Dynamics of Structures

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Forced Harmonic Vibrations
Lecture - 10
Frequency response curves

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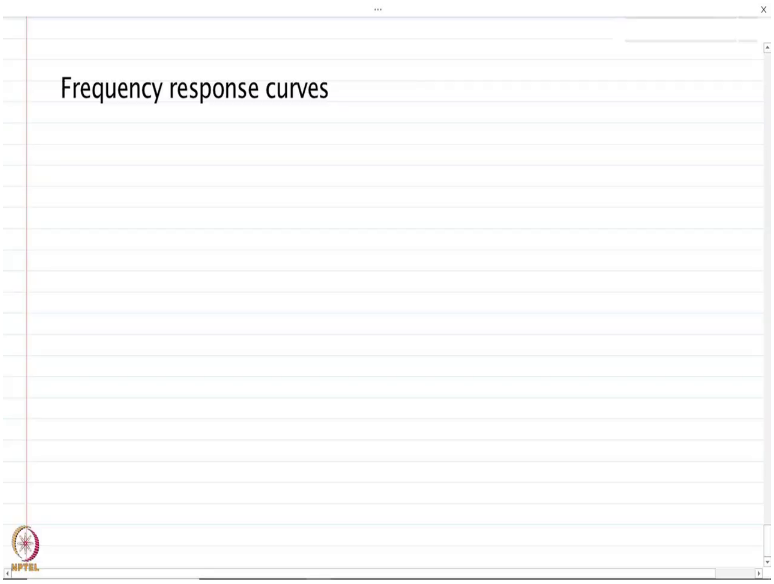

Hello everyone. In today's lecture we are going to learn about Frequency Response Factor. And this is specific to a single degree of freedom system subject to harmonic load, but the concept can be extended to any other loading as well, knowing that any loading to Fourier transformation can be transferred to a sum of systems with different frequencies.

So, one of the ways to measure the dynamic effect of a load is to look at the amplification with respect to static condition and that is what the frequency response factors are, to measure that dynamic effect. Till now, what we have studied is that how to find the solution to the harmonic excitation of an undamped and damped system.

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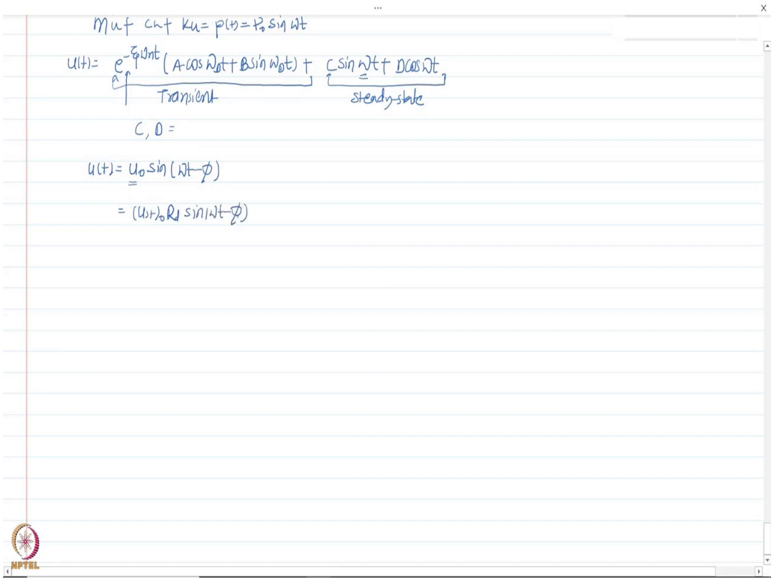

So, let us say, we have the equation of motion. We have the equation of motion for a damped system, which I can write as $m\ddot{u} + c\dot{u} + ku = P(t)$ and this is $P_0 \sin(\omega_n t)$. Now we saw that basically there are two part of the solution: the response for damped system and undamped system.

For undamped system, let us write down the solution for a damped system found out that this is the solution (refer slide). And then there was this part (refer slide) of the solution and we said that, well there are two type of vibration: the first one is basically the transient vibration which dies out over sufficient amount of time because of this damping term that we have here.

And second is the steady state solution and then we turned our focus on this steady state solution. So, in steady state solution, the system is oscillating at the excitation frequency ω and these constant *C* and *D* were basically functions of ω / ω_n, which is the frequency ratio, excitation frequency divided by the natural frequency of the system.

And basically we expressed our solution *u*(*t*) and this is specifically for a steady state solution, *u*(*t*) is equal to u_0, which is the dynamic amplitude here, $\sin(\omega_n t - \phi)$, where ∅ is the phase angle. So, the dynamic amplitude is written as,

$$u_{st} R_d \sin(\omega_n t - \phi)$$

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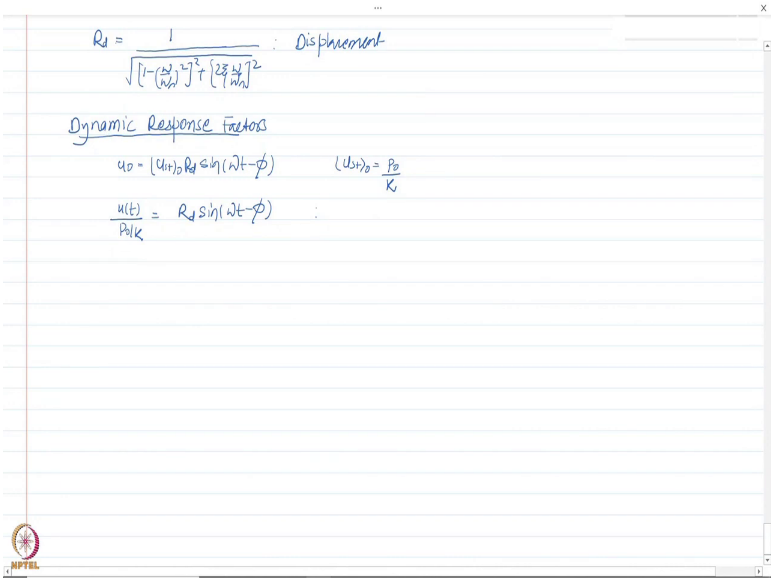

And R_d is nothing, but it is called displacement response or deformation response factor and it is given as this expression here (refer slide) which we can write as this,

$$R_d = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Now, till now basically we have only considered the displacement response factor. But as you could imagine, different kind of systems would have different response quantities of interest.

