Dynamics of Structures
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Continuous systems
Lecture - 18
Generalized SDOF systems

Hello everyone, welcome to the lecture today. We have been discussing so far, a spring -mass-damper representation of different type of system.

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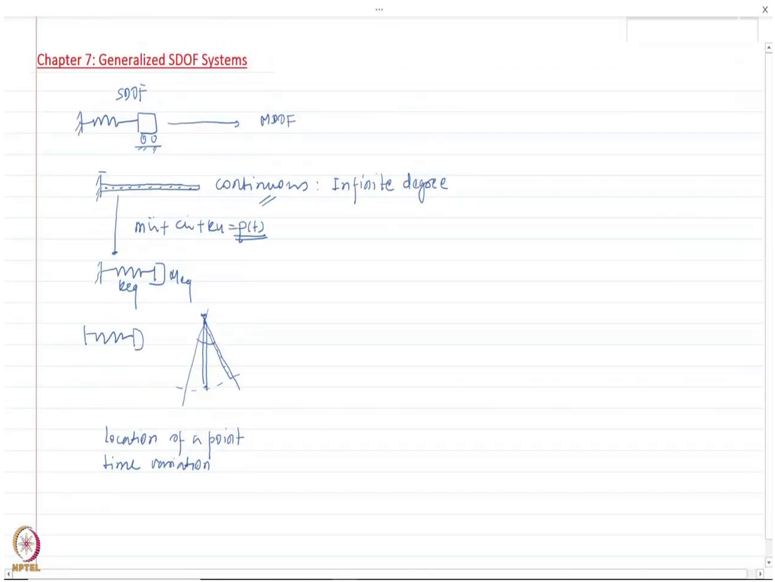

But, as you would realize, in reality most of the structure are not discrete mass-spring and damper. So, you have a structure like beam and column. So, in order to get the response of this system, we need to have a method which can simplify a continuous system to a single degree of freedom system.

So, if you have a beam-column then how can we simplify that to a spring-mass-damper representation? Basically, we have already discussed, how to get the response of a single degree of freedom system (mass-spring-damper) to different type of loading. We can find out the response of a continuous system, once it is reduced to a single degree of freedom system to different type of loads.

We simplify a continuous system to a single degree of freedom system through a concept called shape function. We are going to learn about that today. We are going to see how we can employ the idea or the concept of shape function to convert continuous system to a spring-mass-damper representation of a single degree of freedom system.

So, let us get started. So, till now we have considered a single degree of freedom representation of general structures. So, it could be a spring-mass-damper representation. In reality, most of the system are not single degree of freedom system.

For many systems, single degree of freedom system might not be an appropriate approximation to get the response or desired response quantities of interest. So, we move from analysis of single degree of freedom system to multi degree of freedom system.

Now, to make that transition we first going to learn how do we analyze if we have a continuous system? Let me take the example of this cantilever beam here. Now, how many points we consider on this cantilever beam? In general, a continuous system would have infinite degree of freedom.

Now, it might be appropriate for some cases to analyze this using a single degree of freedom system. For more accurate representation we might have to switch over to multi degree of freedom system. Now, although multi degree of freedom system supposed to be more accurate however, it is computationally expensive as well as the standard solution might not always be available.

But we have seen that for a single degree of freedom system. If we have this $\left(m\ddot{u}+c\dot{u}+ku=p(t)\right)$ equation of motion, we have derived response for different type of excitation and we have learned how to get the response using different type of numerical method. So, we have standard results available for single degree of freedom system.

Now the question comes, is there a way using which I can analyze a continuous system without having to get into the complication of analysis associated with multi degree of freedom system. That is why the generalized SDOF system comes into play.

So, as I said, if you look at any continuous system in general it would have infinite degree of freedom. So, our goal is that we can reduce the system to a single degree of freedom system and then find out *k* equivalent (k_{eq}) and *m* equivalent (m_{eq}). So, although I am analyzing a continuous system, I would utilize some methods through which I can do this simplification so that I do not have to do the calculation for multi degree of freedom system.

Second thing, if you remember till now, we have considered movement of a single mass, so either mass was simply attached to a spring, or it could have been a rigid bar that might be hanging. So, in both cases I need only one degree of freedom to represent this displaced position from its original equilibrium position.

Now, let me take the example of this rigid bar here, if you look at this rigid bar the rigid bar is constrained to move around this point of rotation. Now, on this rigid bar, we consider a displaced position in this fashion. I can have different displacement position on this rigid bar however, I would have the same angular velocity or angular acceleration.

So, two things come into play for a continuous system; the first thing that comes into picture is the location of a point. The second thing that is common to single degree of freedom system or continuous system is the time variation of the response.

So, we are going to consider the time variation of response. So, while for a single degree of freedom system we had considered only a single point. Now for a continuous system there would be several points on the continuity of the body, and we will have to find out the displacement history of each point on that body.

So, that is why the generalized history of system comes into picture. So, what do we do for a continuous system?

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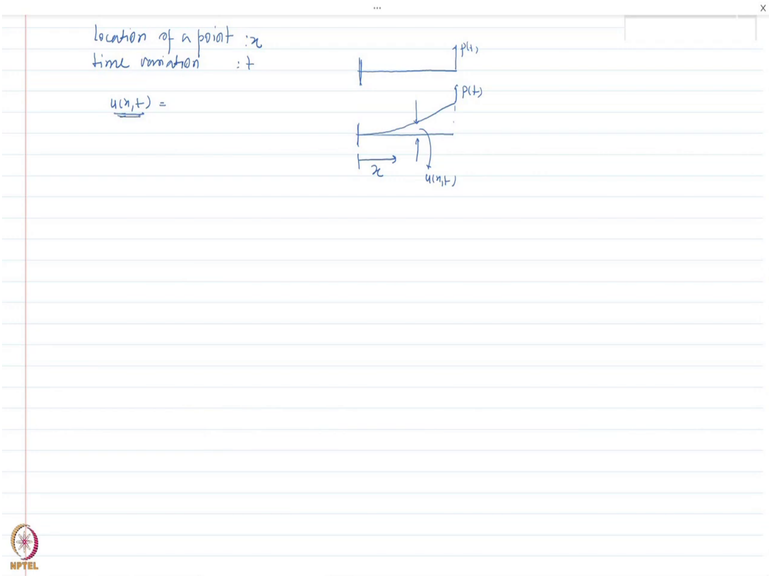

So, let us say I have a deformation $u(x,t)$ which is function of now x and t. This represents the deformation of any point on a continuous system. So, let me take the example of this cantilever beam. Under action of some force, it is deforming, so that in the displaced position this is how it looks like.

