Heat Transfer
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Lecture - 04
Relevant Boundary Conditions in Conduction

So, in the last class we have derived the heat diffusion equation, which is the fundamental equation of conduct heat transfer and where, we have considered the flow of heat, net flow of heat by conduction in the x direction in the y and in the z directions. And then we invoked the conservation of energy, which tells us that for a control volume the rate net rate of energy in the rate of energy in minus the rate of energy out plus any heat, that can be generated that is generated inside the system the algebraic sum of these three terms should result in a change in the energy stored of the system.

Or in other words it is expressed as e dot g e dot in that is rate of energy in, minus e dot out plus e dot g is equal to is equal to the rate of energy stored in the system and we time rate of energy stored in the system. And we know that the energy of a system can be expressed as rho the density cp, that is the heat capacity times the temperature minus the reference temperature. So, when we express that for a control volume and from this equation, which is a difference equation we have obtained the governing equation for conduction in Cartesian coordinate systems.

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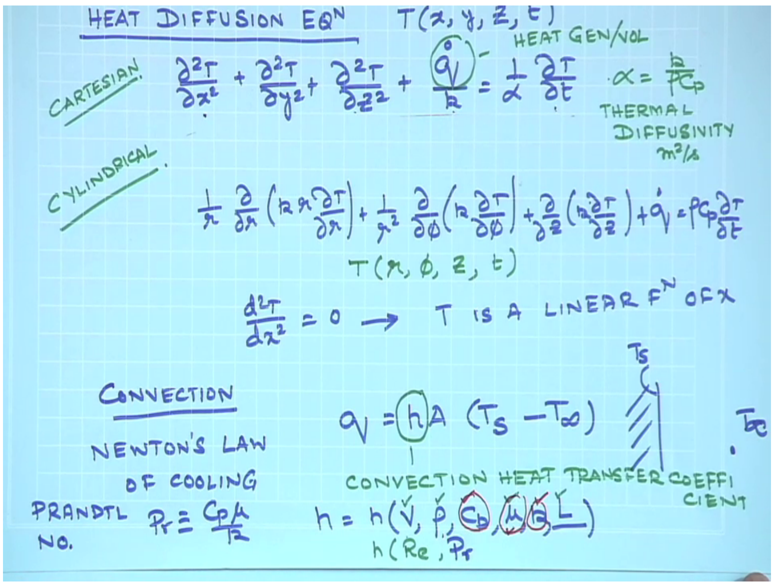

And this is the equation which we have derived last class therefore, Cartesian coordinates systems the temperature change now.

So, in all these cases T could be a function of x, y, z and time. So, the delta T by del x square and for y for z and this q dot is the amount of heat generated per unit volume. k is the thermal conductivity and this alpha as we have as we have seen it is units of meter square per second, it is defined as k by rho C p and its known as the thermal diffusivity.

So, this is the equation that one has to use one this is a starting point, for any conduction analysis for a system, where the temperature could be function of location, where there could be some amount of heat generation and as a result of all these, the temperature can also vary with time at fixed x y and z.

So, this is the fundamental equation, which is also known as the heat diffusion equation. Similarly the similar type of equation can be derived for cylindrical systems as well as for spherical systems. I did not write the spherical systems which fundamentally conceptually there is no difference between these 2 except for the coordinate system. So, here you see that T is essentially function of r phi and axial location z and it could also be a function of time. So, the same way the equation for the spherical coordinate systems can also be written, it is there in your textbooks. So, am not writing it over here once again and we have all discussed that how this equation.

Let say this equation can be simplified when we have for different conditions. For example, let say we have a steady state system. In a steady state system the temperature does not vary with time ok. So, the right hand side would be 0 and let us also assume that we have a situation in which there is no heat generation in the system; so, this term would also be 0.

So, therefore, temperature is a function of x y and z only. Under certain under some conditions it can also happen the temperature is a function only of x and not of y or a z. So, if we think about steady state condition with no heat generation and temperature being a function of only one special coordinates, then your this equation would simply this equation would then can simply be written as d 2 T by dx square equals 0. I do not need to use the partial sign anymore, since t is function of time in which would T is linear function of x.