For example, right now we are only considering displacement here, but it might so happen that somebody would like to know what the velocity modification factor is and other would be the acceleration modification factor. So, what we are going to study now is dynamic response factors. So, these factors combined are called dynamic response factors and then we will obtain each of these one by one.

So, as I said, $u(t)$ I could write as $u_{st} R_d \sin(\omega_n t - \phi)$ and you know that the static amplitude is nothing but the applied the amplitude of the applied harmonic force divided by the stiffness of the system (P_0 / K). So, I can further write this as

$$\frac{u(t)}{P_0 / K} = R_d \sin(\omega_n t - \phi)$$

So, what we want to do now, assume I have the displacement history here expressed as a dimensionless quantity. So, $u(t)$ has the same dimension as P_0/K. So, I am basically normalizing it with respect to P_0/K. So, I want to obtain a velocity history. So, $\dot{u}(t)$ and the acceleration history, which would be $\dot{u}(t)$ divided by some factor and that function we need to obtain.

So, let us see how to do that. Now, we know that velocity is nothing but the differentiation of the displacement. So, what I am going to do here, I am going to differentiate this equation on the left- and right-hand side once and then see what do we get.

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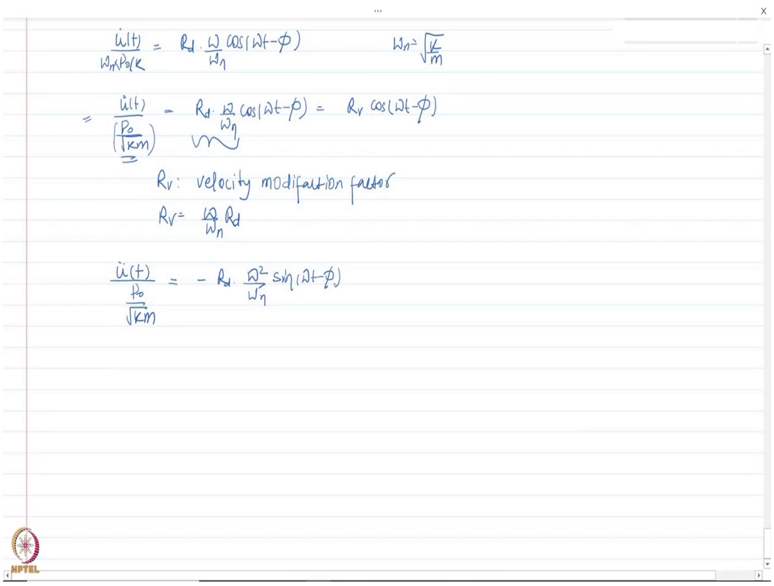

So, let us differentiate it with respect to time. So, that I get here ω and this becomes $\cos(\omega_n t-\phi)$. Now, what I can further do? Divide this by ω_n and on this side I also divide this by ω_n and I know that $\omega_n=\sqrt{k/m}$.

So, if I substitute that, what I will get here is this $P_0/\sqrt{km}$ and that should be equal to,

$$\frac{\dot{u}(t)}{P_0/\sqrt{Km}}=R_d\frac{\omega}{\omega_n}\cos(\omega_n t-\phi)$$

Now this expression here, I can further write as a new expression called R_v. This R_v is called velocity response factor or velocity modification factor.

As you see here R_v is given as nothing but frequency ratio times the displacement modification factor. So, we got our first modification factor after R_d, that is the velocity modification factor. Now, let us again differentiate this equation and see what we get now.

Just to mention, if you see here this is my velocity and if you look at in the denominator this, $P_0/\sqrt{km}$ this is the whole expression as units of velocity. That is why I tried to write it down like that.

So, I am normalizing my velocity with respect to this quantity here. Let us again differentiate this equation and then see what we get. So, I have this expression here, I would have

$$\frac{\ddot{u}(t)}{\omega_n\dfrac{P_0}{\sqrt{Km}}}=-R_d\left(\omega/\omega_n\right)^2\sin(\omega_n t-\phi)$$

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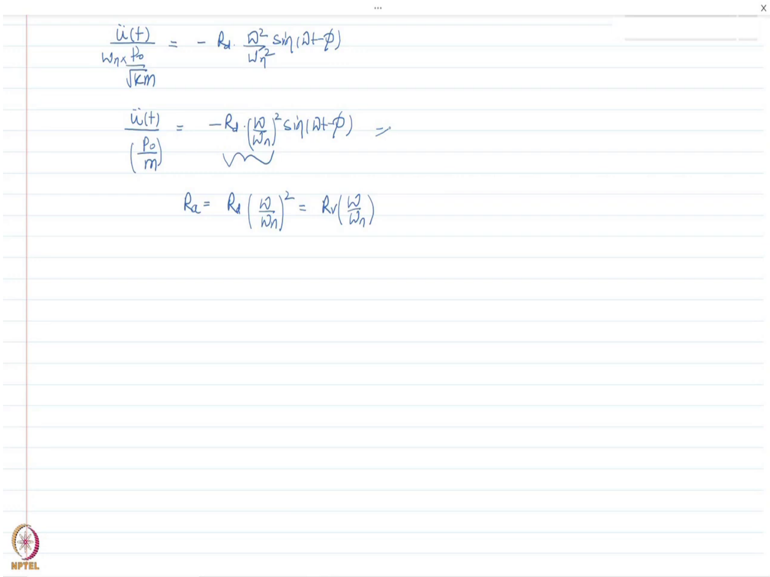

