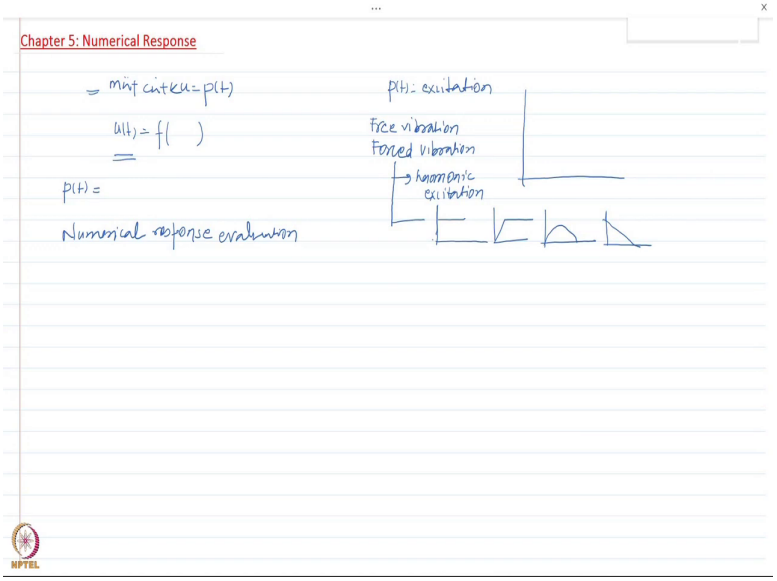
Dynamics of Structures
Prof. Manish Kumar
Department of Civil Engineering
Indian Institute of Technology, Bombay

Module - 01
Numerical Response Methods
Lecture - 16
Numerical Response Evaluation and Earthquake Response Spectra
Numerical Response Methods

(Refer Slide Time: 00:23)



Hello everyone. Welcome to the lecture today. In previous chapters in this course, we have seen that how to obtain analytical response of a single degree of freedom system; subject to either for the case of free vibration or when the loading is harmonic or pulse type loading or step forces.

But we will encounter many cases in which the loading cannot be represented as a closed form solution and even if the loading can be represented as a closed form representation, the solution cannot be obtained as a closed form analytical solution. So, in that case we have to rely on numerical methods to find out the response and we are going to learn few of those numerical methods and see how to utilize those numerical methods to find out the response of a single degree of freedom system.

Now, because this course only focuses on only on structural dynamics and not numerical methods, our discussion would be brief and we are going to discuss one explicit method, one implicit method. And you feel free to explore more, because there is a vast literature available on this topic, but I am just going to show you illustration or examples of those methods and it will give you an idea or a platform that you can use to further explore this topic.

So, let us get started. In today's class, we are going to see how to get the response of a system a single degree of freedom system using numerical methods. So, if we consider our equation of motion that we have been dealing so far, is actually can be written like this.

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

And in the previous chapters we have seen that p(t), which is also the excitation function ok how to get the response u(t) for a given excitation p(t). And basically, we got this response for different kind of situation. So, we got this where only the initial conditions were provided and no force was acting.

So, that was free vibration ok. So, the motion was initiated by initial condition and then, we saw cases in which we had a force acting on the system. So, in forced vibration basically, we saw different kind of forces.

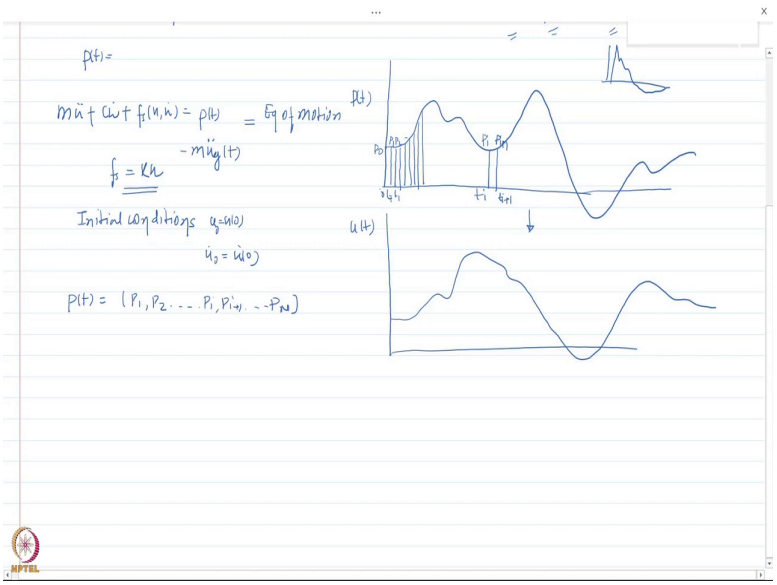
First, we considered harmonic excitations in which the force can be represented as function of sine or cosine. And then, after that we also saw how to get response subject to different kind of arbitrary excitation, but for which we could define excitation as a function of some time variable. So, for example, step excitation or ramp or different kind of pulse loadings with saw how to get the response. So, half sine pulse.

So, the triangular loading. Now, for all these cases, we were able to get u(t) analytically has a closed form solution of some function. Now, it might happen that we can in counter cases in which p(t) cannot be represented as some function or even if it can be represented as some closed form function, it is not easy to derive the solution for u(t) analytically.

And for those situations, it is imperative that we study methods through which we can get the response, even the analytical solutions are not available. So, that is where the numerical

response evaluation comes into picture. So, numerical response evaluation would be utilized to get the response of this single degree of freedom system or they are telling here only with single degree of freedom system, but remember this can also be extended to multi degree of freedom system. So, we would get the numerical response using different methods and we are going to talk about those methods.

(Refer Slide Time: 05:14)



So, let us consider p(t), which is some arbitrary function cannot be write as a closed form solution or even if we consider these cases here, many type many times these loadings that are represented as a nice you know smooth function of are actually simplification of the actual loading.

For example, a triangular loading is a simplification of a blast loading for which the loading actually like you know varies like this is. It is not actually a triangular loading for, but for many purposes assuming or approximating that blast loading as a triangular loading gives us comparable results for you know depending upon the response quantities of interest.

So, the question becomes how to get the response if p(t). So, let me just draw p(t). So, p(t) let us say is given some random arbitrary variation something like that. Now, subject to that p(t), we will have some response right. So, let us say the response looks like this let us say this is u(t). So, the idea is how to get from this point to this point.

So, our problem statement here is that if we have been given the equation of motion here if equation of motion is something like this. Now, what I am going to write down my loading or the spring force as some function of u and $\dot{u}$. Just to emphasize that, when we use the numerical methods, the utility for these kinds of methods so, the applications are not limited to linear systems it can be extended to non-linear system as well. So, in that case my f(s) what we have been assuming as k times u.

It need not be like that. So, we can consider non-linear system like bilinear or hysteretic systems as well ok. So, just giving you like you know what are the potential application; however, for the purpose of this course, we would limit our discussion to only linear systems, but numerical methods can be utilized to get the response for any other type of system as well. So, let us say this is our equation of motion.

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

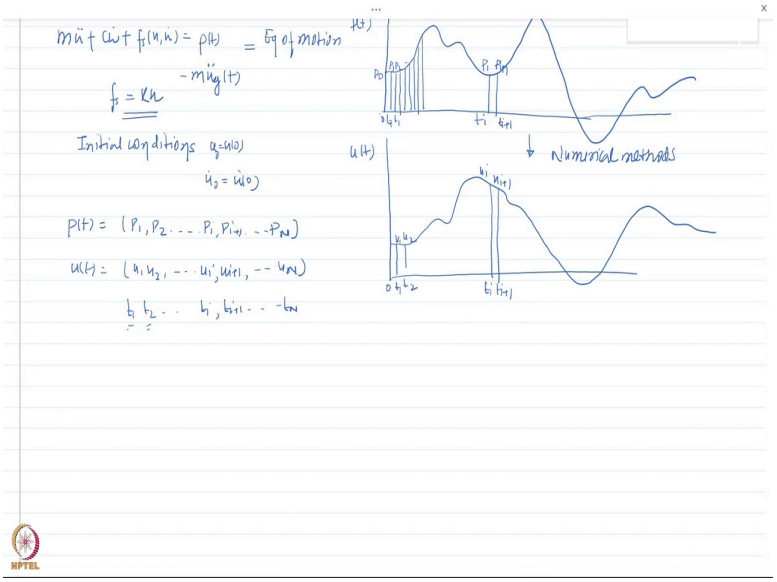
So, p(t) could be either an external force or it could also be ground excitation or support excitation in which case, it can be written as mass times the acceleration of the ground or the support. And these initial conditions are given. These initial conditions are given basically Δt and $\dot{u} = \dot{u}(0)$. So, this is our problem statement.