So, this we have we have discussed and we also understand that this governing equation needs to be can be solved with certain boundary conditions. So, in today's class we will see what those boundary conditions are, which could be used to which can which it may be used to solve for to obtain the temperature profile starting at the governing equation. So, now, we are in a position to know what is the governing equation, what kind of simplifications I can make to that governing equation depending on whether it is a steady state or an unsteady state, temperature is a function of x or temperature is a function of both x and y and so on.

So, for all these cases if you look at the equation once again, you t that you would see that you would require 2 conditions on x, 2 conditions on y, 2 on z and one initial condition on time to solve for the temperature profile. So, in the simplest possible situation of one dimensional steady state conduction, where the governing equation is d 2 t d x square equals 0. So, you require 2 conditions on temperature. When you have its a its an un steady state situation; however, the temperature is still a function only of one special coordinates, then again if you look at the equation you would see only the first term on the left hand side and the right hand side will remain in the equation.

So, therefore, your governing equation would be del 2 T del x square equal to 1 by alpha del temperature by del time. So, this governing equation would then require 2 conditions on x since its del 2 t x square on the left hand side, and it would require one condition on time that is a initial condition when you look at the right hand side. So, for that specific case you need one initial condition at 2 boundary conditions. So, its important to understand important to identify what could those boundary conditions be. And it can the boundary conditions are can also be coupled with some other heat transfer process which is taking place in at the boundary.

For example you have you have a let us say cuboid, which is at temperature high temperature 100 degree centigrade, and which is exposed to a room where the temperature is maintained at 25. So, inside the solid inside the solid cuboid the heat transfer mode is conduction and if we neglect radiation, since the radiation is going to be important only when the temperature is sufficiently high.

So, at the boundary of this cuboid what you get is convection. So, any the cuboid is going to lose heat to the ambient air by a process which is which all of us would know as convection. So, let us say as a small breeze is blowing over the cuboid making it cooler, and in that condition at the interphase between the solid and air and the moving air what you have is convection.

So, inside the solid its conduction once you reach the interphase between the solid and the air what you have is convection. So, in some of the some of the heat transfer cases conditions, you will have the simultaneous present of convection and conduction at the interphase. So, whenever we talk about convection or whenever we discussed about some other modes of heat transfer, we need to identify what would be the governing equation for that mode of heat transfer. Because we understand that in conduction the heat the flow of heat is governed Fourier's law of conduction. So, what governs the convective process? The convection is much more complicated than conduction because you in convection, you are also going to have flow of the fluid over the hot surface.

So, you need to know; what is the velocity profile, what kind of velocity you have on the solid plate, what is the combined momentum and heat transfer process that is taking place close to the interphase. So, you not only have to have an idea of the fluid mechanics the momentum transfer part of it, you would also have to have something which is similar to Navier-stokes equation and applicable for energy transfer, which we would cover later on when we study thorough when we have a more thorough study of convection. But it is suffice to say that the process of convection is complicated because of the because of the presence of fluid motion and the associated momentum and heat transfer process with it.

So, in convection it is customary to define a convective heat transfer coefficient and it has been found experimentally that the heat loss from a heated surface exposed to a moving stream of fluid of different temperature, the amount of heat loss is proportional to temperature difference. This temperature difference is temperature difference between the solid and the fluid which is moving over it. So, the q of convection the q the heat loss or gain depending on whether the solid is at a higher temperature or the liquid is at a higher temperature, this loss or gain of heat in convection is proportional to temperature difference.

So, mark the difference with the conduction heat transfer; in conduction heat transfer its the thermal gradient, which is deciding how much of heat is getting transferred between the between 2 points. Whereas, in convection the heat flow the heat rate is proportional to temperature difference not the gradient and this law which is essentially obtained by looking at experimental results of convection over a wide variety of substrates and fluids at different temperatures, where the fluid is flowing with different velocities on the surface and so on this is known as the Newton's law of cooling.

So, Newton's law of cooling is a fundamental relation of convective heat transfer, which again cannot be derived which is the which is the result of the result of a large number of experimental observations in a generalized form. So, the Newton's or convection we are going to use Newton's law of cooling and this Newton's law of cooling simply tells us that, the heat loss or gain by convection is proportional to temperature difference T S minus T infinity where T S is the temperature of the substrate the solid and T infinity is the fluid which is in which is with which the solid is in contact.