And if I further write down $\omega_n = \sqrt{k/m}$, what do I basically get as acceleration divided by P_0/m and this is equal to

$$\frac{\ddot{u}(t)}{\dfrac{P_0}{m}} = -R_d\left(\omega/\omega_n\right)^2 \sin(\omega_n t - \phi)$$

So, this quantity here and let us neglect the negative term because the amplitude of sin varies between positive one and negative ones, it does not matter anyway. What I am going to write here, my R_a = $R_d\left(\omega/\omega_n\right)^2$ and as you know this is also equal to if you write in terms of velocity this would be $R_v\left(\omega/\omega_n\right)$.

So, now you can see that this has units of acceleration. So, this is a normalized acceleration expression for normalized acceleration that we have obtained. So, as you can see, we have obtained the expressions for the displacement modification factor, velocity modification factor and acceleration modification factor and these quantities can be further written as this expression that you see here.

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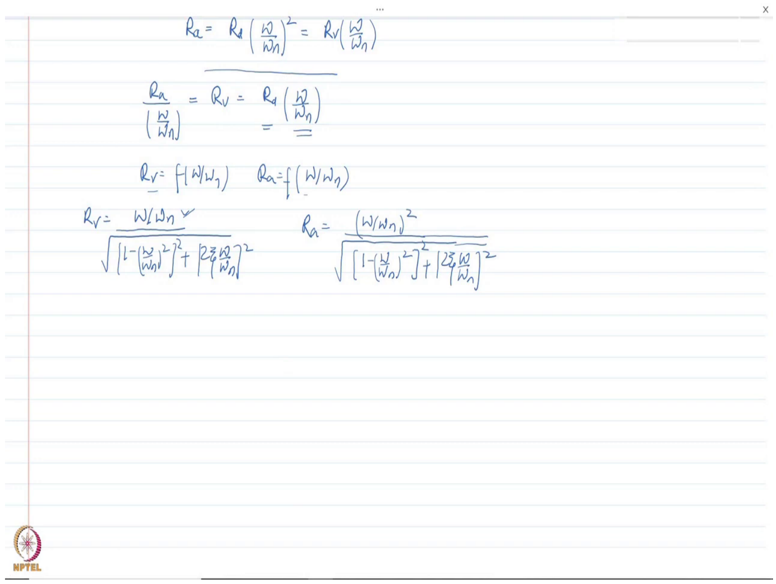

If I write it like this, this should be equal to R_v and this should be equal to R_d times ω_n . Now, we already know what is the variation of R_d with respect to ω / ω_n and we know that what happens when ω / ω_n is very small, when it is very high and when it is closed to 1.

Now based on that we can also obtain the variation of first R_v as a function of ω / ω_n and then R_a as a function of ω / ω_n . So, let us write down these functions and see what we get. So, I have R_v is equal to ω / ω_n divided by same ratio here, that expression that we have been using till now,

$$R_v = \frac{\omega / \omega_n}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

And your R_a is nothing but this quantity here,

$$R_a = \frac{(\omega/\omega_n)^2}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Now, as you can see here my displacement modification factor was 1, when $\omega/\omega_n = 0$; however, if you look at this expression now, I have a ω/ω_n term in the numerator. So, when ω/ω_n would be very small then the velocity would be equal to 0. So, let us see that.

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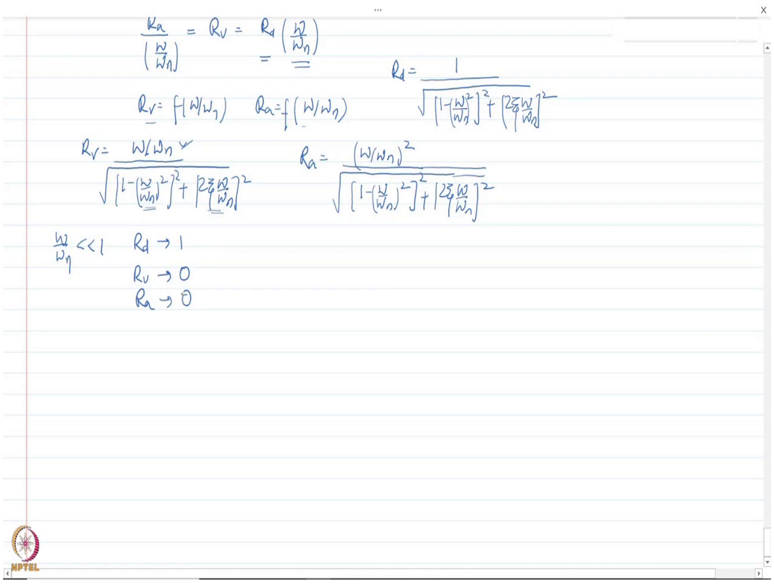

For the case when ω/ω_n is much smaller than 1 remember that R_d, which is basically what you see here. So, R_d as you can see even ω/ω_n is very small, R_d tends to be 1.

However, if you look at R_v and R_a, let us see what we get. So, R_v we get as remember that in denominator this term would become 0 plus now, I have numerator this term ω/ω_n. So, now, this would become 0. Similarly, R_a if you have any ω/ω_n term in the numerator it would again go to 0. So, this is one difference.