Now, this deformation is what we are talking about, this is our $u(x,t)$, this quantity here where x is the distance from the fixed end or any reference coordinate system that you have assumed for that cantilever beam. So, this is what I need to find out and subject to a force $p(t)$ here. So subject to this force I would like to find out $u(x,t)$. So, the first objective of this chapter is to find out this $u(x,t)$ subject to this excitation.

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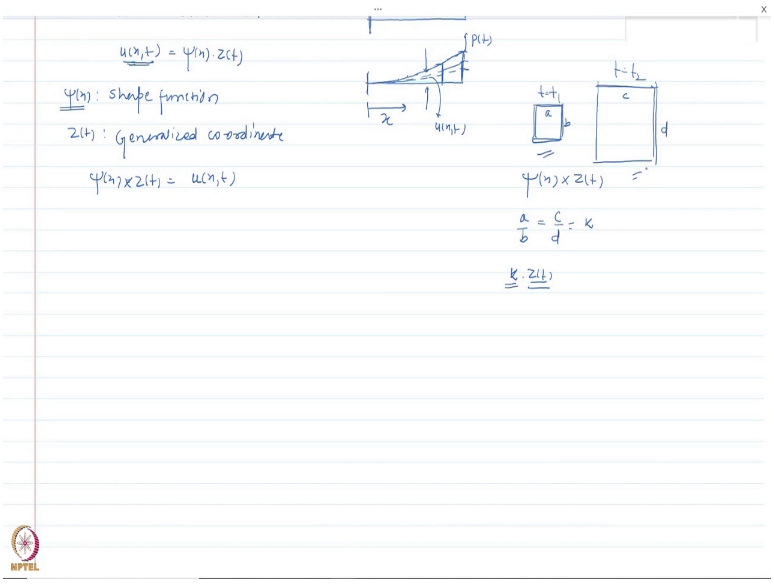

And the way we decompose the deformation in terms of two function; first, is $\psi(x)$ which is only function of the location or x and the second is $z(t)$ which is the only function of time t. $\psi(x)$ basically is called shape function and $z(t)$ is called generalized coordinate. Now, let us see what the significance of these two quantities are.

Now, if you look at the deformation, a different instant for a continuous body. For example, you have this cantilever beam here. So, at different instance of time, it would be something like this. Let us say this is t_1, then this is t_2 here and then so on.

Now, considering that the body is vibrating. There are two things that are happening: first, the movement of each and every point on the body with respect to time. If you look at it, let us see location of this point which is at the foremost end compared to another point which is here, so these two points and then after some another time.

Let us say the location or the ratio of the vertical coordinate, these two are going to remain same and this is the first thing that we are going to talk about is the shape function. So, to look at into the shape function first you need to think about what is shape? Shape basically represents the relative aspect ratio of any body.

So, for example, if you look at these two might have a different size, but they have the same shape. Whether they have the same shape, it depends on the relative aspect ratio. So, the aspect ratio of the sides of these two. So, what happens similarly in terms of deformation, I am going from t equal to t_1 here to t equal to t_2 .

So, while my size is changing my shape is still maintained. So, if I can find out some function that can represents the shape of a structure and multiply this with some other function that represents the time variation then that gives me the overall deformation.

For example, in this case, if I represent the shape as a and b for first one and this as c and d. So let us say $\frac{a}{b} = \frac{c}{d} = k$ where k is some aspect ratio. So, if we can represent the total sizes as, k times some time variation $z(t)$. So, all I need to find out this time variation and if I multiplied this with the shape of the body it is going to give me the overall size. Now, take that analogy and let us revert to the vibration of this cantilever beam that we are considering.

So, this is one dimensional body in terms of coordinate x, let us say I am somehow able to represent location of each and every point on this deformed shape of the cantilever beam through a function $\psi(x)$ and then there is another coordinate $z(t)$ which is basically changing with time. So, if I multiply $\psi(x)$ with $z(t)$, it is going to give me the $u(x,t)$. Which is basically the overall deformation at each and every point on this beam.

Now, shape does not change with time, as you have observed here. What actually is changing the generalized coordinate, so the time variation actually comes from the generalized coordinate $z(t)$ and our goal is somehow is to derive an equation.

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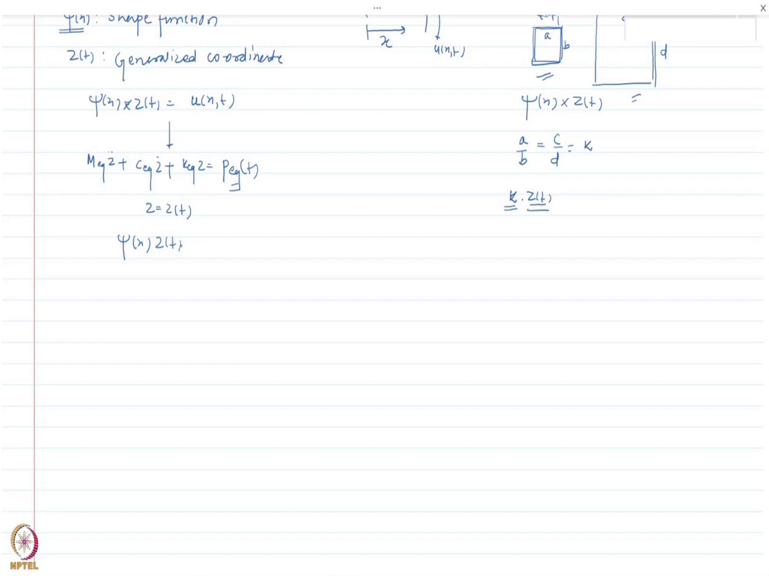

Which let us say may be of this $\left(M_{eq}\ddot{z}+C_{eq}\dot{z}+k_{eq}z=p_{eq}\left(t\right)\right)$ form here. So, we have equation of motion. So, our goal is to somehow use this shape function to find out an equation something like this. If I can find out that, I would be able to get $z\left(t\right)$ as a function of t subject to the excitation $p_{eq}\left(t\right)$, and that would represent how the deformation is changing with respect to time.

