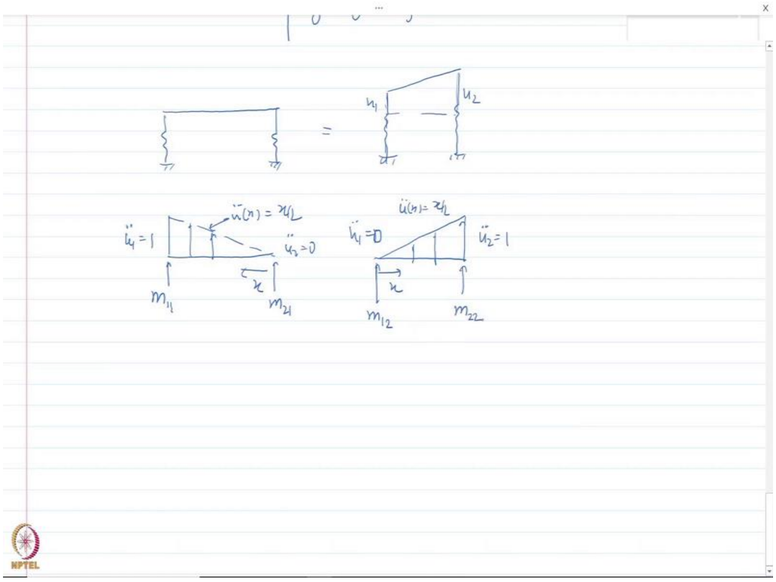

However, if you consider cases where you have the bar, let us take distributed mass and in this it is supported by spring. So, that you are representing it the deformed position through 2 degree of freedom, let us say u_1 and u_2, because it can rotate and translate. So, you need 2 degree of freedom to represent the deformed position of this system with respect to its initial position.

Now, in this case you have degree of freedom u_1 and u_2. So, if you want to get the mass matrix for this, what you will do? In the first case, let us say, you consider $u_1 = 1$ and $u_2 = 0$. So, this is the acceleration profile you will get and acceleration at any distance x_s would be equal to x / L. It increases from 0 to 1.

In the second case $u_1 = 0$ and $u_2 = 1$. Again, if you consider x from this direction, remember I have considered x from this direction, you consider x from other direction your acceleration at x would be equal to x / L.

Now, let us say in these degree of freedom, you have applied force at degree of freedom 1 due to displaced acceleration unit acceleration and degree of freedom 1, then force at degree of freedom 2 due to unit acceleration at degree of freedom 1. Similarly, force at degree of freedom 1 due to unit acceleration at 2, force at degree of freedom 2 due to unit acceleration at 2.

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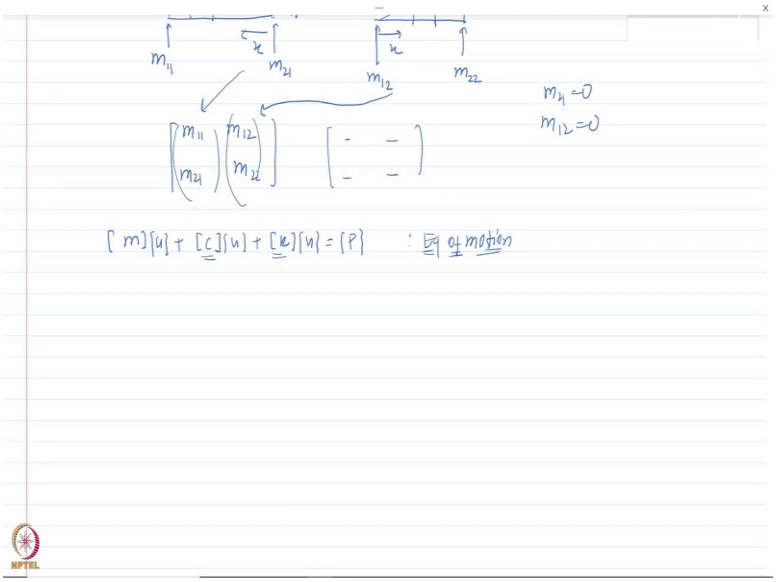

So, basically your mass matrix is m_{11}, m_{12}, m_{21}, m_{22}. So, this system would give you your first column, this system would give you your second column. Now, if you look carefully and if you consider the equilibrium of this, you can never have m_{21} equal to 0 or in this case m_{12} equal to 0. So here, you would have non-zero off diagonal term as well.

So, you would have some term here, some term here and some term here. So, this is a distributed mass system, and, in this case, I do not get a diagonal mass matrix. Even if I employ the influence coefficient. So, this is one of the examples, and you would see similar other example where the mass matrix is not diagonal.

So, using the influence coefficient method, we have been able to derive the mass matrix, then their stiffness matrix or let us say first the damping matrix and then their stiffness matrix. Damping matrix, if it is a shear type building, we just replicate the mass stiffness matrix, and this would be equal to the applied force vector along the respective degrees of freedom.

So, this is another way of finding out the equation of motion using influence coefficient method. And, depending upon the problem you might observe that one method or the other one, it would work better you have to select that matter. But remember one thing, in principle it does not matter what method you use, you would still get the same answer. So, it might be that your selection of method might make your job little bit more complicated. However, the final answer would still be the same.

So, with this I would like to conclude this class. In the next class we are going to see more example and application of these two methods.

Thank you.