So, let us first discuss the fundamental principle of numerical response evaluation how do we do it. So, for numerical response evaluation what do we do? The given function if it is given in a discretized form, then it is fine otherwise, even if it is given in some continuous form ok remember as I said this is like you know more useful for complicated functions.

But it can be utilized for other n any type of function. It is not restricted to a particular set of function. So, what do we do we represent our force in discrete values separated by simple time steps? So, let us call this P_0 here P_1 P_2 and so, on at time t equal t_o=0 t_1, t_2 and so, on and at any time let us say it is step i. So, the time is t_i and the value is P_i let us say after Δt it is t_{i+1} and the force is P_{i+1}.

So, p(t) actually we represented as a set of discrete values P_1 P_2 P_i P_{i+1} and then P_N.

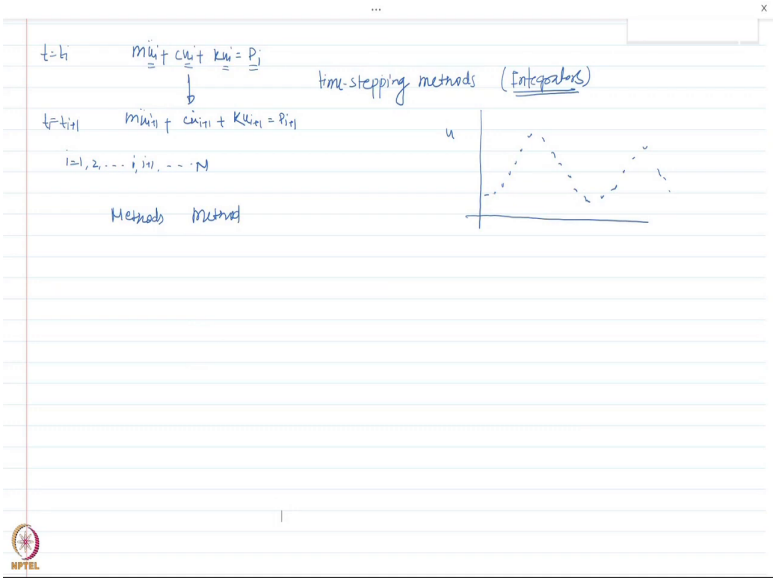
(Refer Slide Time: 10:12)



And our displacement needs to be determined at the same instances of time. So, basically u_1 u_2 u_i u_{i+1} and let us say u_N. Remember, all of these are time instances t_1 t_2 t_i t_{i+1} t_N. So, we discretized our excitation and then, we get the response has discrete values of the response at those points. So, we do not get it as a function or a closed form solution, but for each time instance say, t_1 t_2 we get a value of u_1 u_2 at those instances of time.

So, let us say this is t_1 t_2 this would be u_1 u_2 and so, on. Let us say at t_i and t_{i+1} I get my responses u_i and u_{i+1}. So, this is our goal here, that how from a discretized points that represents the excitation function how from that we can get our response like this. And the way we do it we employ numerical methods to calculate the response. So, using these numerical methods, we can get the response like this.

(Refer Slide Time: 11:46)



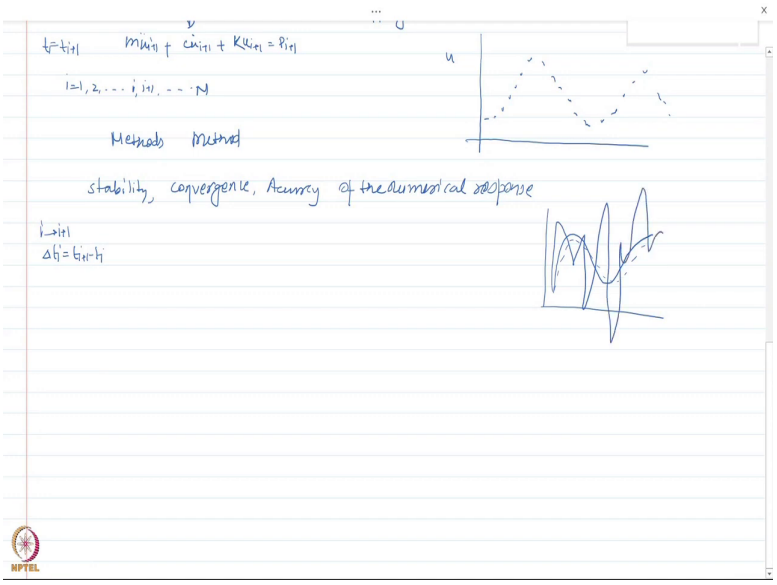
And basically, our idea is that if we know the state of a system at time $t=t_i$. So, let us write down our equation of motion at time $t=t_i$ it would $m\ddot{u}_i + c\dot{u}_i + ku_i = p_i(t)$. If I have somehow found out the values of this parameter. So, the response parameter acceleration, velocity and displacement and force P_i is already given. We want to go from this step to the next step here $m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}(t)$. This is basically called time stepping.

So, numerical methods are also many times referred time stepping methods and many places they are also called as integrators. So, keep that in mind. So, we want to go from this step to this step. So, if somehow, we can devise some numerical method which could assist us in getting the state of system from $t=t_i$ to $t=t_{i+1}$, then i can get the response values at all instances of time.

And if I can get my numerical response, then I can go ahead get the peak response and get the other response quantities as well. So, basically, I would get set of data points representing my response here. And this is whatever goal is. So, through these chapters, we are going to be discussing methods through which we can go from state $t=t_i$ to $t=t_{i+1}$.

And through that we are going to obtain the response. Now, the question comes why there are methods not a single method. If a single method is there that can do the task, then you know why do I need multiple numerical methods? Well, it depends on many criteria.

(Refer Slide Time: 14:09)



Some of those criterions are basically; when we get a numerical solution unlike an analytical solution which is a closed form solution of independent variable, a numerical solution might or might not be accurate it might not be stable or it might not be convergent and it depends on many parameters that leads to those criteria being satisfied.

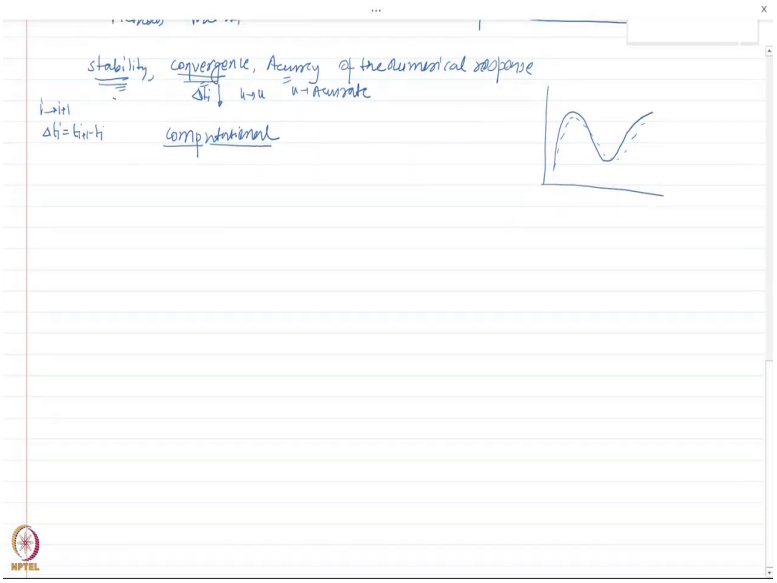
So, the reason for having many methods. Some of those reasons I am listing down here. So, it depends on these criterion stability, convergence and accuracy of the numerical response.

So, let us see what does it mean. Basically, stability means that. Remember, in all these methods, we basically go from one step to next step. Now, a numerical solution is a stable when decreasing the time step actually gives me a value that is now going to a certain value. It is it might or might not be accurate. So, that the numerical solution should be stable.

In the presence of any kind of round off errors or anything like that the solution should not be fluctuating. So, if my actual solution let us see analytically have it obtained the actual solution is like this.

Stability means that there might be inaccuracy like this; however, the solution is still stable ok; however, if I get a solution like this; that means, the solution is not stable. So, stability means that it is basically giving me a realistic value and not fluctuating. Now, second thing is convergence.

(Refer Slide Time: 16:23)



Convergence means that when I decrease the time step Δt_i then, my solution should approach to a particular value at each and every time instant. So, that what convergence means. For example, if I decrease my time step then, it should move towards a fixed value.

So, if I decrease my time step, my solution u should move towards a fixed value which might or might not be an accurate value, but it should at least like you know converging towards that value and that is why we next, we have a accuracy.