And, if I multiply that $z\left(t\right)$ with the shape of the structure which represents the location of each and every point the relative location, then I might be able to find out the overall deformation of each and every point of the body at any time instance t and that is what we will try to do here. Let us try to understand this through two examples. First, I will give you an example of a rigid body.

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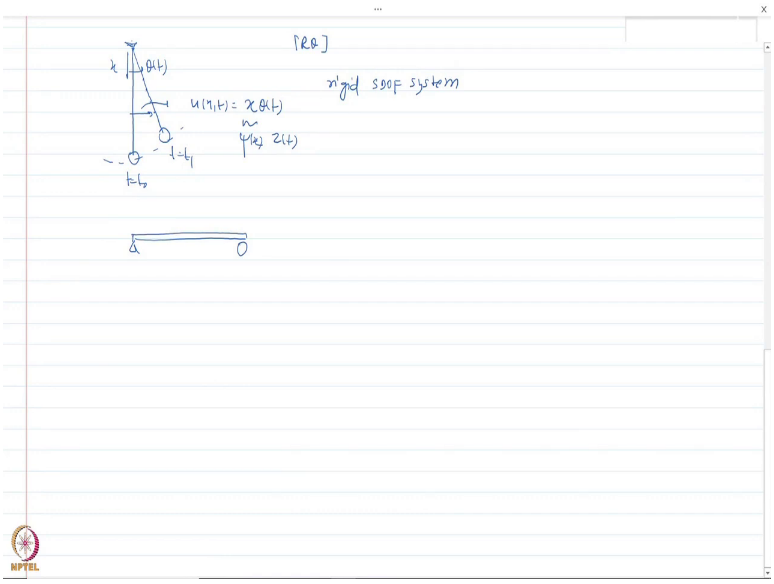

So, if you consider a pendulum. Now, say at any time instant $t = t_0$, the pendulum was in equilibrium position and then at any time instant $t = t_1$ it has come here.

Now, if I consider x to be the coordinate represents in this direction and this to be the $\theta(t)$ which is changing with respect to time. I can find out location of each and every point using this equation $u(x,t) = x\theta(t)$ here. The overall displacement which would be equal to $x\theta(t)$.

This we have already been doing remember $R\theta$ basically represents the location on the radial location or on the circumference of any particle. Now for this case, $u(x,t)$ represents the displaced position of any point. Now, if you consider here, basically the x is the $\psi(x)$ or the shape function. Now this is a rigid pendulum, and the mass is hanging at the end of it.

So here, x represents the shape function, which is a very simple function, and if I multiply this with $\theta(t)$ which represents the $z(t)$, the time variation. The multiplication of these two gives me the displaced position of each and every point on this body with respect to its original position.

So, for rigid system, let us say rigid SDOF system. I can easily find out the shape function and that would be unique, because there would be only one way in which the shape of the structure can be represented. However, if I compare that, to a beam which is say simply supported beam. let us take example of a simply supported beam.

Now, as I said if the system is a continuous system, there would be infinite degrees of freedom. So, by definition, it would have infinite modes of vibration and that you might recall from your knowledge of sound theory or wave theory where you consider different harmonics of vibration it is somewhat similar to that one.

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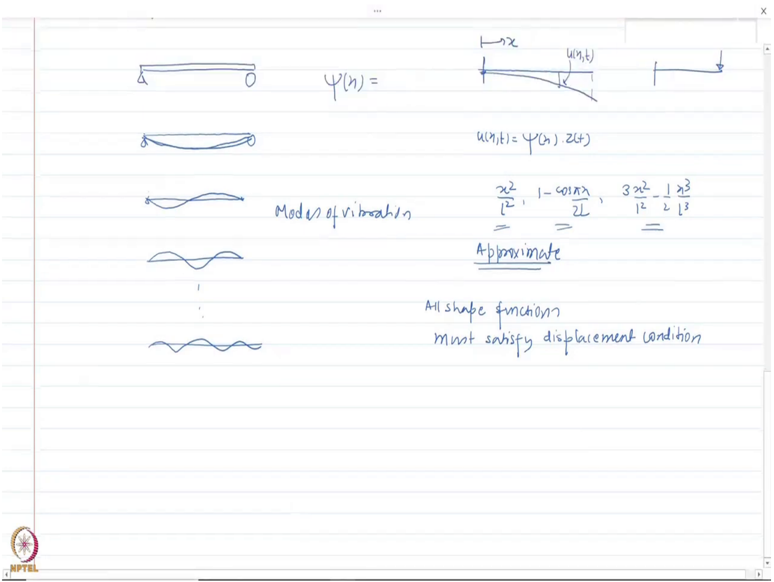

We will look into that more detail when we talk about the multiple degree of freedom system. So, a simply supported beam is a continuous system and it can vibrate like this, or another mode of vibration could be like this, or another mode of vibration could be something like this and so on. So, it can have infinite degrees of freedom or infinite modes of vibration.

Now, the question becomes, if I have to find out $\psi(x)$, which of these shape functions do I need to take, and how can I get those shape function or the mode shapes? Now, we are going to get into multiple degree of freedom system where we are going to do the exact determination of the mode shape; however, for this chapter we do not want to do that, we do not want to do all the analysis to get the exact mode of a system.

What we are going to do though here is assume that the first mode of vibration is represents the deformed shape and then assume any shape function or any function $\psi(x)$, that somewhat represents the deflected shape of the structure. For example, if I consider a cantilever beam, I know that in the deform position it looks like something like this.

Now, this can be represented by various type of expression, so if I consider this horizontal axis to be as x and this deformation to be $u(x,t)$. To represent the deflected shape let us me first write $u(x,t)=\psi(x)z(t)$. Now, the deflected shape can be represented in various ways. The deflected shape represents a parabola. So, the deflected shape can be represented using the expression $\psi(x)=\frac{x^2}{L^2}$. I can also look at it and see well it does look like a function, or a cosine function which is $1-\cos\frac{\pi x}{2L}$.