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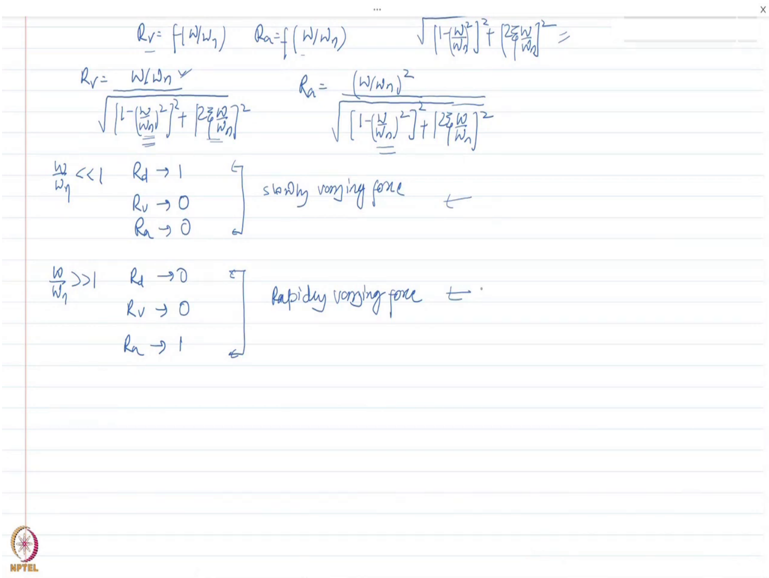

Second difference is when ω / ω_n is much greater than 1. So, it is a very large value, let us see what happens. R_d as we know, if it is a very large value, it goes to 0 that is directly that can be directly observed from here. So, it goes to 0.

However, R_v the velocity modification factor, if we look at it now I have terms in the numerator and denominator as well; however, for very large value of ω / ω_n if you look at the denominator I have a ω / ω_n equal to the power 2, but in the numerator it is power 1.

So, the denominator here, it actually increases at a higher rate than the numerator. So, that is why when the value become very large it actually goes to 0 as well. Now, compare that to third case for the acceleration modification factor, the highest power in the numerator is ω / ω_n to the power 2 and then the denominator also it would be 2 because this is the factor, square to the power is square which going to be contributing to the highest power of ω / ω_n.

So, this one tends to be 1. So, if you have system for which ω / ω_n is very large then R_a approaches to 1. So, these are the conditions for slowly varying force. Because my ω / ω_n is much smaller than 1 and these are the conclusions for rapidly varying force.

And you know we are doing just the here analytically, we could use any of the numerical tools to plot the variation of all these and then see how they look like for different values of damping. So, I have already done that let me just go ahead and copy that here and see how it looks.

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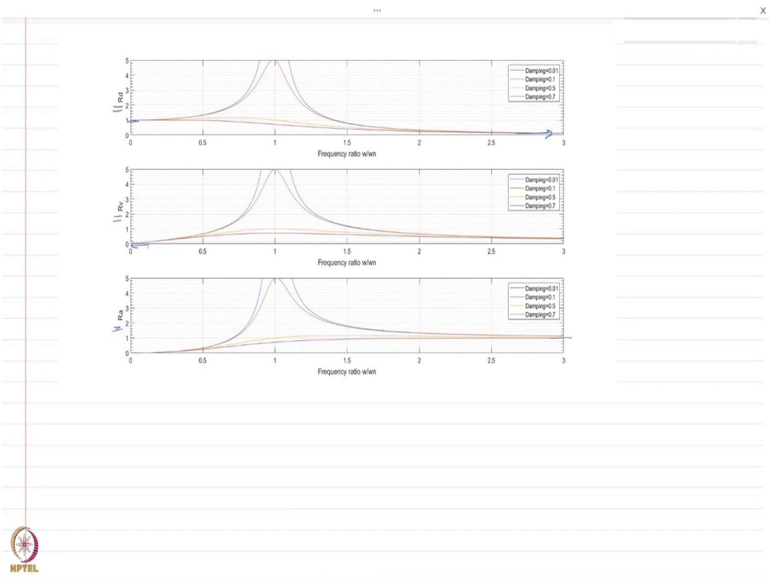

So, going to copy it here. So, if you look at it here, what do we see? We can see here this is my R_d, this is R_v and this is R_a and you can see the variation R_d starts with the value of 1 at a small value of ω / ω_n and then as the you increase the value it approaches to 0.

R_v starts with a value of 0 when ω / ω_n is equal to 0 and then again for a large value of ω / ω_n, it also approaches to 0, but at a slower rate, but at a slower rate than the displacement.

Compare that to the acceleration modification factor R_a here it starts with 0 and that it converges to a value of 1 for very large value of ω / ω_n and these you can think in terms of flexible system, rigid system, slowly varying force, rapidly varying force. So, you can think it in those aspects. So, once we have that figured out, let us see how we find out because now we have three modification factor R_d, R_v, R_a.

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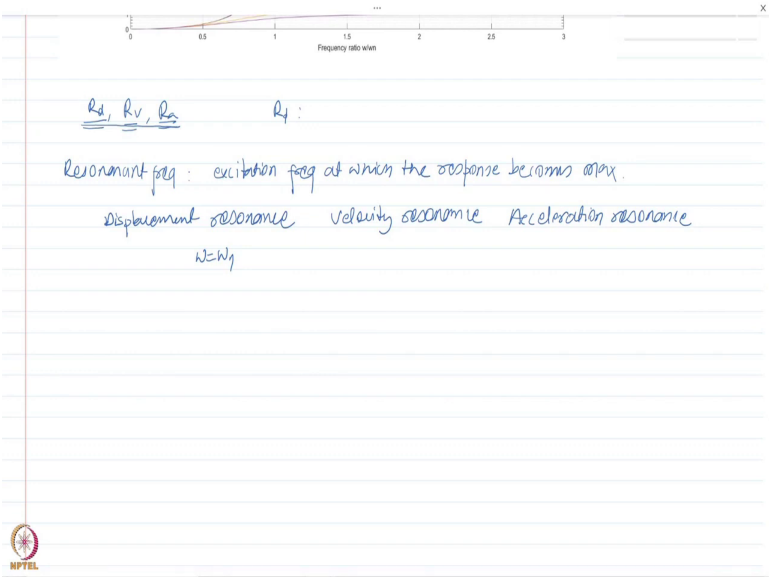

And remember we had used R_d to find out what would be the maximum response, you can use that. So, R_d is basically like an of a different value of frequency ratio what is the displacement. Now in this case we want to find out what would be the frequencies ratio at which these modification factors would have their peak values. Now remember the definition of resonance says or definite of definition of resonant frequency is the excitation frequency at which the response becomes maximum.