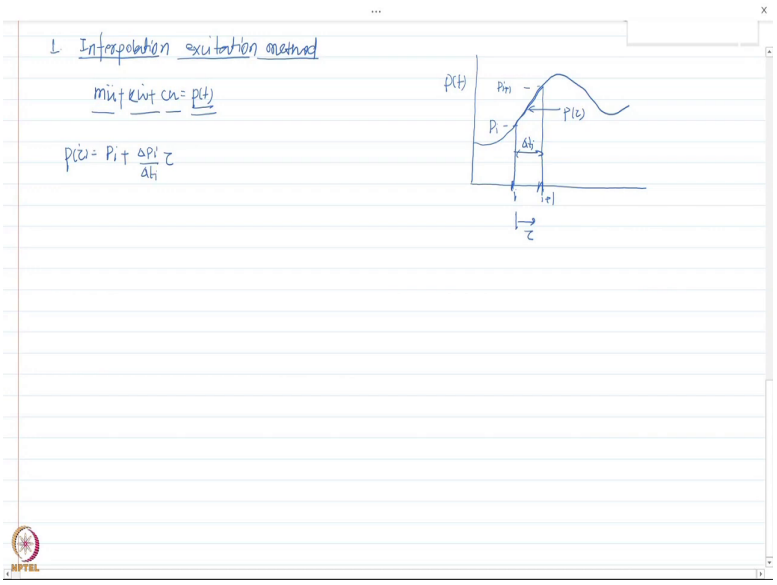
If the value towards a is converging is equal to the analytical solution. It is an exact value, then we say that my solution is actually accurate. So, in terms of basically numerical solution, these three conditions might not always be satisfied. So, that is why we require different numerical methods which can provide appropriate stability, convergence and accuracy for different kind of or different scenarios.

And you will see that some methods numerical methods work better in certain situation and certain other numerical methods work better in some other situations. So, this is from the point of view of the stability, convergence and accuracy, but there is one also other point which plays a big role and that is computational cost.

So, remember each of these numerical solutions when they are determined, computer has to perform certain number of operations to obtain the solution. And how efficient a computational solution is, it depends on the algorithm for that particular method.

So, some method might be computationally very efficient and it might be utilized in certain scenarios, but they might have to give upon certain other features like accuracy. So, that is why different numerical methods are utilized so that computational time is manageable ok with some expectation on the accuracy of the solution. So, these four criteria determines which method or which numerical method to evaluate the response are better.

(Refer Slide Time: 19:07)



If that is clear, let us move to our first numerical method, which is interpolation excitation method. Now, in an interpolation excitation method as the name suggest; if I have the system like this here what I do, I take the function f(t) or p(t) and then, let us say this is the time variation and this is time $t=t_i$.

So, let us say this is state i and this is state i+1. I draw a straight line between two states let us say, this is p_i and this is p_{i+1}. So, we discretized our; discretize our excitation in the interpolation excitation method. So, if our system is linear. This only works for linear system and this is popular, because it is computationally very efficient. So, if the system is linear, I do not have to go and use like you know complicated numerical methods.

If my only issue is that, my p(t) cannot be represented in terms of closed form of function or subject to p(t), my u value cannot be determined as an analytical solution. Then, I can utilize this interpolation excitation method for linear systems and then, it is a very computationally efficient method.

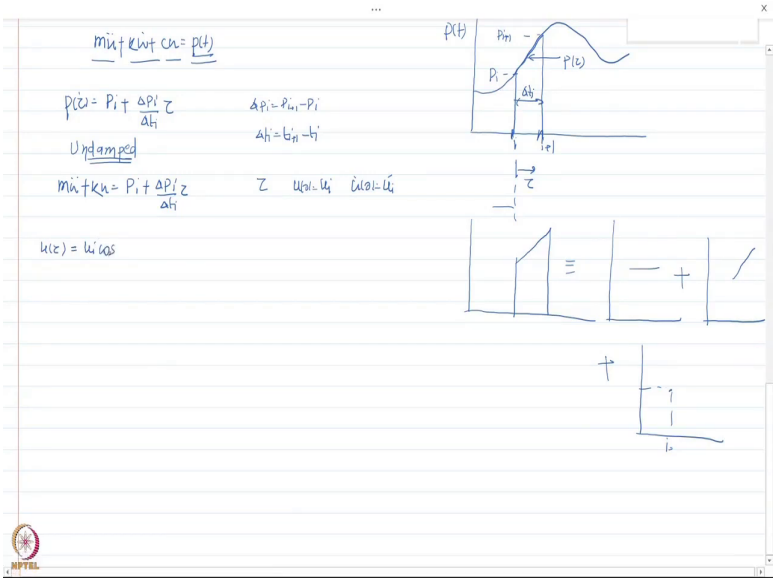
So, let us say what do we do in this method. Given, the state i I interpolate my function excitation function between P_i and P_{i+1}. So, let us say I define a time variable which starts at states i ok and goes up to i+1. So, that τ here represents the time variable between this and this.

Now, this total time step is Δt_i that is already known to me. So, $p(\tau)$ which is at any time τ after the time step i can be written as

$$p(\tau) = p_i + \frac{\Delta p_i}{\Delta t_i}\tau$$

this is the interpolation of the excitation function. So, my equation of motion that I need to solve becomes.

(Refer Slide Time: 21:58)



Remember

$$\Delta p_i = p_{i+1} - p_i$$
$$\Delta t_i = t_{i+1} - t_i$$

So, my equation of motion becomes as

$$\ddot{mu} + ku = p_i + \frac{\Delta p_i}{\Delta t_i}\tau$$

And I am first considering undamped system, but same procedure can be extended to damped system as well. So, this is my equation of motion subject to initial conditions remember my variable τ is starting at time i. So, the initial conditions are basically, conditions given at

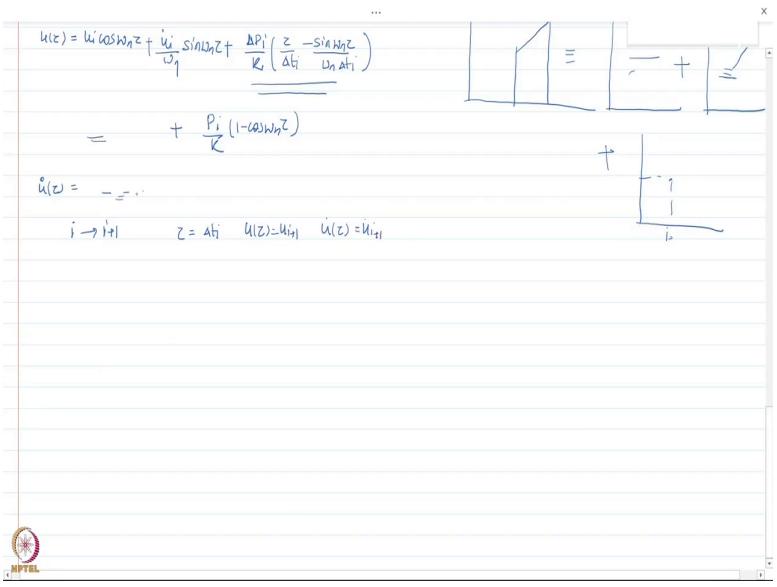
$$u(0) = u_i$$
$$\dot{u}(0) = \dot{u}(i)$$

So, the total response can I say. Remember, I am trying to find out response subject to this is my excitation function. So, this can be represented as response due to initial condition plus response due to this pulse response.

So, a step function, then a ramped function and plus the initial condition; that was a quite up to the step i. This is very important. You have to remember my system is not starting from rest. There would have been some initial condition up to step i. So, I need to include that. So, that total would be sum of response due to all these this solution and it can be written as u ok is equal to first the initial free vibration due to initial condition I can write it as

$$u(\tau) = u_i \cos w_n \tau + \frac{\dot{u}_i}{w_n} \sin w_n t + \frac{\Delta p_i}{k}\left(\frac{\tau}{\Delta t_i} - \frac{\sin w_n \tau}{w_n \Delta t_i}\right) + \frac{p_i}{K}\left(1 - \cos w_n \tau\right)$$

(Refer Slide Time: 24:14)

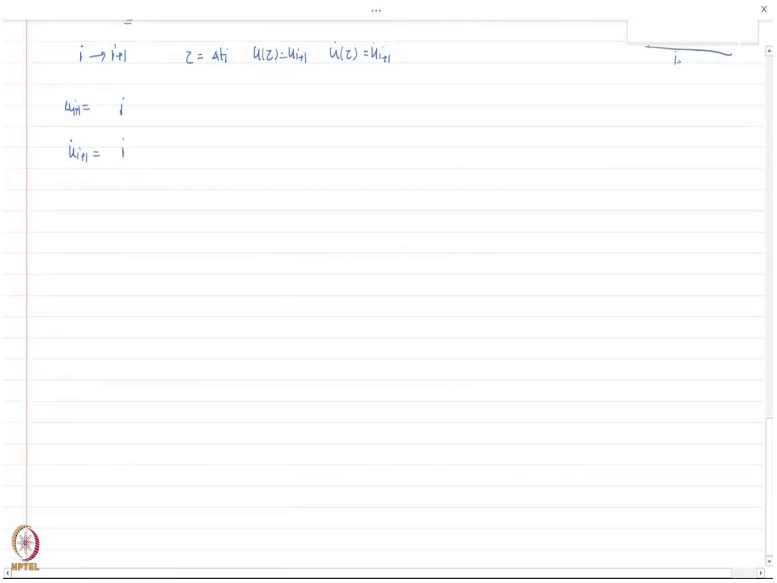


So, the total displacement can be represented as sum of displacement for each of these combination and I could do that, because this is a linear system so, the principle of superposition is valid.