So, if this is the solid and if the temperature on the solid side of the interphase is T S and let us say the temperature of the fluid is T infinity at a distance far from it, then this q is equal to h times A, where A is the area which is in contact with the fluid and T S minus T infinity. This H is known as the convection heat transfer coefficient. So, this H is the conductive heat transfer coefficient convection heat transfer coefficient and most of the studies of convection heat convection is to find what would be the expression of this convective heat transfer. Now let us think what is this H is going to be a function of. It is our common observations that if you want to cool a surface cool a solid faster you have to blow air at a higher velocity.

So, the velocity of the system velocity of the fluid which is flowing over the solid is definitely going to dictate, what is going to be an important parameter in deciding; what is the value of the heat transfer coefficient convective heat transfer coefficient. It is not only the velocity it would also depend on the thermal physical properties of the fluid. What is its density, what is it is a thermal conductivity and since it is a case of flow and heat transfer, whenever there is a mention of flow we cannot neglect we can we cannot set aside the important parameter of fluid flow which is viscosity.

And it is also going to depend on what kind of a thermal conductivity the fluid which is flowing will have. So, it is going to depend on the on the on the density of the fluid, on the thermal conductivity of the fluid, on the specific heat of the fluid and also on the viscosity of the fluid. So, rho, C p mu and k these are the thermo physical properties, which would be relevant to decide what is the convective heat transfer coefficient?.

There is an operating parameter which is the velocity with which these the fluid is made to flow over the solid surface. So, that velocity let us call it as u is also going to be in the functional form of h. So, if we think of the expression of h this functional form of h should contain therefore, it should contain the following parameters. So, h is going to be a function of what is the velocity, which is which is the fluid velocity its going to be a function of rho C p mu k among others and it will be in some cases they are going to be certain other parameters for example, its going to depend on what is geometry of these surface. Is it is a flow taking place over a flat surface, is it on a rounded surface, is it a rough surface, or is it a smooth surface.

So, those extraneous factors that define the nature of the surface will also were also decide what is the value of the heat transfer coefficient. So, if you look at the parameters which I have written over the air, that that is its v rho mu and k, they also going to be this geometric parameters let us I am representing it by L, where L is the characteristic length and I do not talk about anything else right now like for example, the nature of the surface the roughness and all.

So, if you think of these V, rho, mu and L so; obviously, when you think of these four terms V, rho, mu and L you can you can you can you can clearly see that they are they can be combined in form of Renaults number whereas, Renaults number is simply rho V, L by mu. So, h is going to be a function of Renaults number of the system and when you think of C p, C p mu and k when you think of these 3 C p mu and k; you will know that C p cp mu and k they are called Prandtl number.

Where the Prandtl number is defined as C p mu by k this PR is Prandtl number; one of the famous scientists who has done extraordinary work not only on momentum transfer, the concept of boundary layer but also on heat transfer. So, most likely the any co correlation any relation that you would see for forced convection, where you the fluid is by an external agencies is made to flow is going to be a function of Reynolds and Prandtl number.

So, we will see this more when we discuss convection, but right now it would be, but right now it would be sufficient to know that the Newton's law of cooling will decide how heat gets transferred between the solid and the moving fluid, which would help us in deciding what are the boundary conditions for this case.

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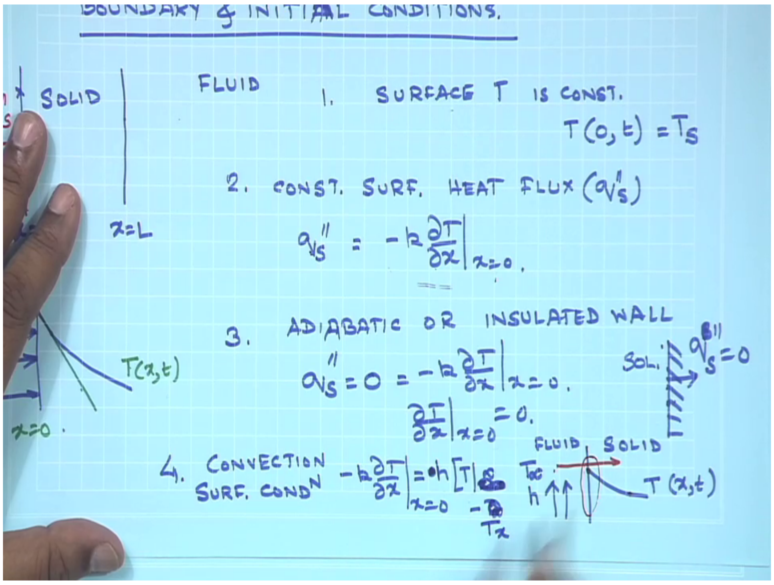

