Dynamics of Structures
Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Introduction to Dynamics of Structures
Lecture – 02
Idealization of Structures

Welcome back everyone. Today, we are going to be taking up from the last lecture, and further discussing how to set up the equation of motion for a dynamic problem. So, in today's class, the first thing that we will going to discuss is basically how to simplify a typical structure or a system to an idealized system that can be utilized for dynamic analysis, and that idealization would help us in obtaining the response of such system.

And first we will do a single degree of freedom system, and we will discuss what a degree of freedom actually means, and then see what is the differential equation for obtaining the dynamic response of a single degree of freedom system. So, let us get started.

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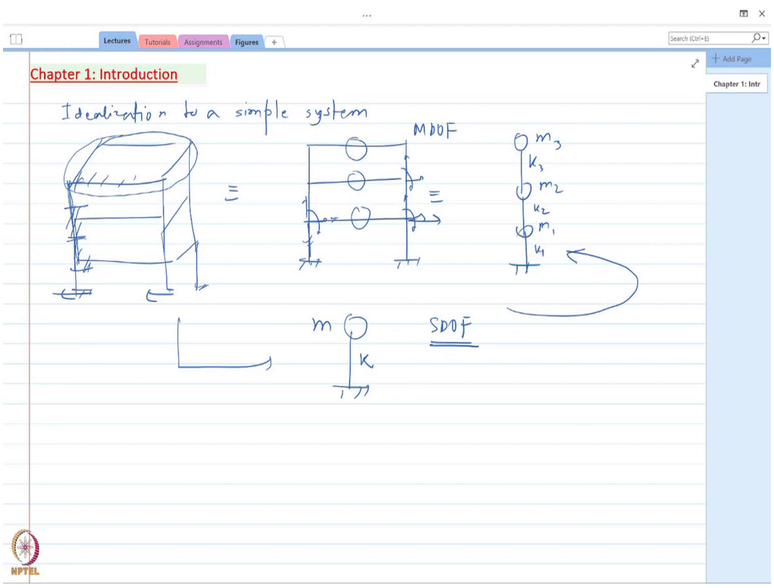

We are going to be starting the first chapter, which basically introduces the basic equation of dynamics. So, we are going to divide this lecture in three parts. The first part would be how do we idealize systems for the dynamic analysis. Then the second part would be the components of a dynamic system, and then in the third part, we would be setting up the equation of motion.

So, coming to the first part, which is basically idealization to a simple system or to a system, which can be used for dynamic analysis. As we discussed in the lecture before this one, if we want to obtain the response of any structure or any system subject to a non-static load, then one of the steps that are required is to simplify a structure which can be used in our numerical analysis.

So, let me just start by taking example of a building. So, in its very detailed form a building might have multiple floors and it might have walls, it might have columns. Now, if we want to simplify this system for analysis, what we need to do depending upon what is the goal of our analysis, we might simplify this to a multiple degree of freedom (MDOF) system.

So, what we are going to do? We are going to consider the tributary allocation of the mass. For example, in this case, we are going to assume that all this slab is situated at this level like this; it is M_1, M_2, M_3. And then we are going to consider this frame type columns and buildings, columns and beams from the component of this MDOF representation. So, this is one representation.

Now, in this case, depending upon whether I am considering my columns and beam to be axially inextensible or not, I can further simplify it. So, I can further simplify the system if these columns and beams cannot have axial compression or extension, these are rigid axially.

So, in that case, what we will have - if you remember from your structural analysis, in general a 2D frame structure - would have how many degrees of freedom? 3 degrees of freedom at each joint. So, I would have three degrees of freedom, so all of these would be 3 degrees of freedom.

However, if I want to represent it with columns and beam, that are basically axial rigid, I can further simplify this, and I can represent it using a single degree of freedom at each level of the building.

So, if I have to represent it, I can represent it something like this. So, in this case, this is would be k_1, k_2, k_3, that represents the storey stiffness and then I have all the mass at different levels, which are represented by M_1, M_2, M_3.

So, what I have done, I have drawn three-storey. So, this much would go into the second floor, this much of would go into the first floor, and this would go into the top floor. I have simplified this model to a system, that can be used for analysis. Now, this was for multi-degree of freedom system.

Now, depending upon what is the goal of our analysis, this can even be simplified to a single degree of freedom system. For example, this can be simplified through a single degree of freedom system. So, the total mass is here and then k. Now, you could argue whether this SDOF representation is an appropriate idealization of that system or not. Well, it depends what is the goal of your analysis.

For example, if it is a rigid building and you need to find out just the base shear, then perhaps this might work out to be reasonably ok, but in many cases, this might not be an appropriate representation. However, what we are going to do? We are going to start our mathematical formulation with single degree of freedom system. Once we have understood the behavior of single degree of freedom system to different kind of loading, then we are going to see how we are going to idealize a multi-degree of freedom system.