Now, I can go ahead and I can differentiate this and I can get certain expression. The next thing that I need to do remember what I want given, the state at i i want to go to i+1. So, what I do. I substitute my time variable $\tau = \Delta t_i$. I basically reach to a step i+1. So,

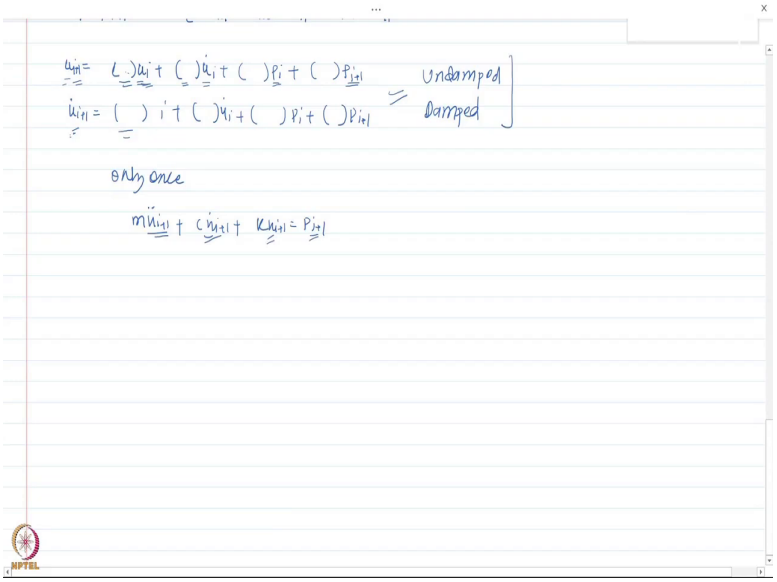
$$u(\tau) = u_{i+1}$$
$$\dot{u}(\tau) = \dot{u}_{i+1}$$

(Refer Slide Time: 26:08)



So, that is I am going to do in the expressions that I have obtained here, ok that I will get my u_i as some function here, and u_{i+1} as some function here, which would be in terms of known parameters of i in terms of previous steps. Now, these can be arranged. These can be arranged and the solution can be written in this form here.

(Refer Slide Time: 26:47)



Some constant times u_i and another constant times. And you can actually derive that and see what are those constants it is not very difficult to do. I am not deriving it here, so that you know not to get bog down by with all the equation in mathematics here. The important thing is to understand the concept here. So, it would come like something like this u_i at the previous step velocity at the previous step force at the previous step and force at the current step.

But remember, I already know the force and the I already know what is the force at the current step. In fact, if the excitation is given to me which according to this problem, then, my force values is it is like they are known at all the time steps ok.

Similarly, my velocity would be another constant which is different from this in terms of

$$u_{i+1} = ()u_i + ()\dot{u}_i + ()p_i + ()p_{i+1}$$

So, although we have derived this for the undamped system, the general form of this equation is going to remain same even for damped system.

So, both this is the equation. Only thing that is going to differ that these constants are going to be different. And you can refer to any text books to have a look at what these constants are. These constants basically, depends on the structural property of the system.

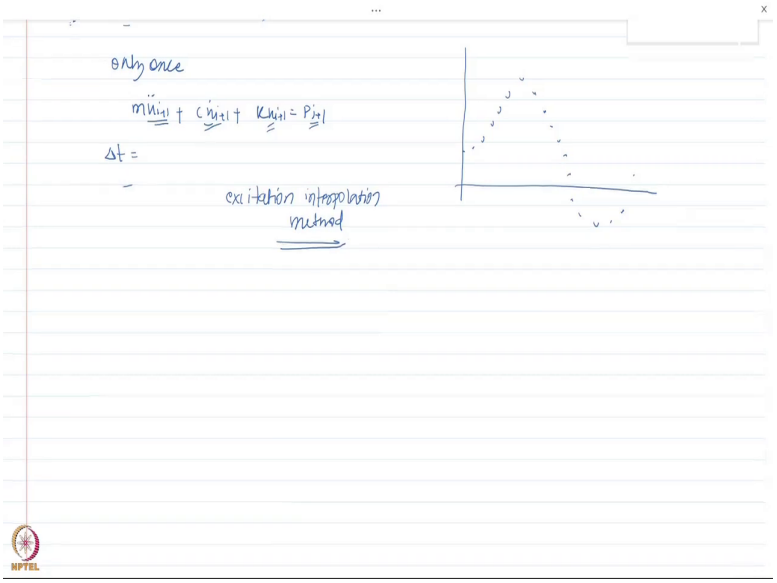
So, it would depend on the time period, frequencies and damping and also the time step end of the system. It will not depend on any other parameter. So, these parameters these constants would need to be calculated only once at the beginning of when the motion starts with the initial condition of course, that as given to us.

And once that is known, I can utilize these expressions to go from step i to i + 1 and then, I will not keep on repeating that. Now, know that we have obtained displacement and velocity here, acceleration we can either differentiate and obtain it or if you consider the equation of motion as this, you can get the acceleration at step i + 1, once the displacement and velocities are known to us.

So, if these are known to us which in fact, they are I can get remember these quantities are known then, we can also get the acceleration. So, generally what do we do? We program it in some sort of mathematical programming software.

So, you can use you can use like you know MATLAB, you could utilize even excel or you can utilize like you know python or any other language of your choice. And you can put this inside a four loop a loop that actually goes you know of from over a certain values certain iteration and then, you can get the response.

(Refer Slide Time: 30:06)

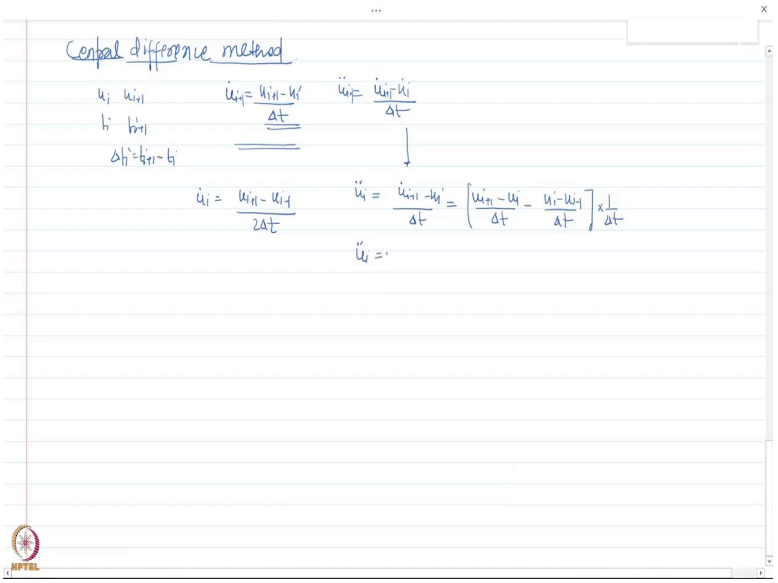


So, basically given; it will start with some initial condition and then basically this is the response you will get you will get the discretized points. And how closely these points are separated ok those depends on the time step that we choose. So, we have considered a constant time step, but remember it does not have to be. Even that if the time step is changing, you can still find out utilizing the same method. Only thing is that in that case, all these constants would need to be calculated at each and average step.

So, this would be changing, because if your delta t is changing in this function, these constants would be changing. So, this is called excitation interpolation method, which is useful for linear system, because it is computationally efficient and works quite well for the linear system.

But you know, I mean that might not be sufficient, because in reality we would we would encounter many systems that are non-linear and I would want to have a method that could be used for linear as well as non-linear system.

(Refer Slide Time: 31:44)



So, from the part perspective; we are going to study two additional numerical methods that can be universally utilized whether the system is linear or non-linear. The first of these methods that we are going to discuss is called central difference method ok it is called central difference method.