Now, one might ask well, what response as I talking about? I am talking about displacement or I am talking about velocity or I am talking about acceleration because for different like you know for different people different response quantities could be of interest. Acceleration might be interest to someone.

If they are studying for a let us say, if you are trying to study the vibration in a vehicle, then acceleration would be of interest. If you are trying to study about the displacement of the shock absorber or absorbing system, then a displacement could be of interest or velocity could be of interest.

So, we need to know, if we need to optimize something that, if we are talking about resonance, then it is with respect to what quantity. So, what we are going to do here, we are going to define three type of resonance: one is displacement resonance, then velocity resonance and then the third one is acceleration resonance.

Now one might ask that, why would they not be at the same frequency? Like you might till now under impression that when $\omega = \omega_n$, you usually get the resonance like situation. However, in reality that is not always the case, especially for system with high damping and I will show you with an example.

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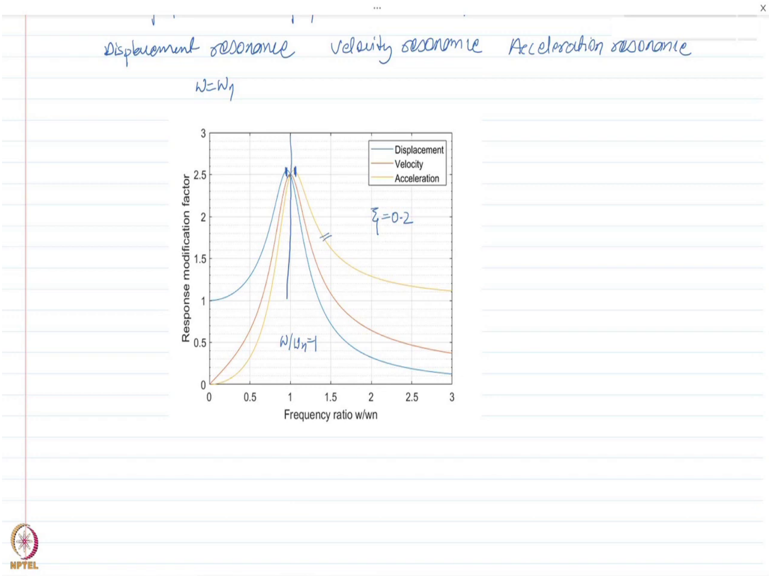

So, what I have drawn here on the same plot, in the same figure I have drawn 3 plots for the response modification factor of displacement, velocity and acceleration. So, using the expression for each of this I have drawn.

Now, if you look at here these quantities and this has been a drawn for ξ of 0.2 or 20 percent. So, the first one is the displacement modification factor, and this is the value of $\omega / \omega_n = 1$. So, let us draw this line here, you can see the maximum for the displacement does not actually occur at the value of $\omega = \omega_n$.

Similarly, if you look at the acceleration here, the resonance does not actually occur at omega by $\omega / \omega_n = 1$, only velocity if you see that occurs at ω / ω_n. So, let us see how we find that out, how do we find out the exact resonant frequency at which these or the response becomes maximum the displacement response, that velocity response and acceleration response.

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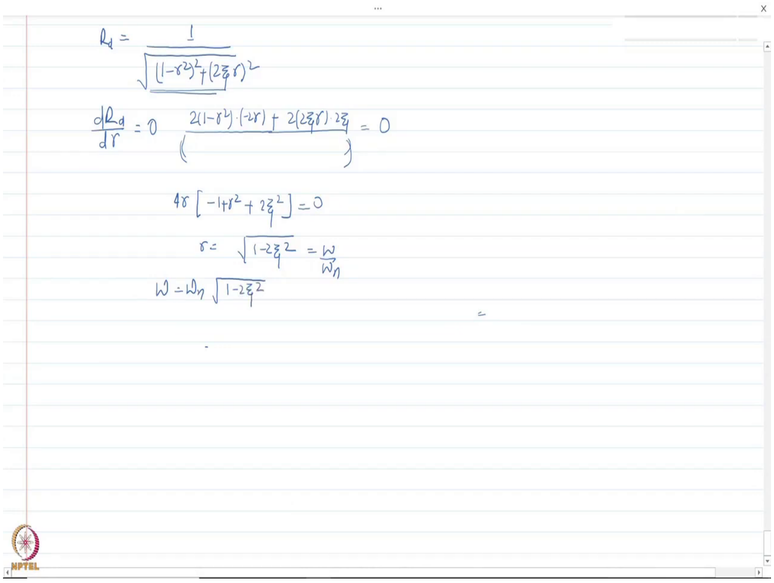

So, the expression for each of these were, let me first assume that I am writing $\omega / \omega_n = r$, some parameter. So, I can write my R_d as

$$R_d = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

So, as you know the maximum of any function can be obtained by differentiating. So, the maximum of any function can be obtained for example, this function R_d can be obtained by differentiating it with respect to the parameter here with respect to r and then setting it to 0. If we do that if you differentiate it you would be able to find out the value of r at which the expression R_d attains it maximum value the peak value.