So, let us see what are the boundary and initial conditions that one would one would get when you come across this. So, let us talk about this is the solid and on this side I have a fluid, and in the solid let us say this is my x equals 0 and this is x equals L. So, through the solid you only have conduction and at this point from the interphase you have convection. So, the first condition possible condition is that, the surface temperature is constant known its known or constant, which would tell you the T at x equals 0 at any time t is maintained at some value of Ts.

So, the if the surface temperature at this point is constant, then the temperature over here would also be constant. And let us also assume that some amount of heat is being added to this surface, which we call as a q s double prime. So, this is the flux and the second condition could be that constant surface heat flux. This constant surface heat flux let us call it as qs double prime, as again double prime refers to the to the flux not the total amount. Now when this heat comes to the solid, when this could be heater this could be some sort of radiation which is following on the surface and its getting absorbed. Once it gets absorbed delivered at x equals 0, the heat then travels through the solid by a conduction process.

So, q s double prime which is the heat flux at x equals 0, must be equal to minus k del T del x at x equals 0. So, this is the this is the second type of boundary conditions which you may see where the amount of heat flux amount of delivered heat flux, which is to be carried into the solid by a conduction process and by invoking Fourier's law, I can write that the heat flow at x equals 0 by conduction is minus k d T dx and this is x equals 0 is a control surface. A control surface does not have any mass of its own.

So, for a control surface the conservation equation takes to form that in is equal to out and therefore, at the control surface the amount of heat which comes in sq is qs double prime and the heat that gets transformed, that gets transported by the conduction is simply going to be a minus k d T dx at x equals to 0. So, how would it look like? Let us say this is my qs double prime, this is the temperature profile and if I draw a tangent to this at x equals 0 ok.

So, this temperature could be a function of x and time. So, this tangent is essentially your del T del x at x equals 0. Multiply that with minus k and what you get is qs double prime. So, this is by a conservation by app invoking conservation at the control surface of the in input heat flux, and the heat which gets transported in the x direction. The third condition could be an adiabatic wall or insulated wall. So, if it is an adiabatic or an insulated wall what you have then here is that, this is the surface on this side its perfectly insulated. So, if it is perfectly insulated then q s q s double prime must be equal to 0. If it is insulated then no heat can cross this surface.

So, if no heat can cross this surface, I am again going to write the equality of the heat that goes out and the equality of and the heat that reaches the surface by conduction. So, this side is solid and I on the on the other side, I have perfectly insulated this. So, since q s double prime due to insulation is equal to 0. So, this must be equal to k del T del x at x equals 0. So, this would give you that del T del x at x equals 0 is 0.

So, perfect insulation or an adiabatic surface would simply show you tell you that the, gradient of temperature the special gradient of temperature at that plane is equal to 0; which comes directly again by a heat balance on the on the surface on the control surface since no heat can cross the control surface, this k d T dx on the other side has to be equal to 0 which would give rise to since k cannot be 0 since d T dx is 0. So, the temperature gradient vanishes, temperature gradient becomes equal to 0 on a surface perfectly insulated or which is adiabatic and I will give you examples of adiabatics of its later on, when we discuss about how what is the what is the amount of heat for a systems in which you have heat generation.

A system in which you have symmetric generation of heat everywhere then, you would see that at certain point inside the inside the solid object, the temperature reaches the maximum. And when temperature reaches maximum then; obviously, d T dx at that point would be equal to 0. So, if d T dx is 0 in that plane, then no heat can cross this plane in either direction and that such a surface is known as the adiabatic surface and the required boundary condition, that prerequisite for having no heat crossing this adiabatic plane from either side is d T dx to 0. So, d T dx 2 be 0 can be a valid boundary condition, if we know that at certain location the its either an adiabatic surface or its a surface which is perfectly insulated not allowing any heat to escape so, that could be the third boundary condition.