Now, a central difference method. It is basically placed on the finite difference approximation of the time derivatives of the displacement velocity and acceleration. Now, you might already be aware that given displacement value.

Let us say u_i and u_{i+1}, if I have to find out and let us say these are separated these are at time t_i and t_{i+1}. So, if these points are given this velocity is basically the slope of the line joining these 2 points.

So, if you want to approximate the velocity approximately calculate the velocity, it would be simply slope of the lining line joining these two points. So, I can write down my

$$\dot{u}_i = \frac{u_{i+1} - u_i}{\Delta t}$$

acceleration can written can be written as.

$$\dot{u}_i = \frac{\dot{u}_{i+1} - \dot{u}_i}{\Delta t}$$

So, this is the concept that we extend in central difference method. So, what do we do in central difference method?

We first consider velocity, joining the displacements which are a part by 2 Δt . So, instead of just consecutive points, I can also say that I would write my velocity

$$\dot{u}_i = \frac{u_{i+1} - u_i}{2\Delta t}$$

And now, remember these points are now separated by 2 Δt , because there are two steps in between. So, the slope joining these two would be 2 Δt .

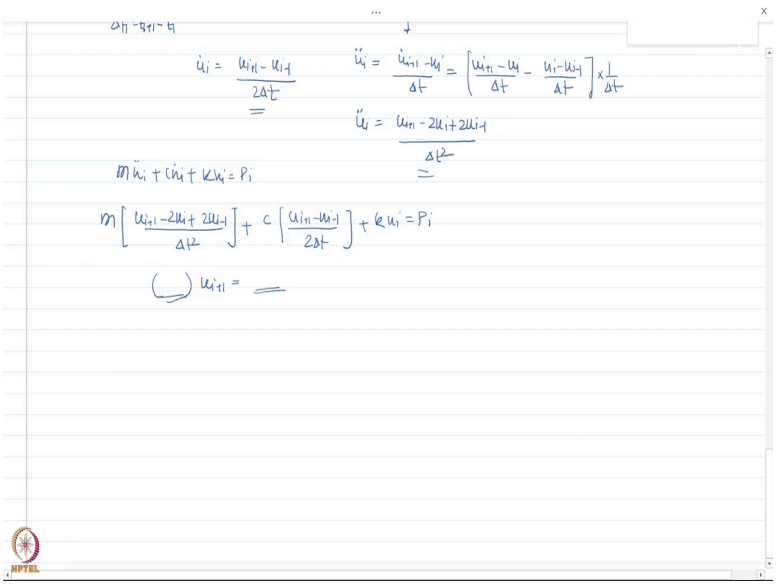
And the reason why I am considering like you know two steps a part, it would clear when I write the acceleration. Remember by default if I try to express my acceleration here as

$$\ddot{u}_i = \frac{\dot{u}_{i+1} - \dot{u}_i}{\Delta t}$$

So, this is the acceleration I am trying to find out which will be derivative of velocity. Let us say, it is the same expression that I have just used above.

$$\ddot{u}_i = \left(\frac{u_{i+1} - u_i}{\Delta t} - \frac{u_i - u_{i-1}}{\Delta t} \right) \times \frac{1}{\Delta t}$$

(Refer Slide Time: 35:18)



So, we can arrange it and we can write down our acceleration as

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2}$$

I have written down my velocity and acceleration which are time derivatives of the displacement. I have written down as simply as an expression of displacement.

Why is it necessary? Well, if I have that, then the differential equation that I have

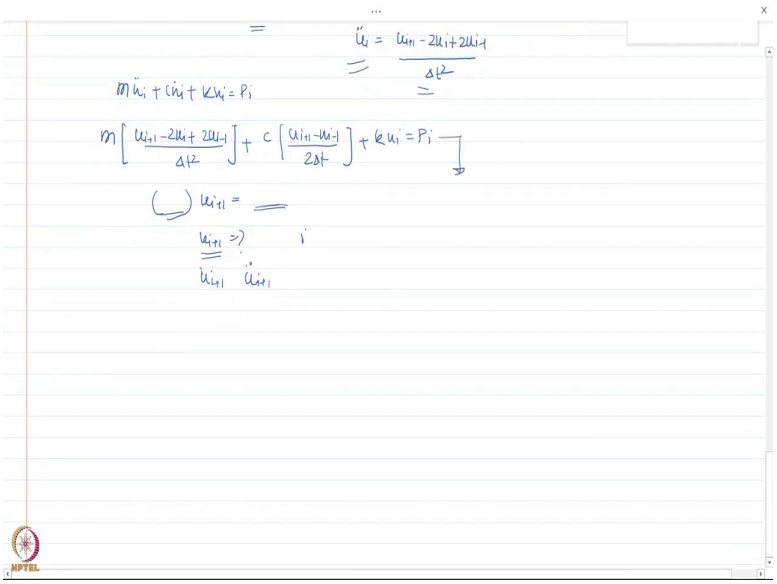
$$m\ddot{u}_i + c\dot{u}_i + +ku_i = p_i$$

$$m\left[\frac{u_{i+1} - 2u_i + 2u_{i-1}}{\Delta t^2}\right] + c\left[\frac{u_{i+1} - u_{i-1}}{2\Delta t}\right] + ku_i = p_i$$

all the derivative terms can be removed, if I substitute these and were written in terms of only displacements and that becomes very useful to us why we will see now.

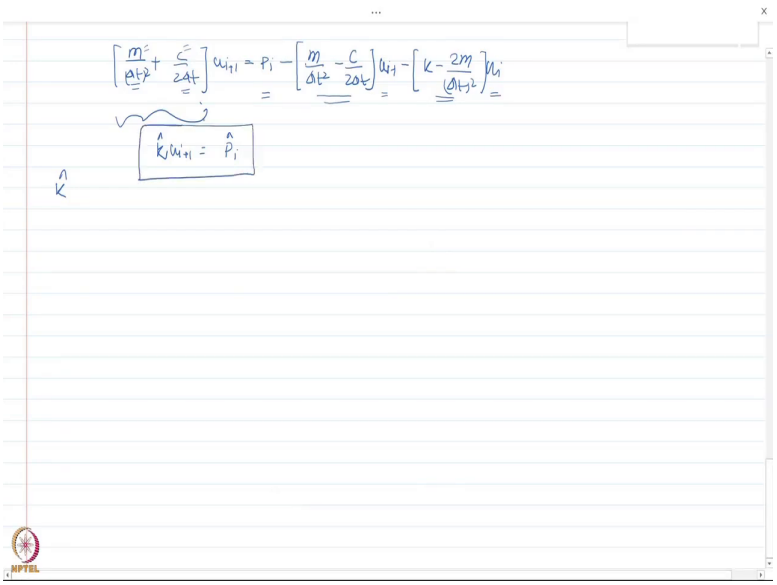
So, the way I want to arrange this equation. So, that I can write it as some constant times u_{i+1} is equal to some functions which are known on the right-hand side. So, that if this coefficient is known and the functions on the right-hand sides are known.

(Refer Slide Time: 36:58)



Then, I can find out this value. And which basically means given the state of the system at i, I can find out the state of system at i+1, because once u_{i+1} is known, then I can find out $\dot{u}_{i+1}$ and $\ddot{u}_{i+1}$ acceleration using the expressions that we have obtained here. So, basically that is our goal from this expression how to get that.

(Refer Slide Time: 37:33)



$$\left[\frac{m}{(\Delta t)^2}+\frac{c}{2\Delta t}\right]u_{i+1}=p_i-\left[\frac{m}{(\Delta t)^2}-\frac{c}{2\Delta t}\right]u_{i-1}-\left[k-\frac{2m}{(\Delta t)^2}\right]u_i$$

So, if you rearrange this equation ok that we have, we will get this.