So, let us do that; in the denominator there would be some terms, terms containing this expression here, we can put it here and in the numerator again we will have the same term, but then we will have a differentiation of the term that is inside and that is exactly what we are going to set equal to 0. So, it would be

$$\frac{2\left(1-r^2\right)(-2r)+2\left(2\xi r\right)\left(2\xi\right)}{\left(1-r^2\right)^2+\left(2\xi r\right)^2}=0$$

So, what do we get as? Let us take 4r outside. So, we get as $\left(-1+r^2+2\xi^2\right)=0$. The value of r that I get from here would be $\sqrt{1-2\xi^2}$. So, this is equal to the ω/ω_n.

So, the excitation frequency at which this is maximum, is nothing but

$$\omega=\omega_n\sqrt{1-2\xi^2}$$

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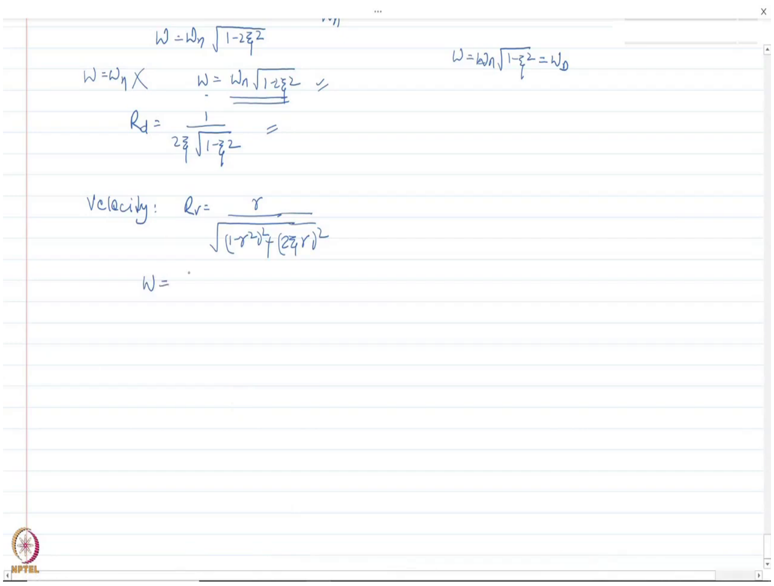

So, for displacement resonance, the applied frequency is actually not equal to $\omega=\omega_n$, but it is $\omega=\omega_n\sqrt{1-2\xi^2}$. And you know one might think that because it is a damped system, the resonant frequency should be $\omega_n\sqrt{1-\xi^2}$ because this is equal to ω_d.

However, as per this expression, it does not happen; it is that the resonance actually happens at this frequency. And once you substitute it back to the expression, you can find out the maximum value of R_d as well.

And you can do that calculation, I am just going to write down the final expression for R_d, which is,

$$R_d = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

So, this is the frequency excitation for resonance of the displacement response quantity, and this is the maximum value of the displacement response in terms of the response modification factor.

So, similarly we can repeat this procedure for velocity modification factor and as well as acceleration modification factor and we will see that for resonance. So, let us now consider velocity, where R_v is nothing but,

$$R_v = \frac{r}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

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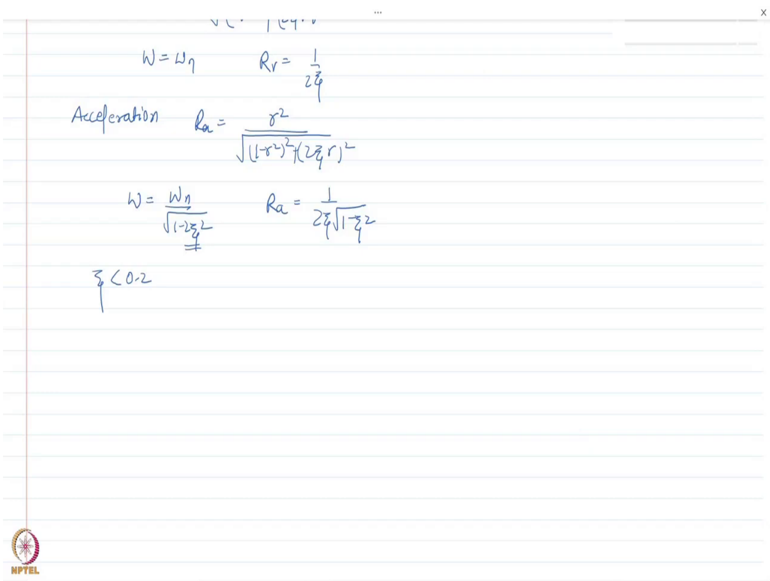

And if you do that, you will get excitation frequency, at the velocity becomes maximum is actually $\omega = \omega_n$ and the R_v or the maximum value of the velocity response in terms of R_v is actually 1/2ξ. Similarly, for acceleration, if $R_a = r^2 / \sqrt{(1-r^2)^2 + (2\xi r)^2}$; so, again follow the same procedure by differentiating it with respect to r and setting it equal to 0.

So, that would give you ω, the excitation frequency at which resonance or the acceleration resonance happens is actually equal to $\omega_n / \sqrt{1-2\xi^2}$. And the maximum value of the acceleration in terms of R_a = $1/2\xi\sqrt{1-\xi^2}$, this expression right here which is nothing but same as the expression we had obtained for R_d.

So, what we have derived in terms of different response quantities, what are the conditions for the resonance. So, what are the excitation frequency at which resonance happens and at that resonance frequency what is the amplitude the displacement amplitude, the velocity amplitude and the acceleration amplitude.

Now if you look carefully for a small value of ξ, if ξ is let us say smaller than 20 percent, which is the case actually for most of the structural engineering systems, you would see that all of these excitation frequencies are actually almost equal, because this term the ξ^2 term is like you know 0.04. So, when you take the square root it becomes further smaller.