And the fourth primary condition we would see that its a case of convection, surface condition. Convection surface condition it would tell you that if we have a. So, if we have a solid on this surface side and let us say fluid on this side and what you have is the, fluid is moving with some velocity and let us say its temperature is T infinity and the thermal and the convective heat transfer coefficient is H and over here the temperature varies as a function of x and time.

So, if I take this as my control surface, and let us say that T infinity is more than T. So, its the solid is exposed to a hot environment. So, heat is going to move in this direction from the fluid to the interphase by convection, from the interphase to the inside of the solid by a conduction process. And at the interphase the convection must be equal to the conduction. So, what is conduction? Conduction is minus k del T del x at x equals 0. So, this is the conductive flow of heat conductive heat flux which must be equal to the convective heat flux, which is T at x equals 0 minus T and minus T infinity sorry. This is T I will put a minus sign over it or rather its T at infinity minus minus T at I will put it in a different page.

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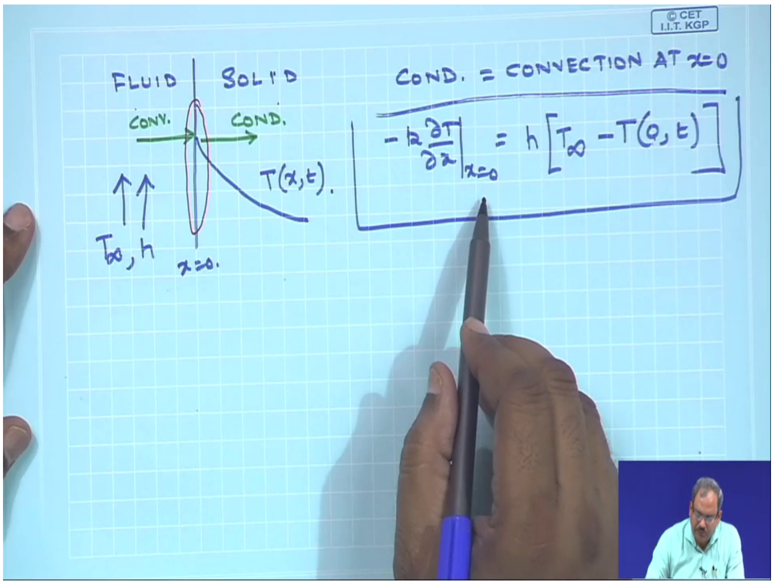

So, what you have then, I will I will I will draw it once again. This left he solid side, this is the fluids fluid, the fluid is moving with a velocity with some velocity having a temperature T infinity and with a convection heat transfer coefficient as h, and let us say that the fluid temperature is more than that of the solid. So, this is how the temperature inside the solid would change with time.

So, if we take this to be our control surface, then the amount of heat which reaches by convection the same amount has to be transported by conduction since this is a control surface and therefore, conservation demands that convection and conduction are equal, and if we write the conduction equals convection at x equals 0. This is let us say this is the location of x equals 0 x equals 0 then what you have is minus k del T del x at x equals 0 must be equal to.

So, this is the conduction conductive heat flux is equal to h times T infinity, where T infinity is the temperature of the fluid at a point far from it minus T at 0 t. So, this is at the at x equals 0 at any given time t. So, these kinds of conduction convection mixed boundary conditions are also prevalent in the study of heat transfer. So, once again if you look at the possible boundary conditions, one is the temperature is known that could be one condition, the heat flux is known where the heat flux is equated to the conductive heat flow. So, that could be another boundary condition, the third could be its the third would be it could be an adiabatic or insulated wall condition, in both cases q dot s q double prime s is 0 which would tell you that del T del x is equal to 0.

And the fourth condition is this is the condition boundary condition 4, which is a convection surface convection of this condition where the equality between conduction and convection at x equals 0 would simply give you that minus del T del x at x equals 0 is this the convective flux convective heat flux. Now depending on what is the condition what is the specific problem that you are dealing with, one or more of these conditions can be used.