Now, what I am going to do. I am going to write this as another expression which I will denote as

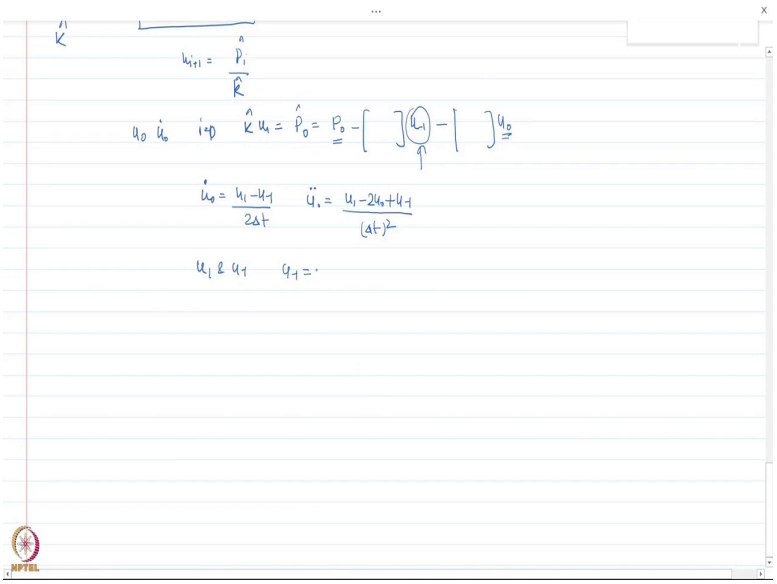
$$\hat{k}u_{i+1}=\hat{p}_i$$

$\hat{k}$ is basically some represents some sort of a stiffness or the equivalent stiffness and this is equal to $\hat{p}_i$. Now, if you look at these carefully; $\hat{k}$ depends on the mass and the damping on the system and whatever the time step that you choose and if the time step is constant, this k needs to be calculated only once not at each and every step.

So, if it is a linear system and the time step is constant, this k needs to be calculated only once. Similarly, on the right-hand side P_i is known to me from the previous step and u_{i-1} and u_i; they are also known to me from the previous steps and the other things are basically

system properties that can be determined using the provided values. So, I get my equation for the differential equation in this form.

(Refer Slide Time: 39:12)



So, that I can find out my as $u_{i+1} = \frac{\hat{p}_i}{\hat{k}}$. So, in this one as well given the state of the system at previous steps, I have been able to find out the response at the next step.

Now, the initial condition would again be the same it would be u(0) and $\dot{u}(0)$. Now, there is one issue here though. If I consider this equation if you look at it here at i=0.

So, that I am calculating this

$$\hat{k}u_0 = \hat{p}_0 = p_0 - []u_{i-1} - []u_0$$

Now, u(0) is known to me as an initial condition P(0) is known to me the force at time t=0. Question becomes how do I get this u_{i-1}

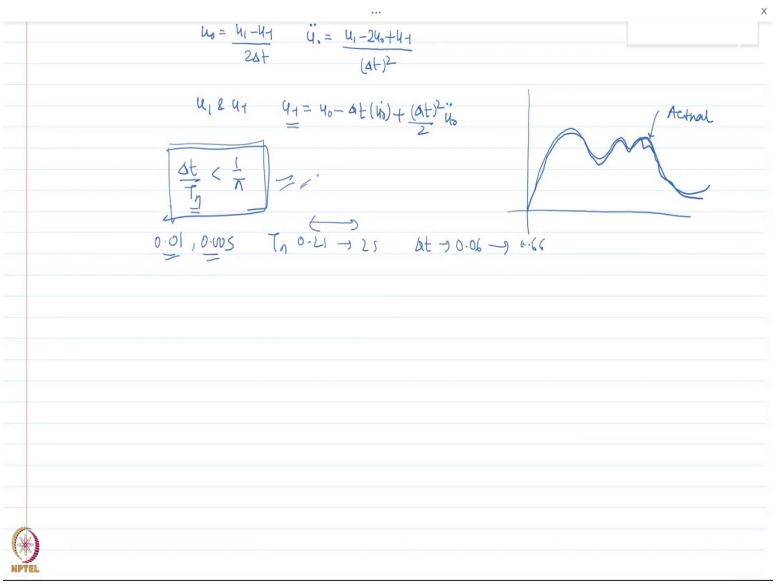
So, to get u_{i-1}, consider it to be another constant that need to be determined. And the way we find out using the initial condition when we where we write down

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t}$$

$$\ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$$

Now, all the terms are known in this 1 except ok u_1 and u_{-1}.

(Refer Slide Time: 41:43)



So, we can solve this and find out u_1 as ok, because there are two equations simultaneous equation. I can solve it and I can this get u_{-1} as

$$u_{-1} = u_0 - \Delta t(\dot{u}_0) + \frac{(\Delta t)^2}{2}\ddot{u}_0$$

And then my solution becomes complete in the sense that I can now, utilize this to start my iterations and get the response. Now, remember how accurate the response would.

So, let us say I have found out analytically that response looks like something like this. Now, whether my numerical response how close it would be to the actual response. It depends on

many parameters and one of those parameters are time steps that we use. So, it can go something like this it can follow like this it is always discretizing is it not always discretized.

So, how accurate it is or how close it is to actual solution; one of those parameters that determines is it is actually the time step and central difference method is stable, if it is satisfy this criterion. So, if you have a single degree of freedom system my

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}}$$

If this condition is satisfied then, my solution would be stable if I determine using the central difference method.

And you know this condition basically comes from you know some stress velocity criteria which is beyond the scope of this course, but basically some condition need to be satisfied for this method to be stable; which is typically not a problem for the like you know some general type of the structures that are there in structural engineering.

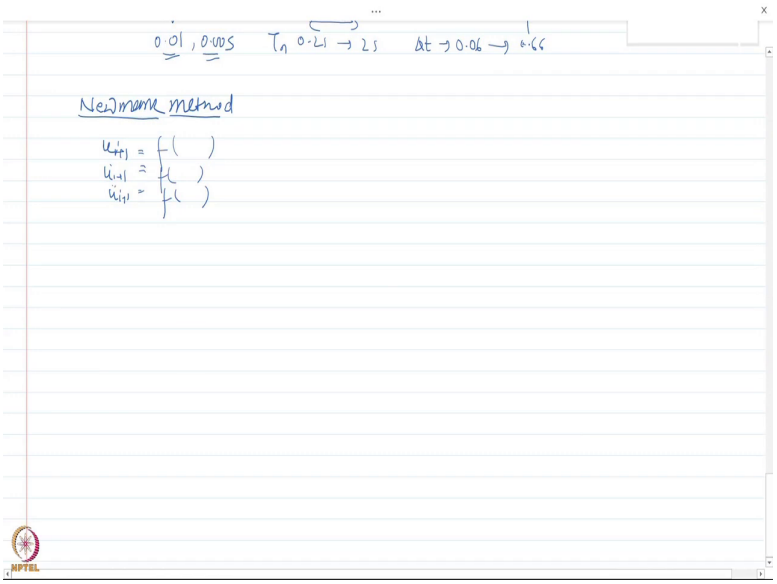
Because, we typically use time steps which are of order of 0.01 or 0.005 like that which would always satisfy this condition, because time period of a typical structure can be between 0.2 second to let us say 2 second or 3 second like that. All the time periods in this zone this condition is satisfied very easily, because it would require delta t to be between 0.06 ok to the value 2 by. So, it would be around 0.66 let us say ok and that is not a problem.

Because typically, the value that we use are satisfy, but remember the time period is for a particular mode and here we are only considering single mode, because it is a single degree of freedom system, but for multi degree of freedom system many modes could be of interest let us say tenth mode and as the mode increase and you will see in the later chapter, the time period actually decreases.

So, depending upon your mode of interest, this condition might not be satisfied. So, it might be problematic for multi degree of freedom of system. So, that need to be kept in mind alright. So, this was the discussion on central difference method which is perhaps one of the

easiest method that could be universally utilized for a linear and non-linear systems altogether. The next type of method.

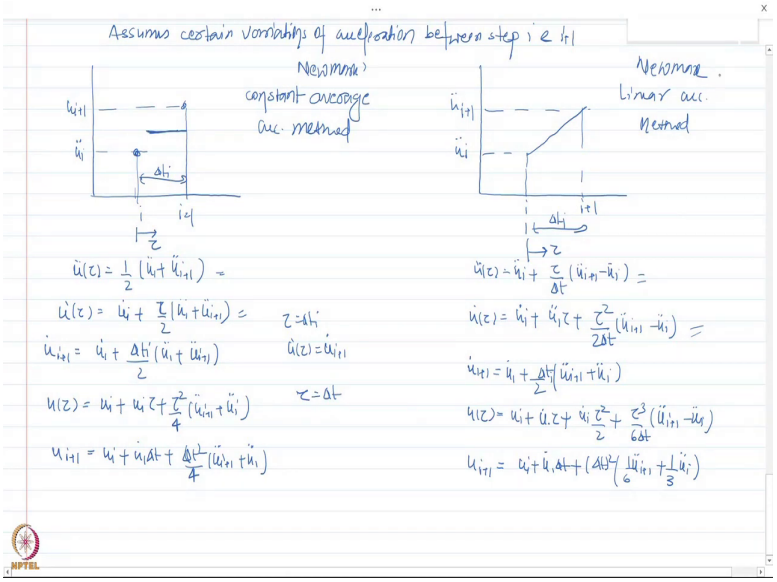
(Refer Slide Time: 45:17)



That we will discuss is called a new mark method. So, new mark method is another set of numerical procedure used to find out the response of a differential or response of basic dynamic system.