So, you do not see much difference in terms of acceleration response factor or velocity response factor or displacement response factor and neither their resonant frequency, they are also approximately equal.

Let us move on to next topic, what we are going to do, we are going to utilize certain property of this response modification factor to come up with the method to obtain the damping from experiments. This method is called half power method or half power bandwidth method.

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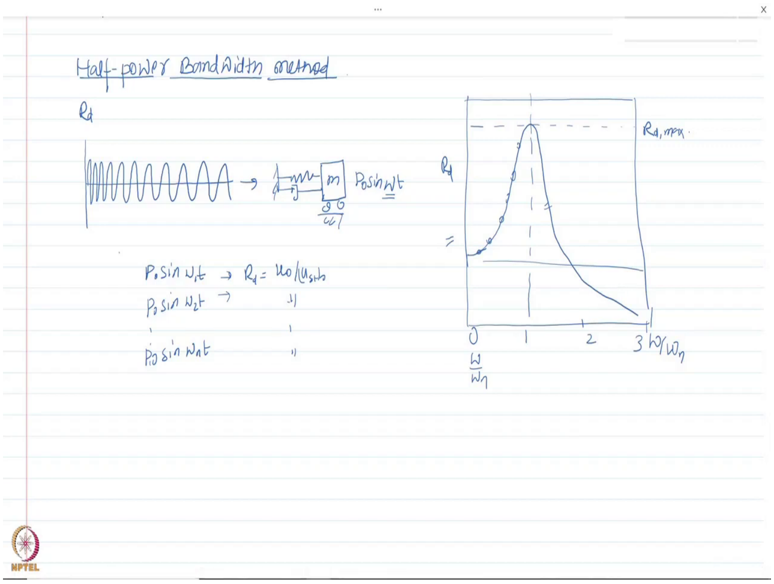

So, let me write it here, half power bandwidth method and let us see what happens now. We know that R_d, the displacement modification factor. This displacement modification factor, this curve if I draw it here for certain value of damping, you know that it looks something like this. So, this is R_d here, this is ω / ω_n equal to 1 or let us say write it here again.

This is the frequency ratio on the horizontal axis, this is 0, this is 1, this is 2 and so on, this is 3 let us say. Now, this is the value or the maximum value of $R_{d\ max}$. Now what happens in experiments, many times we apply what we call a sine sweep. So, what a sine sweep is actually? Sine sweep is it is a sinusoidal function with varying frequency. So, it starts with some frequency and then it the frequency actually decreases or increases. So, sweep means that it sweeps through all the range of frequencies.

Now, what that does? If you have a system, if this excitation is applied to a spring mass system, what will happen. This system now it is not a constant frequency, remember we had this $P_0 \sin(\omega_n t)$. Now this ω is excitation frequency which is varying.

So, when I apply the sine sweep through experiments to a spring mass system or any type of a dynamic system, then the response or R_d, when I calculate it, actually varies with the frequency. So, you can do the experiment and you can obtain a curve like this.

The other way of doing this would be type of experiments, for example, if your machine that is applying the excitation does not have a capability to apply sine sweep then basically what you do, you obtain this curve displacement modification factor, what do you do actually? You first apply $P_0 \sin(\omega_1 t)$, then $P_0 \sin(\omega_2 t)$ and then so on, $P_0 \sin(\omega_1 t)$.

So, that basically you apply this at all frequencies and then for each frequency you basically try to obtain R_d, which is the maximum displacement response and that you can measure from the experiment, what is the maximum deformation in the spring or the system and just divided by this u_0 for all of them. So, you can do for all of them and then you can again obtain points on this and then you can plot this function R_d versus ω / ω_n .

Once have this plot, you know that it would look something like, this at some value of excitation frequency it would achieve its maximum. Now, this response modification factor has a unique property.

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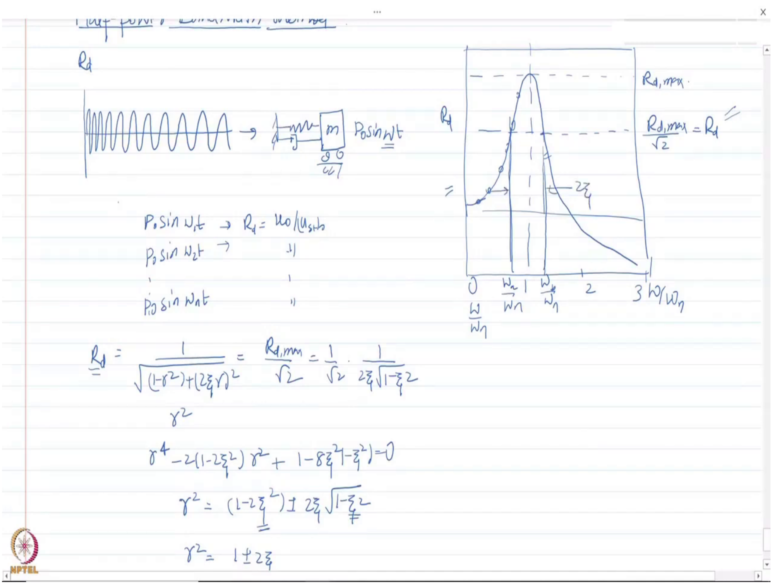

If you consider an amplitude where R_d is actually $R_{d\,max}/2$ and let us say draw horizontal line.

So, you already know $R_{d\,max}$ from experiment, you divide by $\sqrt{2}$, it will cut your response modification factor curve at 2 points.

Let us say, one is corresponding to frequency ω_a and second is ω_b , these are the two excitation frequency and of course, this is divided by ω_n . So, let us see what happens. So, what I am exactly, now let me express this mathematically, what I am saying that R_d is $R_{d\,max}/\sqrt{2}$. So, let us say, this R_d gives me some frequencies.