So, that at $x=0$ it is 0 and at $x=L$, it is 1. This reflected shape can be approximated with the static deflected shape of a cantilever beam subject to the point load at its end, so it can be represented using $\psi(x)=\frac{3x^2}{L^2}-\frac{1}{2}\frac{x^3}{L^3}$.

So, depending upon what shape function we choose, we can have multiple shape function to represent the deflected shape of this structure. Now, as I have told you I am not trying to determine the exact mode shape which I could if I did the multi degree of freedom analysis using large number of modes. But here my goal is to do it without having to get into all the mode determination. So, I am going to assume the deflected shape to be one of these functions.

Now, these functions as you can imagine would provide you approximate results. Whether that approximation is good enough or not depend on the shape function that you have selected. Now, all three-shape function can be used to represent the shape of this deflected shape of this cantilever beam; however, the only condition that all the shape function must satisfy the displacement boundary condition.

What is the displacement boundary condition? So, for example, this cantilever beam has 0 displacement at its left hand (fixed end) as well as 0 rotation. So, whatever shape function I select that must satisfy that.

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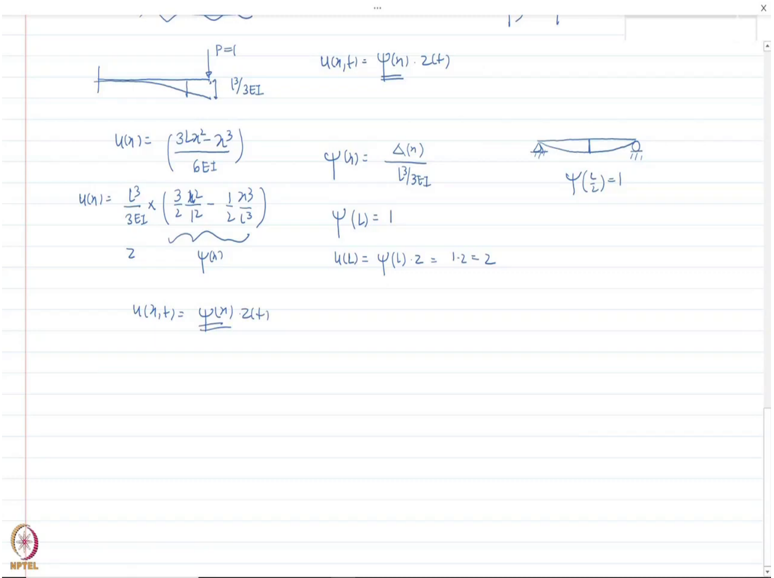

Now, what we usually do? Like in this case, when I write $u(x,t) = \psi(x) z(t)$. In this type of scenarios, we typically write $\psi(x)$ such that we normalize $\psi(x)$ with respect to a point of reference. For example, the actual deflected shape of a cantilever beam subject to a unit point load at its end is $u(x) = \dfrac{3Lx^2 - x^3}{6EI}$.

Now, I am going to write this deflection with respect to some convenient reference point. For example, I know that the deflection at the end is $\dfrac{PL^3}{3EI}$ and for unit force it becomes $\dfrac{L^3}{3EI}$.

So, if I write $u(x)$ as-

$$u(x)=\frac{L^3}{3EI}\left(\frac{3}{2}\frac{x^2}{L^2}-\frac{1}{2}\frac{x^3}{L^3}\right)$$

So, I can normalize such that this $\left(\frac{3}{2}\frac{x^2}{L^2}-\frac{1}{2}\frac{x^3}{L^3}\right)$ represents the shape function and this $\frac{L^3}{3EI}$ represents the z at any time instant and $z(t)$ would be this factor multiplied with something. So that my $\psi(x)$ is now normalized with respect to the deformation at the end of this cantilever beam.

So, usually whatever the shape function $\psi(x)$ that I get, the deformation let us say Δ_x and divide it with respect to some reference point, let us say here it is $\frac{L^3}{3EI}$, when you substitute $x=L$ it becomes $\psi(L)=1$.

If we do that $u(L)=\psi(L).z=1.z=z$. So, if I normalize it then my deformation becomes the generalized coordinate $z(t)$ at that reference point.

So, I have done it with respect to the end deformation here and you can choose any convenient point. For example, if you have a simply supported beam something like this here, you can normalize with respect to deformation at the center and you can select a shape function so that it becomes $\psi\left(\frac{L}{2}\right)=1$. So, it is up to you how to select that. So, once that is known I need to first find out those shape function for different type of structure.

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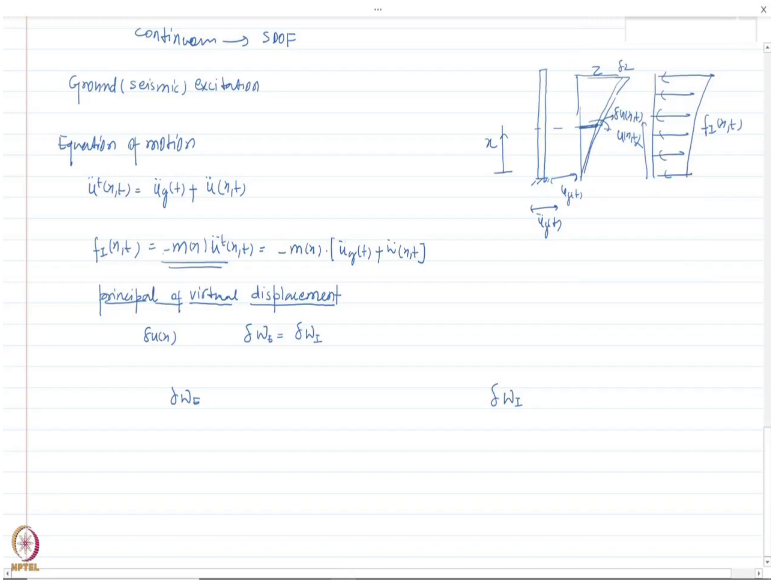

Now, once I have assumed the $\psi(x)$, the question is how I get $z(t)$. To do that I need to simplify the continuous system to a single degree of freedom system. Let us see how do we that? So, we are going to consider an example of ground excitation or seismic excitation.