So, any solution of any heat transfer conductive heat transfer problem is first to identify what is the what is the coordinate system that needs to be used. Is it a cylindrical system or is it a planar system or a spherical system. Then write the governing equation write the governing equation for that situation and based on the physics of the problem, you cancel the terms which are not relevant. If it is one dimensional problem with heat generation, but at steady state then your equation would be if you look at the equation once again you would say that temperature is a function only of x, its not a function of y, its not a function of z and it is not a function of time.

If that is the n q dot is not 0; if that is the case then the equation would be d 2 t dx square the first among the large left hand side, the second and the third term would be 0 since t is not a function of y and t is not a function of z. The fourth term is q dot by k where q dot is the generated heat per unit volume. And the right hand side temperature is not a function of time. So, therefore, the only term on the right hand side is can also be equated to 0.

So, your governing equation in that case would be d 2 t dx square plus q dot by k is equal to 0. So, this way looking at the problem at hand, you would be able to simplify the governing equation to arrive at the final form of the governing equation. Then see out of the four boundary conditions that I have discussed so far, which are relevant in for this specific case. Is temperature at a given location known, is the heat flux known or does the physics of the problem tell you that the flux at some x location such some location would be 0 because either it is perfectly insulated or it is an adiabatic surface.

So, you know which boundary condition to use or is it that at some location, this solid is exposed to a fluid and the solid is losing or gaining heat by convection. So, using

Newton's law and the equality of conduction and convection at that specific location, you should be in a position to write the mixed boundary condition which is the boundary condition 4 that I have discussed.

So, that is how a conduction problem is to be attempted and if you can solve the governing equation with the boundary conditions, what you get as a result is temperature as a function of location as well a function of time. And once I have temperature known at any point inside the control volume, then I can do all other possible analysis to obtain the flux at a point and so on. So, what I would do now is, I will give you just a problem for you to practice and then the in the tutorial part of it the problem can be analyzed in greater detail. So, very quickly the problem that I would like to give you is about a solid wall, which is generating some amount of heat.

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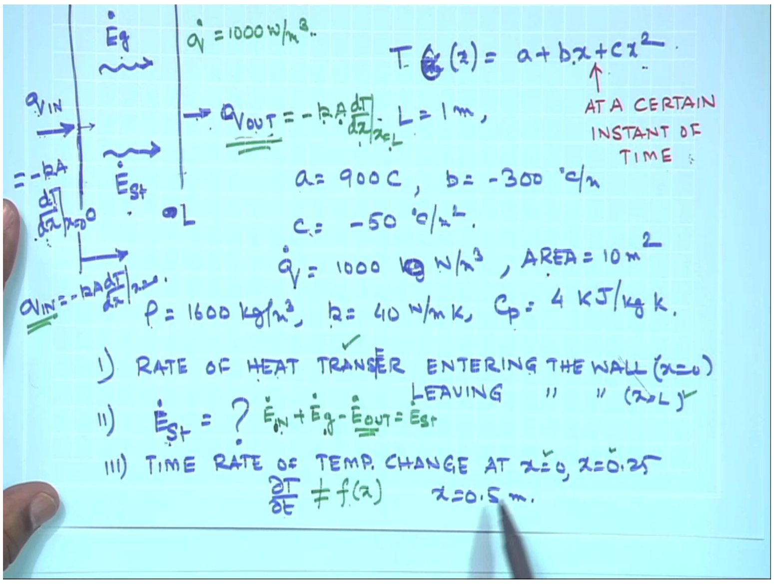

So, this solid wall there is some amount of heat which is generated. So, E dot g and its not a steady state. So, there is some E dot stored in it, some amount of heat comes into the system, some amount of heat goes out of the system. The x starts from here and x this is from 0 to L, and it is known that the temperature inside this is as a function of x, T as a function of x is equal to a plus bx plus cx square. So, this L is equal to 1 meter and this is the temperature profile at a certain instant of time.

So, that is why you would see that the temperature function, the temperature expression does not contain any time, because it is measured and expressed in this form at a given instant. And the values of a b c which are constant these are, a is 900 degree centigrade, b is minus 300 degree centigrade per meter and c is minus 50 degree centigrade per meter square.