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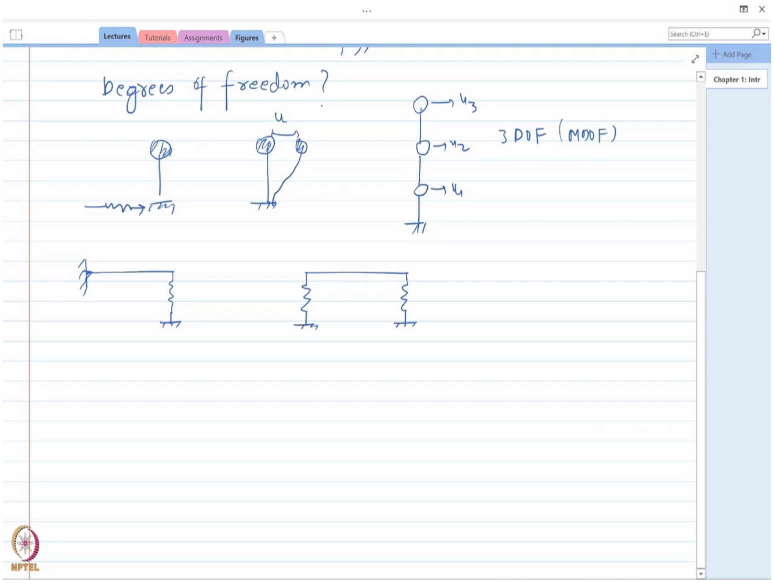

Now, I have mentioned many times degrees of freedom. So, I have said, degrees of freedom. So, the question comes what exactly degrees of freedom is. Now, for our course or in dynamics, in general, degrees of freedom represent the number of independent displacements of the mass that are required to represent the displaced position of a body with respect to its original position.

For example, if I have something like this and how many degrees of freedom would I need to represent the displaced position? So, let us say there is a horizontal earthquake and I want to represent the position of this one. So, in general, I would only need one degree of freedom to represent the position of the mass in the deformed position with respect to its original position, so you must keep in mind. So, this is degrees of freedom.

Similarly, if I have multiple degree of freedom, remember there could be how many degrees of freedom here? One would be for this, one would be for this, and one would be for this. So, I could have single degree of freedom system, or I can have in this case three degree of freedom system or multi-degree of freedom system. So, you must keep in mind that, how do we represent degrees of freedom.

So, now I am going to show you some examples in which you have to find out how many number of independent degrees of freedom you would require to represent the displaced position of masses in the body with respect to its original position.

So, let us take our first example. In our first example, I have a beam which is pin connected at the left end, and then there is a spring at the right end. In the second example, I have a beam which is supported on two springs. Now, in the third case, I have a pulley mass system.

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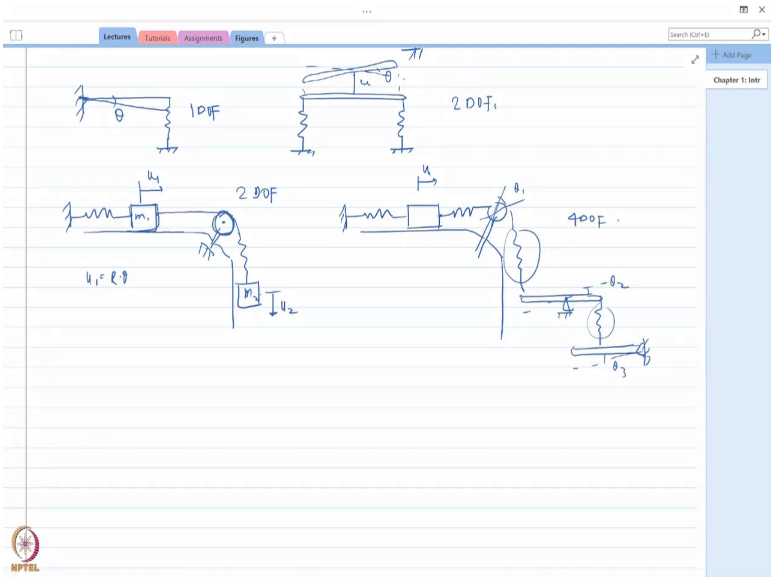

So, I have mass M here, and then it goes over pulley, which is fixed here, just assume this to be under top and then I have another spring coming out to be. So, let me just redraw it again. So, I have this spring in between, and I have then another mass here.

Now, similar to this, there is also fourth case in which I have similar kind of set up, but there is a one small difference or in fact, there is a big difference. Just draw it again, and then you would know initials this part of the setup is same.

So, this part is same and there is another spring here which goes over this. Then there is another spring which comes like this, then there is a rigid bar here, and that is pin connected at this point. Then there is another spring, and then there is another bar which is connected here, and which is pinned at the this support.

So, there are four examples in front of you, and you have to find out how many number of degrees of freedom or independent degrees of freedom required for each case to represent the displaced position or body or simply, it would be just finding out what is what are the degrees of freedom of this system for dynamic analysis. So, at this point of time, I would like you to take 5 minutes, and just go through each setup one by one and find that out.

Now, let me discuss each problem one by one. So, what I have here, there are all of these are rigid bars. So, in the first case if this is constrained to rotate about this point be. So, it can only rotate about this point and there cannot be any other movement because bar is rigid. So, the degree of freedom for this is one degree of freedom.