So, basically what I am saying here, I have the same cantilever beam and it is undergoing vibration due to ground motion. Now due to this it would have some displacement due to ground motion which is let us say $u_g(t)$ plus it would have its own deformation, which is relative deformation, and it would be different at different location.

Let us say this is coordinate x and at any time instant the deformation is u here. So, this represents the deformation of a structure. So, to get the equation of motion, I know the total acceleration at any location in this structure which can be represented as-

$$\ddot{u}^t(x,t) = \ddot{u}_g(t) + \ddot{u}(x,t)$$

Where, $\ddot{u}(x,t)$ is the relative acceleration of that point.

Now, knowing that, let us first find out the inertial force due to this acceleration. Inertial force will look like something like this (refer slide time 30:50). I am applying it opposite to the direction of motion. So, at any height *x* inertial force I can write it as $f_I(x,t)$ and this would be $f_I(x,t) = -m(x)\ddot{u}^t(x,t)$.

Here the mass is a function of *x*. So, it does not have to be a constant mass. Further this can be written as-

$$f_I(x,t) = -m(x)\left[\ddot{u}_g(t) + \ddot{u}(x,t)\right]$$

Now, this is effectively the external force that is acting on this cantilever beam. To get the equation of motion we are going to use a method called principle of virtual displacement and you might have already come across this in different courses.

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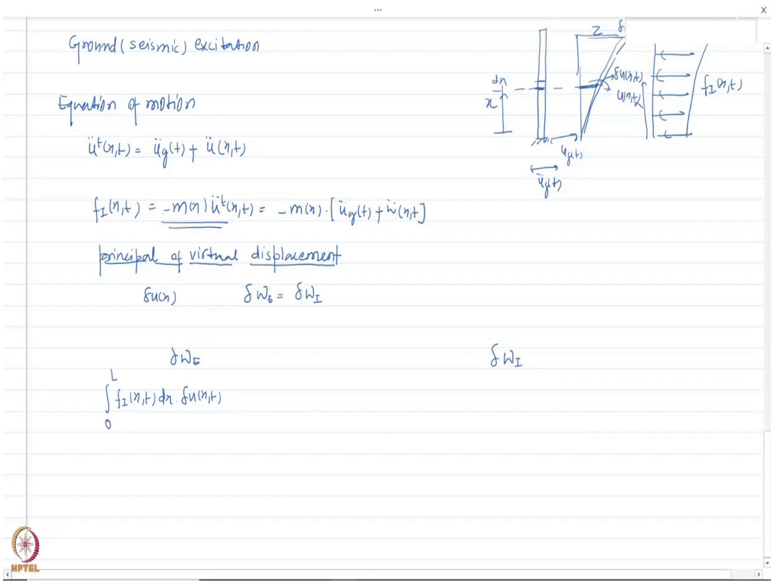

So, we are going to utilize principle of virtual displacement. Now, principle of virtual displacement basically says, if a system is in equilibrium and if you apply a small virtual displacement let us say $\delta u(x)$ then work done by the external forces δw_E are equal to work done by internal forces δw_I due to virtual displacement.

So, let me repeat it again, if a system is under equilibrium given a small virtual displacement $\delta u(x)$, then work done on the system due to external forces is equal to work done by internal forces due to this virtual displacement.

Now, the external force as you know can be computed as force times displacement.

So, let us say this is the deformed position and it is given a small virtual displacement which is again a function of *x*. So, this small virtual displacement is $\delta u(x,t)$ the function of *x* and *t* and at the top it was *z* because I am normalizing it with respect to the displacement at the top and let us say this is δz.

So, let us see due to that how much is the work done. So, work done due to the external forces would be $\delta w_E = \int_0^L f_I(x,t)\,dx\,\delta u(x,t)$. Where $f_I(x,t)$ is force at any height *x* and if you consider the force over the height *dx* it would be $f_I(x,t)$ times *dx*. So, this is the total force over the height *dx*.

Now, this is undergoing a virtual displacement of $\delta u(x,t)$. And I need to integrate it over the whole height to get the total virtual work done by the external forces. Now, for the internal forces, if it is a flexure system and I am assuming that there is no significant shear deformation.

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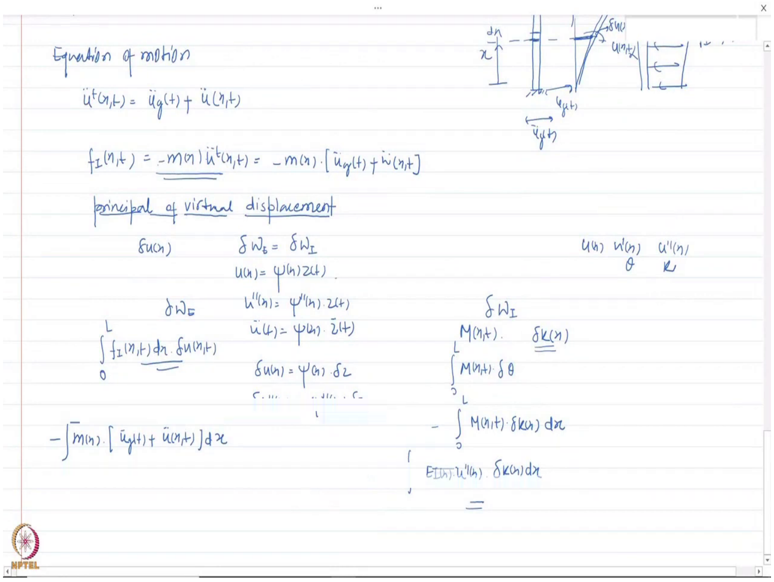

The work done due to flexure; you might already know this is done by the internal moment $M(x,t)$ due to curvature $\delta k(x,t)$. And I know, if $u(x)$ is my deformation then $u'(x)$ represents the angle θ and $u''(x)$ represents the curvature *k*. So, the internal work done would be work done by this moment $M(x,t)$ times θ or let us say it is $\delta\theta$ and this need to be integrated over the whole height.