Now, new mark method; as we discussed in the central difference method as well. The idea is to find out variation of u(i) or the response parameter at i+1 in some sort of known functions. So, that is the goal. Once we can find that out, then I can go from step i to i+1.

(Refer Slide Time: 46:11)



Now, what new mark method does? It assumes certain variation of acceleration between step i and i+1 and depending upon those variations, I get the variation of velocity and displacement.

So, let us see what those variations are. So, in the first type of new mark method basically, if I go from let us say step i where the acceleration was $\ddot{u}_i$, I go to the next step i+1.

So, the first set of numerical is the first set of numerical procedure it is assumed that acceleration is constant between the step i and i+1. So, it is a constant value and the value of that acceleration actually is let us say, i represent in terms of variable τ . Remember, this one total is Δt_i .

And if I have to represent my acceleration for any point in between i and i+1, it would be simply a constant value average of these two values. So, I will write it as

$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_i + \ddot{u}_{i+1})$$

So, the acceleration is assumed to be constant. In the second set of in the second set of numerical procedure, what is assumed that again let me draw the step i and i+1 here basically a linear variation of acceleration is assumed. So, it is assumed that between the step i and i+1. My acceleration is not constant anymore it is varying linearly between the value $\ddot{u}_i$ to $\ddot{u}_{i+1}$.

So, in that case again let me say this is Δt_i and i am representing the time between ok i and i+1 is τ. So, i will write my acceleration

$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$$

which is basically linear interpolation between this these two values. So, the acceleration is varying linearly. So, this is called linear this is constant acceleration this is linear acceleration.

Once the variation is known, I can go ahead and I can integrate both these expressions here to get the velocity as a function of τ, is not it? So, I can if I integrate it, I will get it as initial velocity plus the constant average acceleration times τ.

$$\dot{u}(\tau) = \dot{u}_i + \frac{\tau}{2}(\ddot{u}_i + \ddot{u}_{i+1})$$

Here, If I integrated, I will get a velocity as

$$\dot{u}(\tau) = \ddot{u}_i + \ddot{u}_i\tau + \frac{\tau^2}{2\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$$

So, if I substitute it here, I can write that as

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t_i}{2}(\ddot{u}_i + \ddot{u}_{i+1})$$

Similarly, it can be done for the linear acceleration as well.

$$\dot{u}_{i+1} = \dot{u}_1 + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

And then this value here ok which is sort of same as what we have obtained here the left-hand side and the velocity have follow the similar distribution.

I can again go and then integrate these expressions for the velocity as a function of τ to get the displacement as a function of τ and I can go ahead and write this as

$$u_{i+1} = u_i + \dot{u}_1 \Delta t + \frac{\Delta t^2}{4}(\ddot{u}_{i+1} + \ddot{u}_1)$$

Similarly, for this I would write my displacement as

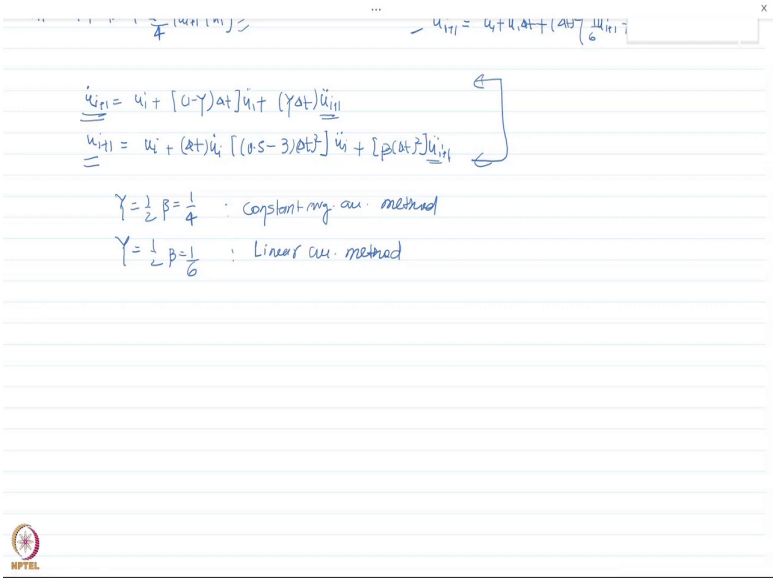
$$u_{i+1} = u_i + \dot{u}_1 \Delta t + \Delta t^2 (\frac{1}{6}\ddot{u}_{i+1} + \frac{1}{3}\ddot{u}_1)$$

So, what we have seen here, that given or assuming a distribution of acceleration between the step i and i+1, I am able to derive ok the expressions for velocity and displacement at step i+1 as a function of velocity and displacement and acceleration at previous steps.

Now, in this method. The first method is called constant average acceleration method The second is called linear acceleration method. So, this is called new marks constant average acceleration method and again, new mark linear acceleration method.

Now, the expressions that we have obtained for velocities here, and displacements here, can be written as a combined expression in terms of some constants. And depending upon values of those constants, I can either get this constant average method acceleration method or linear acceleration method.

(Refer Slide Time: 54:27)



So, if I write down my velocity as

$$\dot{u}_{i+1} = u_i + \left[(1-\gamma)\Delta t\right]\ddot{u}_1 + (\gamma\Delta t)\ddot{u}_{i+1}$$

Similarly, I would combine both this expression. Write down my displacement as

$$u_{i+1} = u_i + (\Delta t)\dot{u}_i\left[(0.5-3)\Delta t^2\right]\ddot{u}_i + \left[\beta\Delta t^2\right]\ddot{u}_{i+1}$$

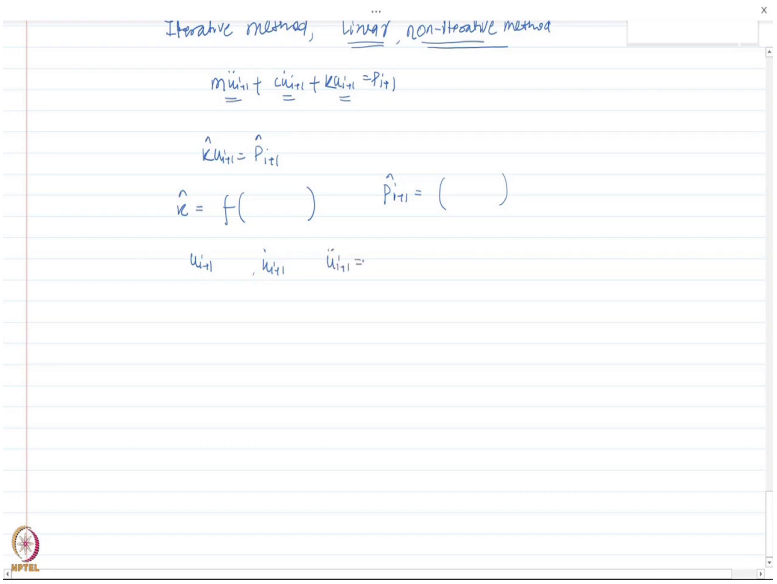
So, I have now written in terms of constant gamma and beta. And what happens when gamma is equal to half and beta is equal to 1 by 4, these expressions transformed to the expressions that we have obtained for constant average acceleration method average acceleration method.

And if I utilize gamma equal to 1, but beta equal to 1 by 6 then, I get the same expressions for velocity and displacement as linear acceleration method. So, in general, new marks method numerical procedures are written like this in which ok I we have assume or we assume certain variations of acceleration and then we get these expressions here.

Now, if you look at carefully here velocity and displacement at step i plus 1 depends on the response parameter at the previous step, but I also have these parameter here, which are

basically acceleration at the current step. So, remember unless I can write down the expression; such that, my current step can be calculated wholly based on the previous step.

(Refer Slide Time: 57:13)



I would need to use iterative method to solve new mark equation; however, if the system is linear for that special case there is non-iterative new mark method, then I do not have to iterate it, because I can solve the system for u_{i+1} and then substituted in the equation of motion.

So, only for linear system I can use non iterative new mark method. So, for linear system all I need to do is basically, write down my equation of motion, add the step i+1 and then substitute all the variations that we have obtained the expression for

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}$$

whatever the expressions that we have obtained here and when we substitute it and eliminate u_{i+1} double dot acceleration and the current step.