Now, let us consider the second example in which I have this bar which is supported on these two springs. Now, in general, this bar can translate, and this can also rotate. For example, bar it rotates about its center of mass, and then it also translates by this much. So, this is the one degree of freedom, and then it rotates about this point. So, this is another degree of freedom. So, for this to completely represents the displaced position, we need two degrees of freedom.

Now, for the third example, I have this mass here. Let us call it M_1 and this is M_2. Now, for this mass the degree of freedom is u_1. Now, remember this pulley can rotate about this point, however, if you consider the rotation of this pulley to be independent, it is not because there is no spring in between. So, it is inextensible between these two points. So, whatever is u_1, by the same amount the pulley rotates related by u_1 equal to radius of pulley times the rotation of pulley. So, there is not an independent coordinate here. However, there is a spring between this point and this point. So, M_2 need to be represented by another degree of freedom. So, in this case, I have two degree of freedom system.

Now, let us come to the fourth example. In this example, start by saying this is u_1, my first degree of freedom system. Now, in this case, if you look at it, this is not inextensible between the mass and the pulley. So, in that case, the rotation of the pulley would be another independent degree of freedom needed to describe the displaced position. So, this pulley is rotated by θ_1. Now, whatever θ_1 is here, again whatever is rotated by this amount, there is a spring in between and this the first bar is constrained to rotate about its support. So, I would need another θ here or let us call it θ_2, and then there is again a spring in between. So, for this case, again I need an independent degree of freedom θ_3. So, in general, I have one u_1 and then three degrees of freedom θ_1, θ_2, θ_3. This is four degree of freedom system.

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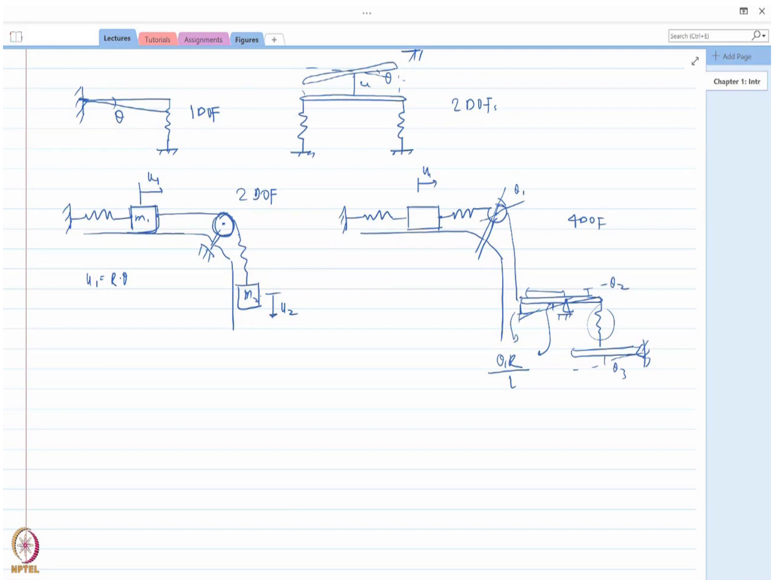

You could modify this problem a little bit and say if there is no spring in between, for example, right here then the degree of freedom would reduce by 1. Because whatever is θ_1, this displacement here would be θ_1 times the radius of this pulley. So, it is pretty much determined here.

The rotation at this point would be θ_1 times the radius of pulley and this rotation would be θ_1 divided by the whatever the length here is, which is let us say, L. So, it is again not independent. So, it would reduce a three degree of freedom system. So, I hope now it is clear how do we represent degrees of freedom in a system.

So, now, let us move on to different components of a dynamic system. Let us see how we simplify a system to a mathematical model, which is amenable for mathematical analysis or setting up the equation of motion. And in this case, I will simply take example of a single storey building of frame type building.

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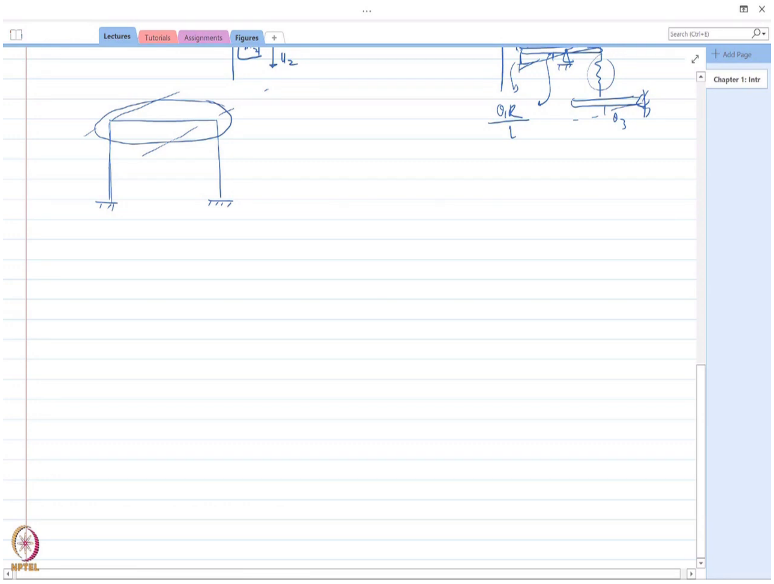

So, what I want to draw here, a single storey frame here something like this. So, as you could imagine, it would be in three dimensions, but I have just considered in frame saying that all the mass is concentrated along this line. So, basically what I am saying, it would be in three dimensions, but I am assuming that all of this is concentrated here. So, let me just go ahead and delete this.