$\delta\theta$ is nothing but curvature times the height $(\delta k(x)dx)$ that is being considered. So, I can write this as-

$$\delta w_I = \int_0^L M(x,t)\delta k(x)dx$$

And you might remember from your undergraduate classes from beam theory $M(x,t) = EI(x)u''(x)$ and it becomes $\delta w_I = \int_0^L EI(x)u''(x)\delta k(x)dx$.

Now, before going further, let me point out that if I consider $u(x,t) = \psi(x)z(t)$ then,

$$u''(x) = \psi''(x)z(t)$$

$$\ddot{u}(t) = \psi(x)\ddot{z}(t)$$

Further, $\delta u(x) = \psi(x).\delta z$ and $\delta u''(x) = \psi''(x).\delta z$

So, we are going to utilize this and substitute in these two expressions here δw_E and here δw_I and let us see what we get. So, in the expression for δw_E is-

$$\delta w_E = -\int_0^L m(x)\left[\ddot{u}_g(t) + \ddot{u}(x,t)\right] dx \delta u(x,t)$$

$$\delta w_E = -\ddot{u}_g(t)\int_0^L m(x)\delta u(x)\, dx - \int_0^L m(x)\psi(x)\ddot{z}(t)\delta u(x)\, dx$$

$$\delta w_E = \delta z\left[-\ddot{u}_g(t)\int_0^L m(x)\psi(x)\, dx - \ddot{z}(t)\int_0^L m(x)\left[\psi(x)\right]^2 dx\right]$$

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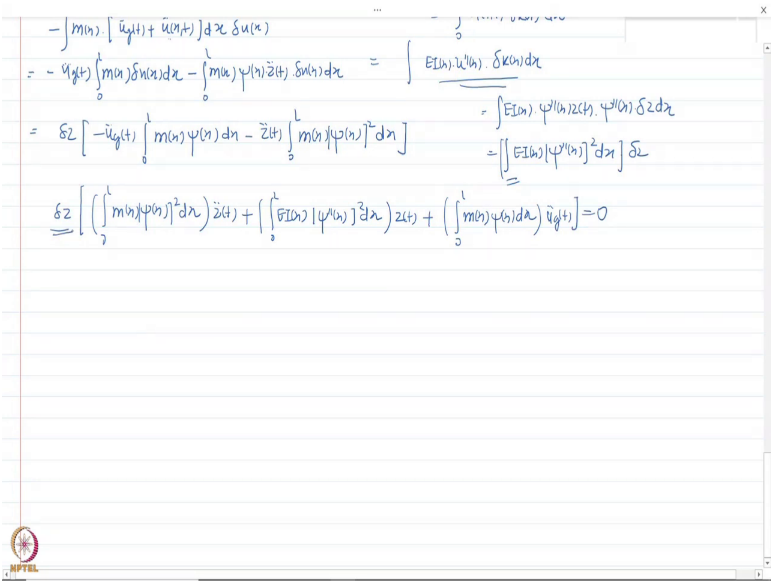

Similarly, I am going to further expand this δw_I term here.

$$\delta w_I = \int_0^L EI(x)u''(x)\delta k(x)\,dx$$

$$\delta w_I = \int_0^L EI(x)\psi''(x)z(t).\psi''(x)\delta z dx \Rightarrow \left[\int_0^L EI(x)\left[\psi''(x)\right]^2 z(t)\,dx\right]\delta z$$

So, if I equate these two equations expression and then bring it to right side.

$$\delta z\left[\left(\int_0^L m(x)\left[\psi(x)\right]^2 dx\right)\ddot{z}(t)+\left(\int_0^L EI(x)\left[\psi''(x)\right]^2 dx\right)z(t)+\left(\int_0^L m(x)\psi(x)\,dx\right)\ddot{u}_g(t)\right]=0$$

Now, I know that my virtual displacement $\delta z \neq 0$ because I have assumed it to be a finite virtual nonzero displacement. So, what I have left with the expression inside the bracket to be equal to 0.

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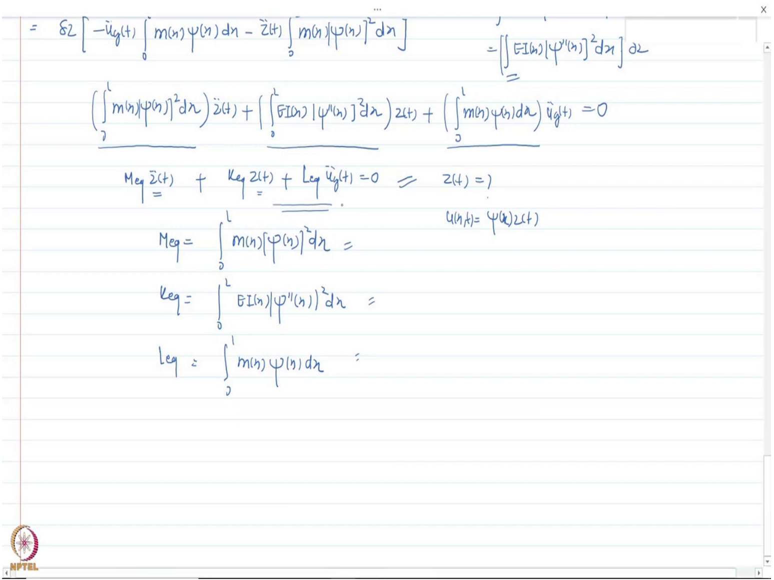

So, let me do that.

$$\left(\int_0^L m(x)\left[\psi(x)\right]^2 dx\right)\ddot{z}(t)+\left(\int_0^L EI(x)\left[\psi''(x)\right]^2 dx\right)z(t)+\left(\int_0^L m(x)\psi(x)\,dx\right)\ddot{u}_g(t)=0$$

Now look at carefully these terms that are inside this bracket. These terms only depend on the parameter *x,* or the location and it is not a function of time. So, if I consider this expression as

$$M_{eq}\ddot{z}(t)+k_{eq}z(t)+L_{eq}\ddot{u}_g(t)=0$$

And this is somewhat familiar to you from the expression of a single degree of freedom system. Now, I have a second order differential equation in the generalized coordinate *z* and that is what I need to solve.

$$M_{eq}=\int_0^L m(x)\left[\psi(x)\right]^2 dx$$

$$k_{eq}=\int_0^L EI(x)\left[\psi''(x)\right]^2 dx$$

$$L_{eq}=\int_0^L m(x)\psi(x)dx$$

So, I have been utilizing the shape function $\psi(x)$, I have been able to reduce a continuous system to a single degree of freedom system and then my job becomes very easy, because I can solve this system and I can find out what is $z(t)$. Once I find out $z(t)$, which is the time variation of the response, I can multiply with $\psi(x)$ as I have told numerous times before to get $u(x,t)$.