Let me write here, this is as $1/\sqrt{\left(1-r^2\right)^2+\left(2\xi r\right)^2}$ and this would give you $R_{d\,max}/\sqrt{2}$ and what is the value of $R_{d\,max}$. You have previously obtained it as the value of $R_{d\,max}$ as

$$R_{d\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

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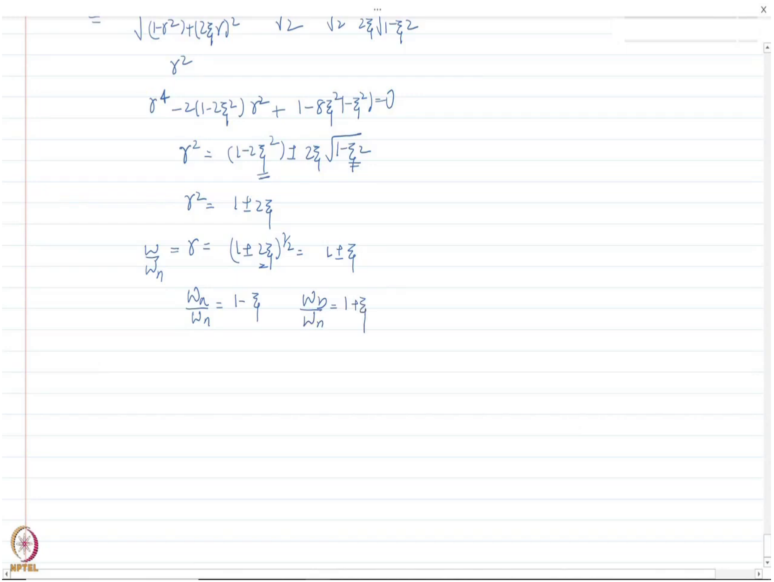

So, basically when you solve this, you will get a quadratic equation in r^2, it would have some r^4 terms, then r^2 terms and then some constant equal to 0. So, that would be basically a quadratic equation in r^2. So, let me write that out, if I expand this, I would get basically r^4 and you can do this calculation yourself and check if you are getting the same expression or not, r^2.

This is the expression that you will get and when you solve this equation, what you will get as r as equal to, remember r, I have considered as or let us say here we will get $r^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1-\xi^2}$.

Now in this expression you can neglect the ξ^2 term here with respect to 1 and same inside this root, so that you get r^2 as $1 \pm 2\xi$. So, r would be equal to $\sqrt{1 \pm 2\xi}$ and through power expansion, we know that, if the second term here is very small, I can write this as $1 \pm \xi$.

So, I have now 2 roots, which this is nothing, but ω / ω_n . So, this gives me 2 roots, ω_a and ω_b . The smaller value, let us say is ω_n which is 1–ξ, the higher value is 1+ξ and you can subtract from the second term to the first term, to get $\omega_n = 2\xi$.

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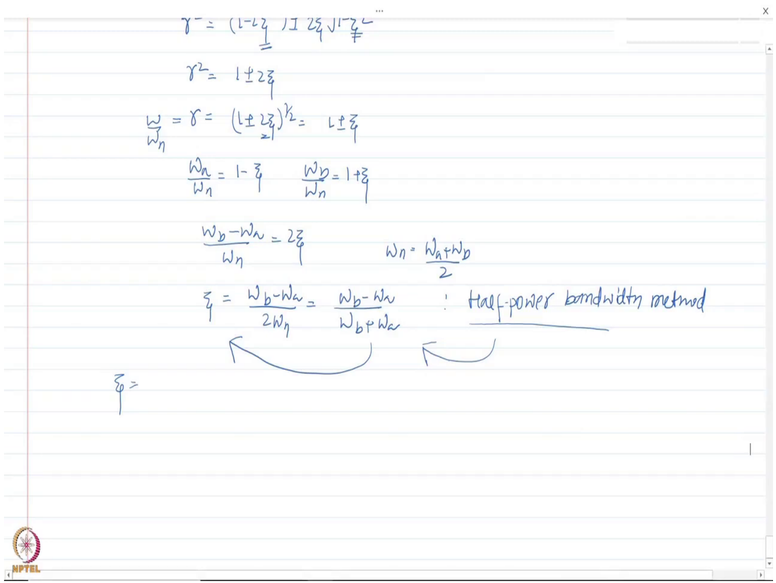

So, basically once you draw horizontal line and it cuts at two frequencies, which are basically the roots of those, this width here is nothing but 2ξ here you have just proved that here. So, this is here 2ξ and you can read this from your graph.

So, you know ω_b and ω_a, ξ you can easily calculate and ω_n you can also find out right here. For a small value of ω_n it would be simply the value at which it is maximum or many times what you might also do, you can approximate $\omega_n = (\omega_a + \omega_b)/2$.

We assume that this frequency is symmetrically located. So, you can write this as ξ calculate this as $(\omega_b - \omega_a)/2\omega_n$ and if you want you can further write this as $(\omega_b - \omega_a)/(\omega_b + \omega_a)$. So, once you cut this curve through a horizontal line at $R_{d\,max}$ / 2, it will give you two values of frequencies and utilize those two values of frequency to get the damping in the system.

So, this is called half power bandwidth method, which is utilized to get the basically damping in the system for some harmonic excitation. So, remember that, there was another method that we had done in the damped free vibration in which we had used logarithmic decrement method and we had utilized the logarithmic decrement method to get the damping in the system.

Here, another method is there which we use in utilizing the harmonic excitation of single degree of freedom system to obtain the damping in the system using this method.

So, these two methods can be conveniently utilized to experimentally obtain the damping in the system. So, I hope this method of obtaining damping is clear to you. So, we are going to conclude our lecture.