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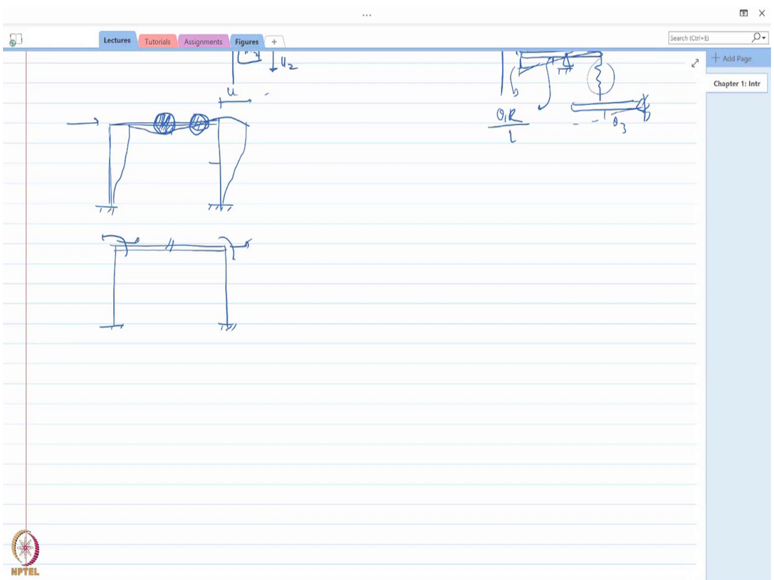

Now, as you can imagine, this would have certain mass, even the column would have certain mass, but if you consider a single storey building or multiple storey building, most of the masses are situated where in a building? Can I say it would be at the floor level? Ok.

And of course, you can say that there are also masses concentrated at the wall, but we say that half of the mass above and below the floor, I can just lump it at that floor level. So, our representation for the single storey frame building would be, I take all the masses and lump it at here, this is just one way to represent it.

Now, this system has mass, which is I am assuming it to be here. Now, if I apply a lateral load to this system, what will happen? The system would of course, deform. So, let us say the system deforms like this, and the displaced position become something like this. This mass moves from here to here.

Let us say, this is the displacement with respect to original position and the same would be the displacement of the mass because this representation is inherently assuming that these columns and the beams are inextensible axially. So, this is a simplified representation of single degree of freedom representation.

Now, if you remember from your undergraduate, if I had a frame like this, how many degrees of freedom you needed to represent this system? Well, in this case, 3 degrees of freedom here and 3 degrees of freedom here. Now, I could only reduce this system to a single degree of freedom system. Why? Because I assume if this is inextensible, so this degree of freedom, this degree of freedom would be same.

If columns are inextensible, then there are no vertical degrees of freedom here. And if I assume that this beam here or the slab here cannot bend, then I can represent this with a single degree of freedom without any rotation here. So when I redraw it, I will get the same figure what you see above.

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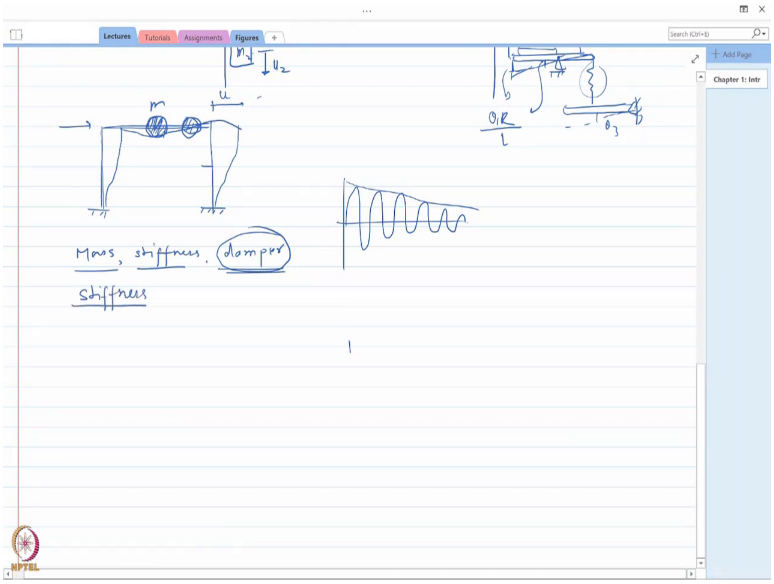

So, we have a mass here. If I am applying a force, it is deforming. So I have a stiffness and there is also a third component. So, we have mass, we have stiffness, and there is a third component which is called damper. Now, if you pull an elastic system or like a flexible system and let it vibrate. Would it keep on vibrating for infinity? It is not possible.