So, displacement of each and every point on that continuous system as a function of time. And that is how we solve any type of continuous system. We assume a shape function, utilizing that shape function we find out what is the M_{eq}, k_{eq} and L_{eq}, once that is found out, we solve this second order differential equation to get the value of set of *t*. So, this is the procedure that we would be following to solve a continuous system.

Now, you note one thing, here we have not considered damping. And as I have previously mentioned, damping the c_{eq} . There is no term such as c_{eq} that I can find out from the geometrical and mechanical properties of the structure.

So, what do we do? Because damping is a very complicated mechanism. So, typically if we find out damping ratio from experiment for different type of structure and then use that to solve the system or include the damping in the system. So, first let us write down this expression here that we have in this form.

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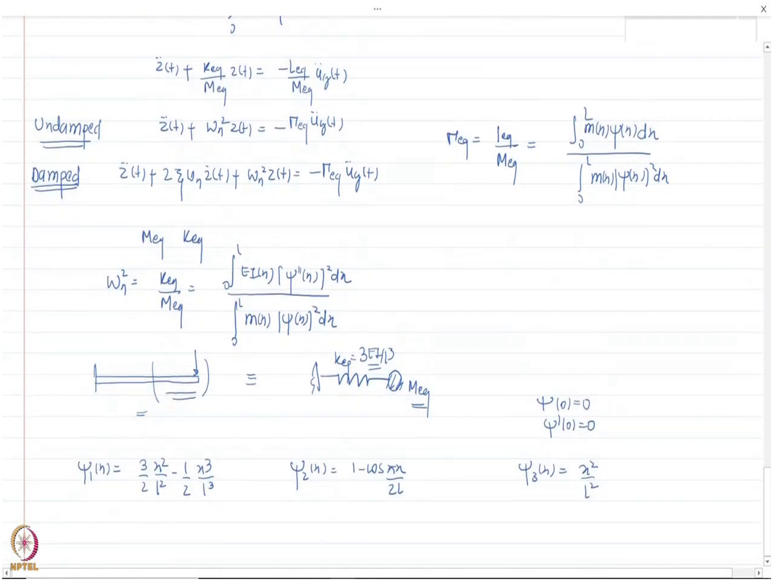

So, let me rearrange that so that I can write it as first for the undamped system.

$$\ddot{z}(t)+\frac{k_{eq}}{M_{eq}}z(t)=\frac{-L_{eq}}{M_{eq}}\ddot{u}_g(t)$$

$$\ddot{z}(t)+\omega_n^2 z(t)=-\Gamma_{eq}\ddot{u}_g(t)$$

Now, if I have a damping in the system-

$$\ddot{z}(t)+2\xi\omega_n\dot{z}(t)+\omega_n^2 z(t)=-\Gamma_{eq}\ddot{u}_g(t)$$

And, if I know the value of damping experimentally or let us say assuming some ratio, I can formulate this equation like this.

And this gamma equivalent Γ_{eq} is-

$$\Gamma_{eq} = \frac{L_{eq}}{M_{eq}} = \frac{\int_0^L m(x)\psi(x)\,dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx}$$

So, we have been able to reduce our continuous system to a single degree of freedom system and utilizing that to find out the response of the system.

Now, once we find out M_{eq} and k_{eq} it is very easy to find out the frequency of the system as before.

$$\omega_n^2 = \frac{k_{eq}}{M_{eq}} = \frac{\int_0^L EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx}$$

Now, the next step is, how do we get the response of the system. Or before going into that let us first see some examples. For example, if we have a cantilever beam something like this and a load is being applied; let us say at one end. We used to simplify the system to a spring mass system saying that I am going to assume my $k_{eq} = \frac{3EI}{L^3}$ and then mass to be represented by tributary.

But many times, it will not be as simple as that what is my M_{eq} and k_{eq}, and this method that we have just described is going to assist us in finding out this M_{eq} and k_{eq} and how accurate it would be it depends as I said previously on the shape function.

So, here what we are going to do now, we are going to consider this cantilever beam and then consider 3 shape functions. The first shape function would be basically $\psi_1(x)$ which represents the static deflected shape of the cantilever beam subject to a point load at one of it ends. So, the first shape function as $\psi_1(x) = \frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}$.

The second is another shape function which is $\psi_2(x) = 1 - \cos\frac{\pi x}{2L}$. And then there is a third shape function which I am going to write as the parabolic function $\psi_3 = \frac{x^2}{L^2}$. Now, if you look at all these shape function, all shape functions satisfy the displacement boundary condition for this cantilever beam which is $\psi(0) = 0$ and $\psi'(0) = 0$.