The value of q dot which is the amount of heat generated in inside the system is 1000 watt per meter cube. So, this is heat generation per unit volume and the area of the wall is 10 meter square. The material of the wall the rho is 1600 kg per meter cube, the thermal conductivity k is 40 watt per meter Kelvin and C p the heat capacity is 4 kilo joule per kg Kelvin.

What you need to find out is; what is the rate of heat transfer entering the wall entering the wall which is at x equals 0 and leaving the wall which is at x equals L. What is the rate of energy storage in the system at this instant. So, what is the value of E dot ST and third one. So, you have to evaluate what is the rate of energy stored in the system at the given instant of time instant, where the temperature can be expressed in this form. And the last part is; what is the time rate of temperature change at x equals 0, x equals 0.25 and x equals 0.5 meter.

So, once again the problem I described the problem once again, where we have a wall in which some amount of heat is generated, some amount of heat comes into the system and it leaves out of system, and as a result of imbalance of these three quantities the energy stored in the wall will also change. However, the temperature at a given instant of time can be expressed in the polynomial form of a plus bx plus cx square. The relevant quantities for example, L a b c are provided as well as the amount of heat generation and the area through which the wall is the area of the wall that is also provided.

So, this area cross sectional area is 10 meter square, the density, thermal conductivity and specific heat of the wall material is also provided. What you have to find out that what is the heat transfer rate of heat transfer entering the wall that is at x equals 0, leaving the wall which is at x equals L and what is the rate of energy stored at this for this system at that that instant of time and time rate of temperature change. So, del T by del time at these three locations.

So, how do you solve the problem? As I mentioned the first thing that you need to identify is, when you look at x equals 0 plane, the amount of energy which enters the wall as a result of which the temperature profile is provided. So, if I take my this as my control surface, then q dot in must b equal to minus k A d T dx at x equals 0. So, at this location the amount of heat that comes in is going to be taken up by conduction on this side, and the conductive heat flow is k A d T dx at x equal to 0. So, by equating q in with the conductive heat flux and knowing temperature as a function of time, I can find out what is d T dx and evaluate what is d T dx at x equals 0 and therefore, find out what is what is the value of q in.

So, my q in would simply be minus k A d T dx at x equals 0, and the temperature profile is known to me. Similarly my q dot out would be e q q dot out would be equals minus k A d T dx at x equals L. So, the T as a function of x is known to me.

So, I can find out what is d T dx, what is d T dx at x equals L, the value of k and the value of A are known to me. So, I would be able to find now what is q dot out. So, these 2 it should be able to evaluate. Now what is e dot stored how do I write E dot stored. So, from my energy equation E dot in plus E dot generated minus E dot out is equal to E dot stored that is your that is your energy conservation equation.

So, my E dot in is nothing, but q dot in, my E dot out is nothing, but my q out what is E dot g? The value of E dot g is is is provided as E dot g equals where the q dot is provided as 1000 watt per meter cube. So, when everything is expressed on a per unit volume basis, my E dot in is known, E dot g is known E dot out is known. So, I should be able to find out what is E dot what is E dot stored. And I leave this part for a you to find out for you to think and solve and what I can tell you is that this del T by del temperature is you will see that its not a function of x value of temperature change with time that you would determine, it will not be a function of x.

So, the time rate of temperature change at x equals 0.25 or 0.5, all are going to be equal. What I am what I am going to do is I will I will give you the values the answers to this problem and you can try it on your own.

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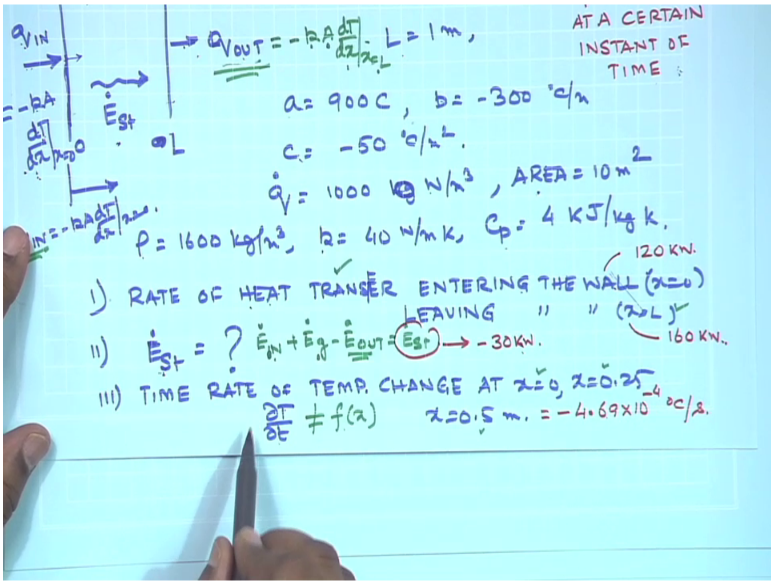