So, while for some system that idealization is assumed, what happens if you pull any system, there is always some dissipation of energy. And if you let a system vibrate, after some time any system would come to rest. So, it would not keep on vibrating infinitely, it would ultimately come to rest. So, damping represents the energy dissipation in the system due to which the amplitude of vibration of a system reduces to 0 over a time.

So, we have three components of this system and we are going to see how we can derive the mathematical formulation for each of these three components. So, the first representation we are going to talk about is actually for the stiffness.

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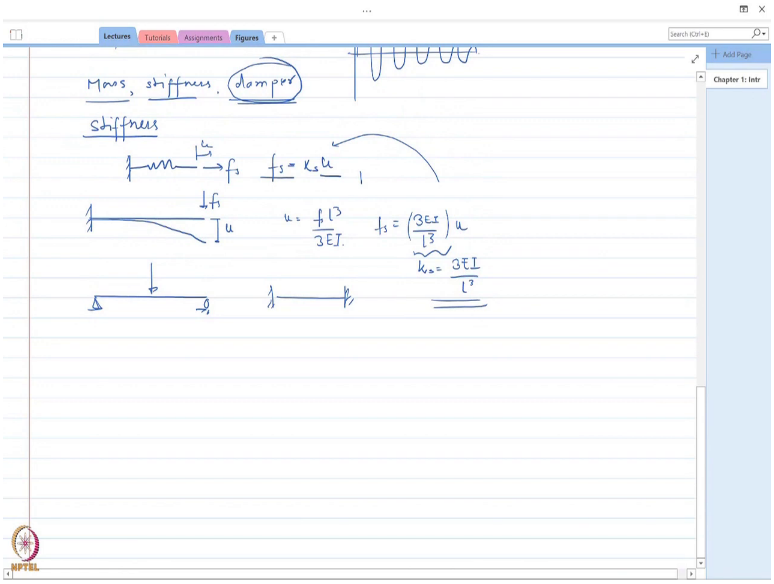

Now if you consider any system that can be deformed, for example, the simplest one is what if I take a spring like this and there is no mass here. If I apply a force which is the force in the spring, you know that it will deform and how is the displacement in this is related to f_s, I can write $f_s = k_{spring} \times u$. So, this relationship you know, and nothing is complicated about this one.

Similarly, for any other type of structure as well, for example, a cantilever beam. So, what I have here, it is a cantilever beam and when I apply a load f_s here, you know that it will deform. What is this deformation here? If you remember from your structural analysis class, can I say $u = f_s L^3 / 3EI$ or I can write it in other way, saying $f_s = 3EI / uL^3$, which is similar to this relationship, except now here my stiffness is represented by $3EI / uL^3$ and this I can do for different type of structure.

For example, I can do it for simply supported beam, I can do it for fixed beam. I can do it for different type of system, and then equivalently I can derive this relationship. So, when we apply a load on a deformable body, they deform. Then the system can be idealized through a spring with a stiffness and for which the applied force f_s and the displacement u is related through this relationship.

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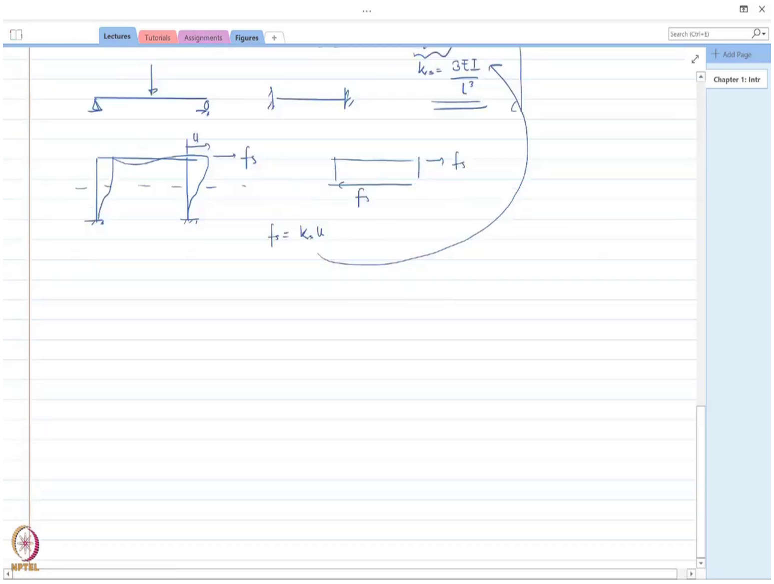

If that is clear, let us come back to our example that we were discussing, the single frame building here. In this situation, I am not considering any mass. If I apply a force on this, it will deform. Let me redraw it again, it will deform, let us say, the force applied is f_s, it due to this applied force there is a deformation of u. And if you take the free body diagram here, how does it look like if you have the system here. The resisting forces whatever force you apply here.