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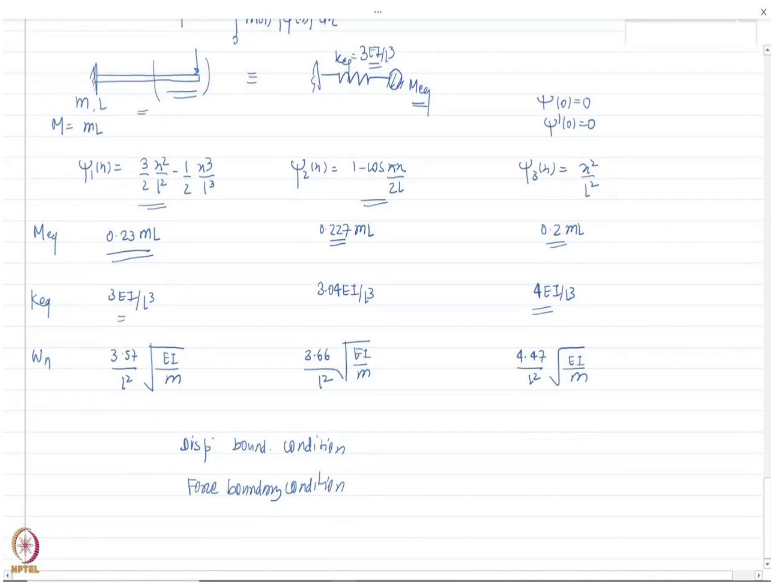

So, all I need to do, substitute the shape function in the equations that we have derived for mass and stiffness here, to get the expression of M_{eq}, k_{eq} and then ω_n and compare to see that using different shape function how do they actually affect the equivalent properties and the modal properties of this cantilever beam.

So, the job is simple basically substituting this $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ that expression for finding out the equivalent mass and equivalent stiffness. So, for the first case when we do, this value comes out to be approximately as $(M_{eq})_1 = 0.23mL$, where m is the mass per unit length and L is the total length such that the total mass is m times L. The $\left(k_{eq}\right)_1 = \frac{3EI}{L^3}$ and $\left(\omega_n\right)_1 = \frac{3.57}{L^2}\sqrt{\frac{EI}{m}}$.

Now, for the second case when you substitute that, you will see $\left(M_{eq}\right)_2 = 0.227mL$, $\left(k_{eq}\right)_2 = \frac{3.04EI}{L^3}$ and $\left(\omega_n\right)_2 = \frac{3.66}{L^2}\sqrt{\frac{EI}{m}}$. For the third case if I do that, this comes out to $\left(M_{eq}\right)_3 = 0.2mL$, $\left(k_{eq}\right)_3 = \frac{4EI}{L^3}$ and $\left(\omega_n\right)_3 = \frac{4.47}{L^2}\sqrt{\frac{EI}{m}}$.

Now, if you look at that we see that the mass that is participating in the dynamic response is not 0.5M; remember mL is equal to the total mass M. So, it is not actually $M/2$, but it is actually $0.23M$. So, approximately one-fourth of the mass, which makes sense, because the mass which are further away from the fixed support is going to participate more compared to the mass which are closer and that is why the effective mass that participate is $0.23M$.

And the k_{eq} that we get is actually $\frac{3EI}{L^3}$, which is expected because we had assumed the static deflected shape to be the shape function in this case, and for the static deflected shape the equivalent stiffness is actually $\frac{3EI}{L^3}$. For the second case, this is again any arbitrary function to represent the deflected shape.

So, when I use this, I look at here again, the mass is around 0.23M the stiffness is $\frac{3.04EI}{L^3}$ which is again close to the first shape function and the frequency is again 3.66 times that same common factor. So, first and the second shape function produce result which are quite comparable.

However, if you look at the third one, there are not significant, but the differences which are much more than the first and the second one. So, you have 20 percent of the total mass; however, the stiffness now increases to four times $\frac{EI}{L^3}$ and even your frequency is 4.47.

So, the question comes, how do I know which shape function is appropriate. So typically, a static deflected shape of any structure can be assumed to represents the dynamic deflected shape. Although, it will not be exactly equal; however, I know that a static deflected shape would always because it constrained by the same boundary condition and it will satisfy the displacement boundary condition. And then we can check.

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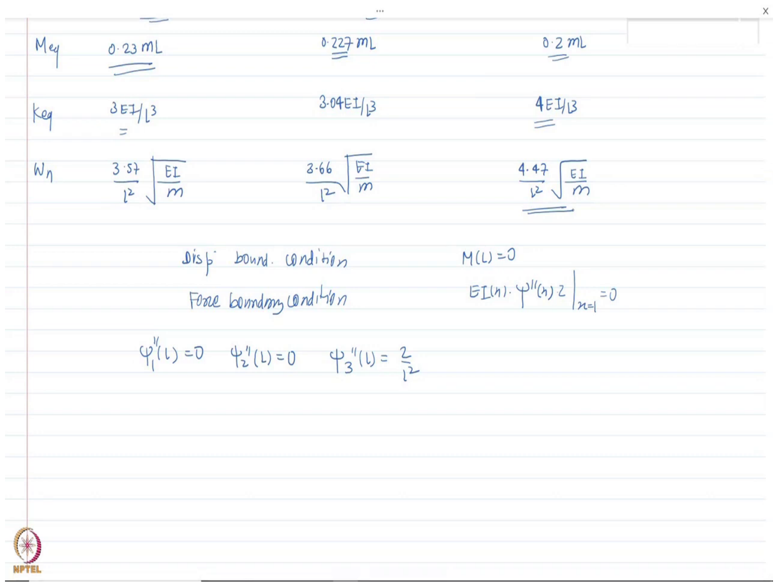

Now, in this case what I see here, all three satisfy the displacement boundary condition; however, only the first two satisfy the force boundary condition. I will show you how. Now, the force boundary condition here is for cantilever beam at the free end, the moment should be equal to 0. So, the moment at length L should be equal to 0. Which means, this expression

$$M(L) = EI(x).\psi''(x)z\Big|_{x=L} = 0.$$

Now, if I look at it $\psi_1''(L)=0$, $\psi_2''(L)=0$ and $\psi_3''(L)=\frac{2}{L^2}$. So, it $\psi_3''(L)$ is not equal to 0. So, while all three satisfy the displacement boundary condition. The first two also satisfy the force boundary condition.

So, it represents the deflected shape that also satisfy the additional force boundary condition. It is more likely to represents the dynamic deflected shape compared to the third one and that is why third one produces large deviation compared to the first and second.

So, this is how we select the shape function and shape function is very critical to the accuracy. If it satisfy the displacement boundary condition, usually the results are reasonable, but if it also satisfy the force boundary condition your results would be very close to the actual solution.

So, with that, we would like to conclude our lecture today and in the next lecture we are going to further study how to apply this to get the response of a continuous system in terms of internal forces and moment.

Thank you.