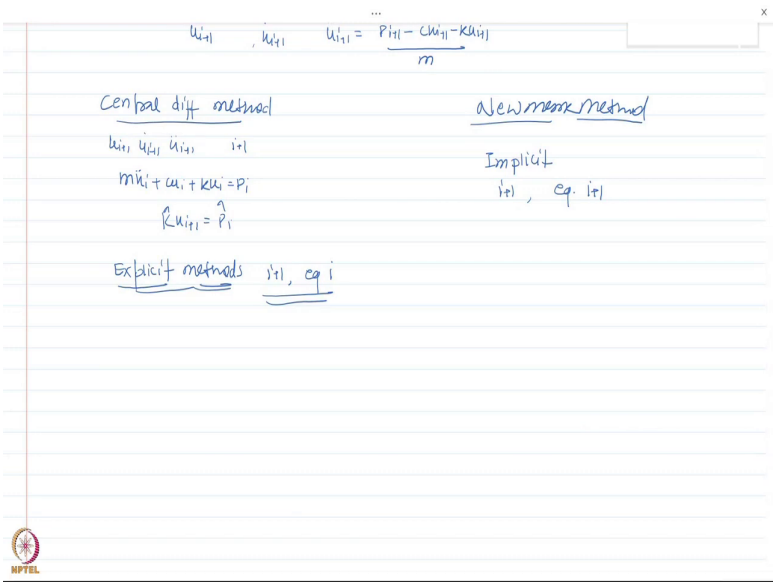
Then, I can write down my expression again in the similar form that I have obtained for that we had obtained for central difference method. So, for new mark method, I also get again expression like this

$$\hat{k}u_{i+1} = \hat{p}_{i+1}$$

where again k is some function of system parameters and P_{i+1} is again system of or like you know expressions of known parameters from the previous steps.

So, through this we can calculate u_{i+1}. If u_{i+1} is known, then $\dot{u}_{i+1}$ can also be found out and then, the acceleration can also be found out either using these expressions here.

(Refer Slide Time: 59:11)



Or we can just utilize the equation of motion and get and write this as

$$\dot{u}_{i+1} = \frac{p_{i+1} - \dot{c}u_{i+1} - ku_{i+1}}{m}$$

And then, we keep iterating ok going from step i to i +1 where i varies between 0 to n or the steps that we are of inter interested. Now, if you look at here, there is an important difference between the central difference method and new mark method.

If you remember, in the central difference method even to get the response parameters $u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1}$. We never utilized the equation of motion at i+1. The equation of motion that

we utilized was actually this $m\ddot{u}_i + c\dot{u}_i + ku_i = p_i$. we arrange this substituted the term and then, wrote as $\frac{\Delta t}{T_n} \leq 0.551$.

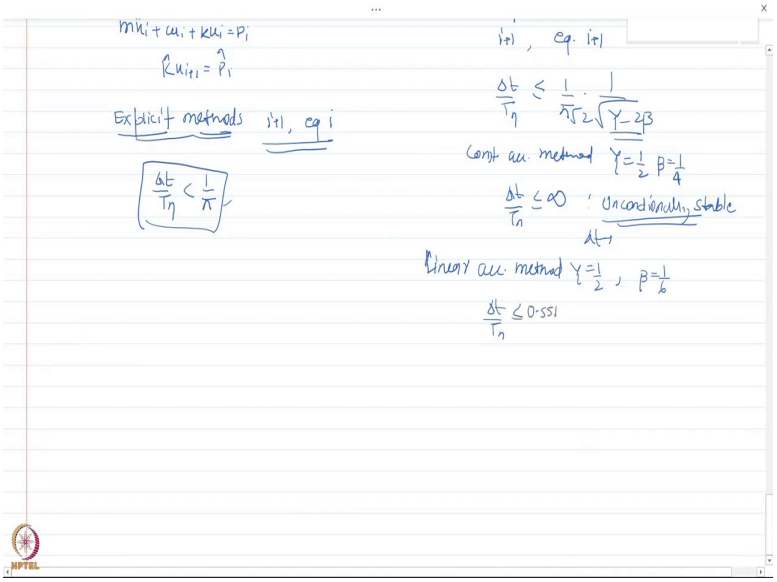
So, if you look at let us go back, where we discussed central difference method and let us see, what is the equation of motion we utilized is this step i or i+1. So, even to get the value of the response parameter at the next step, we never satisfy the equilibrium equation at the next step.

We got everything from the current step ok and the next step was just found out using the previous step. These kinds of methods where the state of the system is actually obtained wholly from the previous steps it is called explicit methods.

Because it is explicitly determined from the previous steps and the equilibrium at the current step is not utilized; however, in new mark method if you look at it, the equilibrium at the current step was utilized and that must be satisfied to get the response at i +1.

These kinds of methods are called implicit method, where response at step i+1 is determined using the equilibrium at i+1. In this case, response at i+1 is determined using the equilibrium at i. So, these are also popularly referred as explicit methods and implicit methods of numerical procedure to get the response.

(Refer Slide Time: 62:16)



Now, we saw that for a central difference method the criterion for the stability was

$$\frac{\Delta t}{T_n} < \frac{1}{\pi}$$

Now, for new mark method again, similar procedure can be followed that was followed to determine this which we did not discuss. So, that it can be discussed considering the stress propagation theory, but finally, we get this condition for the stability.

Now, new mark method is a stable, if

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}}$$

If this condition is satisfied, then my new mark method is a stable. Now, if you consider constant acceleration method. What was the value of gamma equal to 1 by 2 and beta 1 by 4 and if we substitute this, we get here as this term becomes 0. So, this becomes infinity. So, delta needs to be smaller than infinity which would always be the case. So, that is why the constant acceleration average acceleration method is unconditionally stable.

So, it is always stable ok does not matter what is the value of Δt ; however, it might only be accurate remember there is difference between stability and accuracy. So, it might still be stable, but it would only be accurate if Δt is very small.

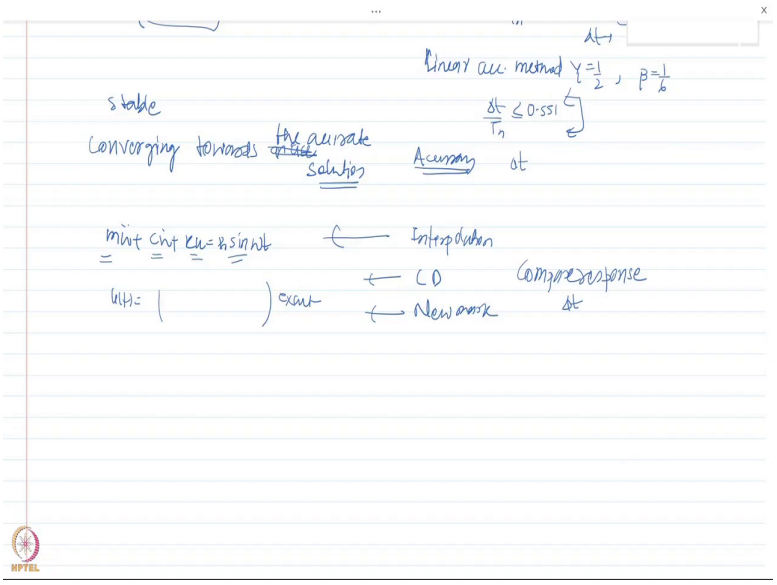
Now, for average for the linear acceleration method as opposed to constant average acceleration method. So, for linear acceleration method as well, we can substitute the value of gamma equal to half and beta is equal to 1 by 6. So,

$$\frac{\Delta t}{T_n} \leq 0.551$$

So, this must be satisfied for the solution or the response to be stable and you know for most of the systems actually, this is not an issue.

So, this is satisfied for the civil engineering structure for most of the cases anyway considering typical values of properties of the structures.

(Refer Slide Time: 64:38)



The issue here is accuracy. So, while they might still be stable, but for to be accurate, the Δt has to be sufficiently small. And usually what happens whenever we utilize these integrators,

we perform some analysis to see. We do some sensitivity analysis to find out if the Δt that I am using for the analysis is a small enough that it is stable and it is converging towards an accurate solution.

So, remember I used all three property that we discussed that it should be stable and it is it should be converging towards an accurate solution or the accurate solution. So, these are the three typical you know numerical procedures that are used and there are so much of additional things to learn for these types of methods you know stability, convergence.

You know all those things you should recognize that there are lot of material that can be studied for this topic; however, we would limit our discussion only to these three just to get an overall idea what you can do. You can go ahead and you can try out some of the excitations that we have already considered for example, harmonic excitation like this.

Consider some typical values of the properties and the excitation. For this, we already know analytically what is the response. So, we know the exact response from the solution of differential equation, but you can employ the interpolation method, excitation interpolation method or central difference method or new mark method and then compare the response ok compare the response for different values of the time steps and see how accurate they are alright.

So, with this we conclude our chapter here today.

Thank you very much.