Now, we have to find out how is this force related to this displacement. So, we have to find out this k_s factor here. Now, if the system is elastic, we saw that we can directly find out using the principle of a structural analysis. So, $k_s = 3\ EI\ /\ L^3$, depending upon what kind of setup you are considering. So, for small deformation usually, this relationship is linear.

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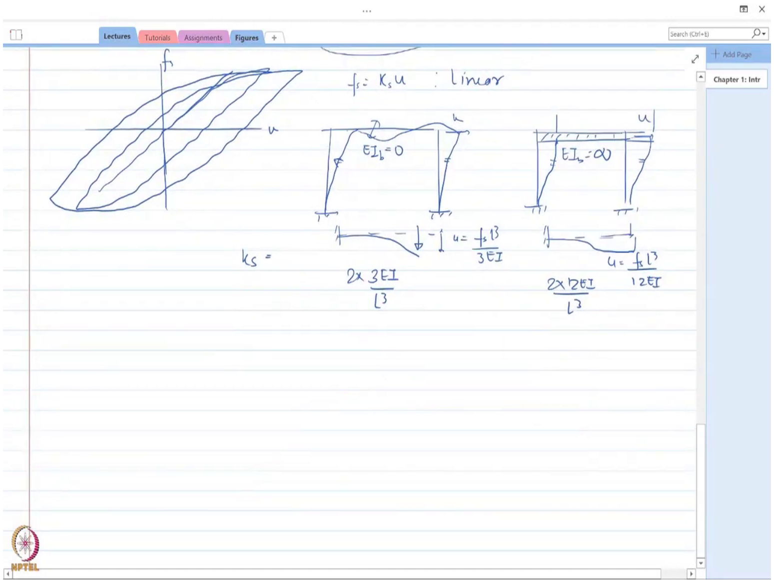

So, if let us draw f_s versus u, it would be usually linear. However, if you keep applying the force or keep increasing the force, there will come a point at which the system would start to break or yield. Whatever you want to say, and then it would not remain linear, there would be energy dissipation. And then you must consider the system something like a non-linear system.

So, we do see that the structures are in general non-linear. However, for this structural dynamic course we would only be consider considering linear relationship of the force versus deformation.

Now, let us consider two extreme cases for this frame here. Two extreme cases by what I mean, in this case in one case, this is very flexible so that the modulus of rigidity is 0; in the second case it is very rigid, so that modulus of this one is equal to infinity. Now, if you apply the force and if you are asked to find out equivalent stiffness of this system, how would you do that?

As you can imagine, if there is no flexural rigidity here, so if the column wants to bend here, there is nothing to prevent that rotation. So, it would be just the displacement you are applying like this and the columns looking simply like this, because there is no restraint at rotation here. So, it would look something like this. In this case, because it is so rigid, these columns cannot rotate at these points of rotation. So, the deformed shape would look like this. So, it would simply become case of a cantilever here, so two cantilevers here.

In this case, it is also cantilever, but it is restrained again rotation. So, basically this look like cantilever, which is not restrained against rotation. So, if you remember from your structural analysis, what is the displacement or u here? u is nothing but $f_s L^3 / 3EI$, and here this $u = f_s L^3 / 12EI$. Now, there are two columns in each of these.

So, k_s for this system would be $2 \times 3\ EI / L^3$ and for this it would be $2 \times 12\ EI / L^3$. Now, in reality the situation would be somewhere in between and for that there is a way to derive the relationship in terms of ratio of flexure rigidity of the beam and column, but we are not going to get into that.

I am just demonstrating you two very simple cases. So, this is how the force and displacement relationship would be in this case. Now, let us come after discussing their stiffness. So, in this case, we have discussed stiffness. Now, let us discuss damping in the system the second component.

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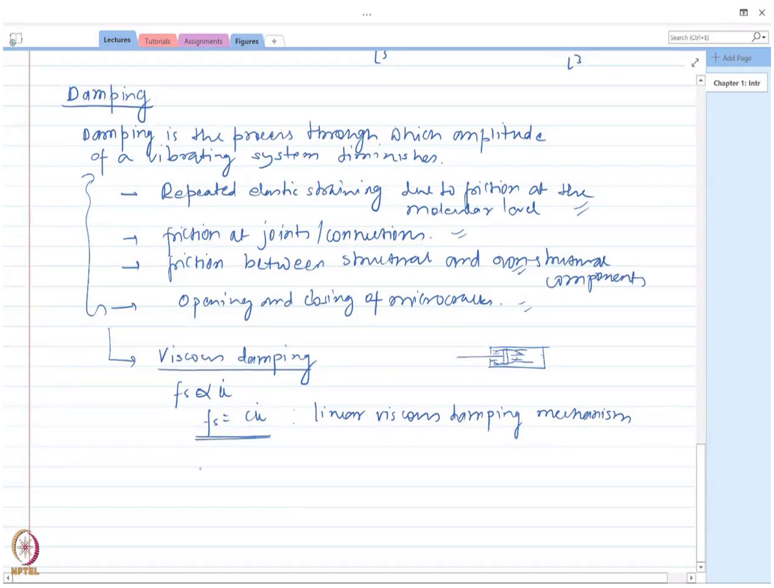

So, we are going to talk about damping and as I mentioned before damping is the process. So, damping is the process through which amplitude of a vibrating system diminishes.