So, q dot in is going to be 120 kilo watts, the q dot out is going to be 160 kilo watts and the energy stored in the system is this part is going to be equal to minus 30 kilo watt so; that means, the system is losing this much of energy at that instant of time and you would say that del T divide the temperature change with time, the value of this would be minus 4.69 into 10 to the power minus 4 degree centigrade per second.

So, the answer to part the first one the in is 120, the out is 160, the stored is minus six minus 30 and the time rate of temperature change is going to be minus 4.7 into 10 to the power minus 4 degree centigrade per degree centigrade per second.

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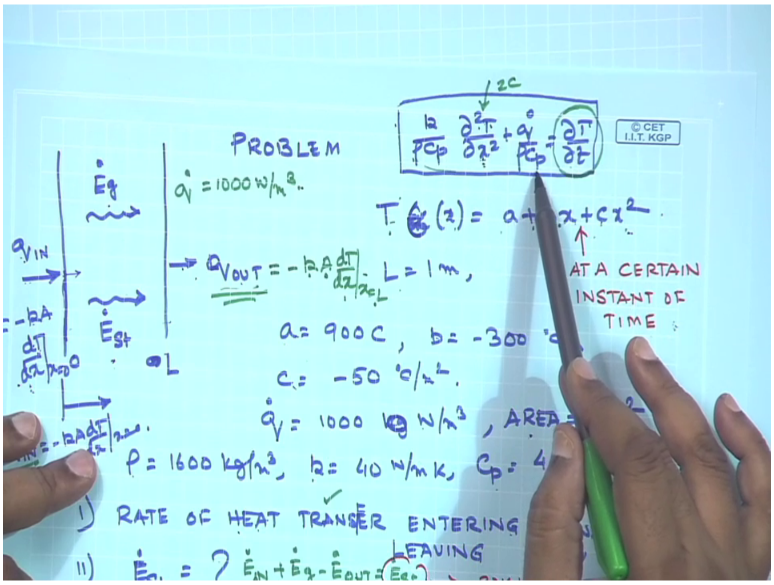

And the way this part is going to be solved, I will simply write it once again the governing equation for this case would be k by rho C p, the del 2 t by del x square plus q dot by rho C p is equal to del temperature by del time. So, this is the form of the governing equation, which you would obtain by writing the energy equation and cancelling the terms in T is a function of x and time.

So, all the terms that those are del 2 T del y square and del 2 T del z square will be of will be set to 0, temperature as a function of x is known to you. So, you should be able to find out what is del 2 T del x square. The values of k rho C p q dot are known to you. So, of only unknown here del temperature by del time. And looking at the expression of T x T as a function of x you would see that del 2 T by del x square is going to be a constant that simply going to be 2 C.

So, del 2 T by del x square is simply going to be 2 C. So, the entire left hand side which is time rate of change of temperature is going to be a constant. So, it is immaterial at which value of x you are measuring the time rate of temperature change, it is not going to be a function of x and when you plug in the values you would see that, the time rate of change of temperature would be minus 4.7 into 10 to the power minus 4 degree centigrade per second.

So, this completes our today's lecture, where we have seen what could be the relevant boundary conditions and we have we have discussed a problem where a direct application of Fourier's law and the conservation of energy can be used to arrive at numbers for the heat, which comes into the system going out of the system and time rate of change of temperature and so on.

So, I believe if you go through this your concept of conductive heat transfer must be clear right now and the use of use of the use of the heat diffusion equation. In the next classes we will see how this heat diffusion equation can be modified to obtain special cases, solutions for special cases and what form would it take for, not only for Cartesian coordinate systems, but also for cylindrical and spherical systems.