Now, if you have a building and if you apply a load, there is various ways in which the energy could be dissipated. If you apply the displacement to a system and damping represents here in this case what are the different type of mechanisms.

So, let us say one mechanism could be repeated elastic straining. So, if you have repeated elastic straining leads to energy dissipation due to friction at the molecular level. Other could be, if you have a building – it is a steel building or a wood building or any type of building – at joints, you could have bolted connection or welded connection. So, the second could be friction at joints and connection of a building. What other mechanism you think could be there? There could be friction between structural and non-structural component.

If it is a concrete building, you apply a deformation, there could be closing and opening. So, opening and closing of micro cracks. So, many of these mechanisms involve you know friction and generation of heat due to which the energy is actually lost and that is why you cannot have in reality a perfectly a perfect elastic system, you would have non-conservative forces.

Now, let us say we have to model damping. Now, for the simple mathematical formulation modeling each of these technique is very difficult and it is not required, I mean if I want to get into that much of detail, then I would not do it for just a simple problem like this.

So, how do we do, what do we do? We combine all these damping mechanisms into a simple mechanism. And we are saying that we are going to represent all these energy dissipation mechanism to a damping called viscous damping. So, viscous damping is there.

What happens in viscous damping, let us say, I have a cylinder in which I have a piston like this, and there are fluid in each of these orifices. So, fluid in each side of this piston.

Now, what happens if you push this piston, the fluid will flow around these orifices. So, there would be resisting force. Now, can you imagine if you tried to push it very quickly, the force would be more and if you try to push it very slowly, then the force would be almost 0.

It is like in real life let us take an example. You have a fluid and then you try to push a flat object onto the fluid. If you do it very slowly, it would not offer any resistance, but if you try to do it very quickly it offers lot of resistance. So, a damping force, can I say it is proportional to not the displacement because there are displacement in both cases, the whether you apply slowly or whether you apply suddenly, however, it is better correlated to velocity.

So, the damping force is proportional to velocity. And the coefficient of proportionality is called damping coefficient and represented by $c\dot{u}$. So, this is called linear viscous damping mechanism. So, although in a building or in a structure or in a system, in reality the nature of energy dissipation or damping is not actually viscous damping. It is just mathematically easy to combine all these energy dissipations in a single mechanism and represented through viscous damping.

And then we will see in a later chapter how do we get this viscous damping coefficient or how do we get equivalent viscous damping to represent all the system. Thus, right now this is the formulation, or this is the expression for viscous damper. So, you will see that later what we are going to represent a damper or viscous damper mechanism to this notation here, which basically like you know a simplified notation of this damper here.

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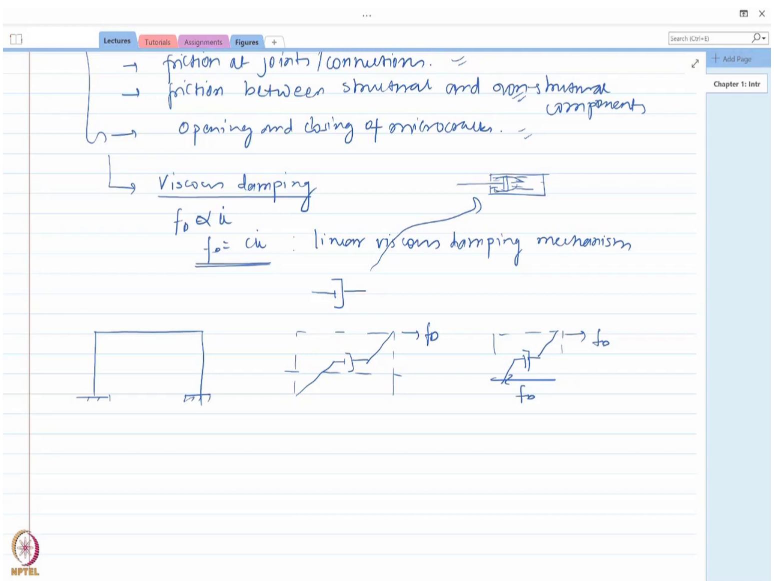

So, now, again let us come down to this one, which we are considering. So, if I have to just represent the energy dissipation mechanism in this frame building, what I am going to do, I am going to connect a damper between two points of this structure.

So, let us say, I do not have anything, I do not have mass, I do not have its stiffness. Just to represent damper I have something like this here. And the force in this damper can be represented with f_D here.

So, if we apply a force f_D here, the resisting force in this frame would be what? Consider something like this, can I cut the structure at some point, here it would be f_D, same force would be applied here. So, this is the damping mechanism that we consider. So, once that is clear, now let us get into the third part which is the equation of motion.

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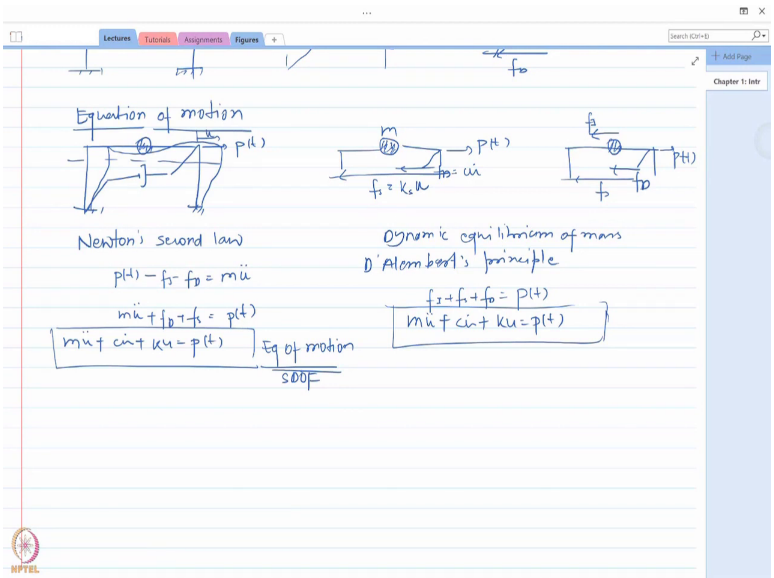

So, let us see how do we setup the equation of motion. So, as you know our system was something like a frame here on which there was this mass right here, and then to represent all damping mechanism I have this damper.

So, this is how I am representing mathematically, including all the components of our dynamic system. Now, if I apply an external force here, what my goal here is to find out the equation of motion for this mass or for this single degree of freedom system subject to this external force $p(t)$.

Now, there are two ways in which we could do it, and this comes from the course of engineering mechanics. If I have to set up equation of motion either I can consider Newton's second law, which is resultant of all the force on a body equals mass times acceleration.

So, I can use Newton's second law. So, as per Newton's second law, I have to first draw the free body diagram. So, let us consider equilibrium of this mass, and I am going to cut my system here. So, what are the forces acting on this system? If you consider here, I have force $p(t)$ which is acting here.

And if I cut my system as we have saw previously, whatever the stiffness of this system is, I can write $f_s = k_s \times u$, and remember that there is this damper here as well. The force in this damper would be $f_D = c\dot{u}$, and then there is this mass here. Now, what is the resultant on this mass m here? It is $p(t)$ which is actually deforming something like this here.

So, I am assuming that it is deforming by amount u. So, $p(t)$ minus f_s minus f_D, the stiffness force and the damping force, this is the net resultant force – this mass should be equal to mass times acceleration which is $m\ddot{u}$. And if I bring it on the same side, I can write it something like $m\ddot{u} + f_D + f_s = p(t)$.

And if I consider this to be made up of linear viscous damper and a linear system the stiffness is linear, then I can write f_D as $c\dot{u}$ and then f_s as ku. They should be equal to $p(t)$. So, this is my equation of motion.

I will keep coming back to this equation of motion again and again so many times in subsequent chapters. You will see that how to solve it, how to use this for undamped system and damped system. So, this is the equation of motion for a single degree of freedom system for a SDOF system. So, Newton's second law was one way to do it.

There is other way you can do it and that is basically using dynamic equilibrium of the mass. So, I need to use the dynamic equilibrium of the mass. So, remember till now you have only considered, in your engineering mechanics as statics. You did not consider dynamic equilibrium.

And here we use what is call D'Alembert's principle, which you might also know through the terminology called pseudo force, which says that if a mass is accelerating, I can apply an inertial force on the mass which is opposite to the direction of acceleration, and then solve the equilibrium of the system as a simple static problem.

So, what it says in this case I will again have the same forces f_s, I will have here f_D due to this damper, this $p(t)$ is here. On this mass, I am going to apply an inertial force f_I, which is opposite to the direction of motion. And then I am going to write it as $f_I + f_s + f_D = p(t)$.

So, I did not use that resultant force should be equal to mass time acceleration. I just apply on the mass and opposite equal basically force that is opposite to the direction of motion. And this is equal to f_I, which is basically equal to mass times acceleration. And then I can have f_s and f_D, let me just bring it here, thus, $m\ddot{u} + c\dot{u} + ku = p(t)$.

So, it is up to you which approach do you want to follow. I have found out that for a complicated problem in multi degree of freedom system, this second approach might work out to be better, but you just need to be consistent and both of them would give you similar result. So, remember setting up the equation of motion is the first step in a structural dynamics problem.

And you need to be very accustomed how to do that. So, we have done it through an example of a frame structure. You could have different type of structure; you could have one of the examples that we just saw here. That four of the examples that I showed you and you could set up the equation of motion for those examples alright.

Now, after we are done with this, remember the same thing mass, damping, and stiffness can be obtained considering a representation which is a simpler representation, a mass – spring – damper representation and the spring – mass – damper representation have been typically used in courses like mechanical vibration or physics.

So, the formulation that I have described till now considering this is intended for structural engineers. But the spring-mass-damper is a more commonly adapted representation in mechanical vibrations and other physical problems. So, we will be going to discuss that representation as well.