Heat Transfer

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Lecture - 15
Fins and General Conduction Analysis

We will continue with our study of the Heat Transfer from extended surface, towards the end of last class I have introduced the general conduction analysis for an extended surface. However, I feel that it need to be, I need to go through it once again because probably I was little bit too fast in that class. So, what I am going to do is, I will start with the derivation of the generalized conduction analysis and then see how they can be applied for specific boundary conditions, all of which are realistically possible depending on what is the size of the fin. What is the material of construction of the fin and what kind of conditions I have at the end of the fin; that means, at the tip of the fin.

Once we covered that and once we have an idea of the form of the temperature distribution that one can expect in an extended surface, then we will solve one problem to demonstrate how this conduction analysis can be applied for, for practical problems. So, the first 15 minutes is going to be a recapitulation of what we have done in the last class with extensions and more insights, which should help you in understanding the analysis and the second is going to be about problem solving.

So, as I have mentioned in previously that an extended surface is a one in which, which is added to a hot surface in order to extract more heat, in order to dissipate more heat from that surface; So, fins can come in different shapes, in size, but no matter what we need to justify the effectiveness of the fin, the performance of the fin whether and not we should go for attaching fins which are costly and requires additional fabrication to the hot surface. So, for that we have defined the two quantities, effectiveness and the performance and their, we have seen that there is a certain numerical value of the effectiveness, which you have to cross in order to prescribe the use of the fin.

But let us look at the mathematical side of it, how we can develop an expression that would give us an idea of the temperature distribution. The one dimensional temperature distribution in a solid fin where we are going to have conduction in let us see the x direction through the material of the fin, through the cross section of the fin and through the periphery of the fin we are going to lose heat by convection. So, it is a case in which both conduction and convection are present and we will have the more general situation in which the cross sectional area is allowed to vary with x. So, will not first assume that is a constant cross sectional area fin, where the A c the cross sectional area can also be a function of location. In similarly the area, the peripheral area which is available for heat transfer will also, can also be different.

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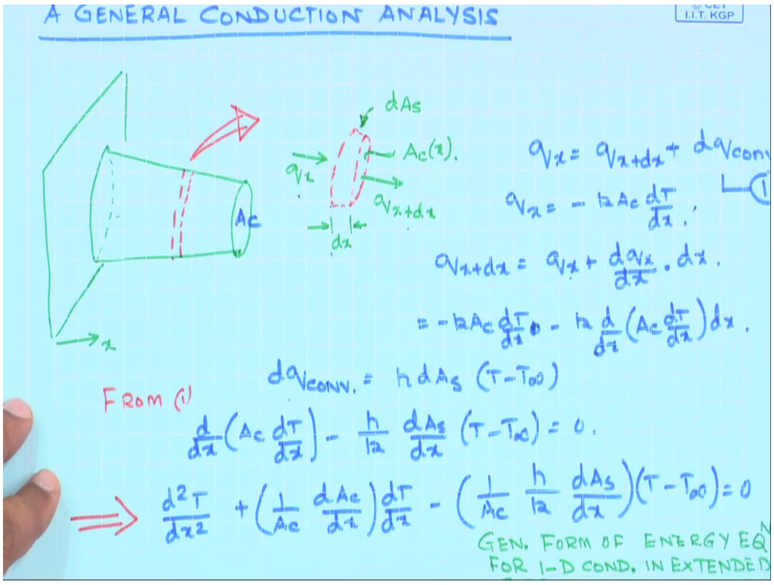

So, let us look at the figure which we have drawn in the last class where a fin; a circular fin of variable cross section. So, this A c what you see you over here, this A c is a function of x and this area the surface area of the fin is available for conduction. So, what I have done is, I have taken a small section of lengths dx and enlarged heat over here, writing down identifying the terms the by which heat can come into the control volume and it can leave the control volume.

So, what you seen here is that q x is the heat flux which is coming at x equals to 0, at x equal to some x and qx plus dx is the heat conductive heat which is leaving the surface at x plus dx. So, this is conductive heat in and conductive heat out, we realize as we have stated before that the cross sectional area could be a function of x, this area denoted by d As, the surface area that is the entire area is available for convective heat transfer. So, if I write a balance, heat balance across this volume element, what I am going to get is that the heat that comes in by conduction is equal to the heat that goes out by conduction and the heat that goes out by convection. So, qx must be equal to q at x plus dx plus dq conviction.

So, this is simply from energy balance of, energy balance for this volume element, we also know that from fouriers law, the heat flow rate can be by conduction can be expressed as minus k A c dT dx, remember that A c in here is can be a function of x. So, using a Taylor series expansion of q x and neglecting higher order terms, I can write that qx plus the dx in the specific form in therefore, when you when you expand this then you are going to get in a this qx is simply Fourier's law and dqx dx is simply d dx of this entire thing. So, this is the heat that comes in by conduction, this is the heat that goes out by conduction. So, what we have left with is, what is the heat that goes out by convection and I invoke Newton's law of cooling, which tells me that convective heat transfer is h area, surface area, times the temperature difference. So, this T is the temperature of the surface at the point where you are calculating the convective heat loss.

So, we realize that this t can also be a function of location or it can be a function of x. So, now, I have identified each of these terms in equation 1 which is the conservation equation. So, when I substitute the expressions in here, what I get from equation 1 is this form. So, this is A c and this is A s in, once I expand this in identifying or recognizing that A c can also be a function of x. This is the general form of the energy equation for a, for one dimensional conduction in an extended surface.

So, what we have here, once again I expanded this term. So, d A c dx I am not going to set it equal to 0. So, I am allowing it to for a general condition in which the area of cross section can be a function of position. So, this equation this energy which is nothing, but an energy equation, this equation if it can be solved it is going to give us the variation of temperature with location and here we have correctly identified the conductive heat transfer and convective heat transfer.

So, it is the limitation of this equation is its valid for one dimensional conduction, but as I have mentioned before most of the fins, the cross sectional area is generally small, they can be long with very small cross sectional area. So, end the one of the requirements of fin material is that they should have high thermal conductivity. So, if you have high thermal conductivity with a small cross section, then it can safely be assumed that the temperature is going to be a function of x, that is his actual location and not of the direction, perpendicular direction perpendicular to the heat flow. So, the fins more or less ideally, a fin should look like this paper which has a significant cross sectional area, which are the significant cross sectional area and therefore, its temperature is going to vary with x.

But at any location since the fin is thin its temperature is not going to vary with, let us say y. So, one dimensional conduction is a good approximation to express the heat transfer, the temperature distribution in such situations. So, what I have been is a general equation, now this general equation since it is a second order equation, it may requires two boundary conditions. So, what are the two possible boundary conditions? One boundary condition for the fin would be where it is attached to the base and the temperature at the base is known, temperature at the base is let us say it del x, denoted as t b. So, the temperature at the base of the fin is known, that is one condition which is fixed and then I have to think of what is going to happen to the other end of the fin.

In case the, what is happening at the other end of the fin could give us different expressions for temperature distribution in the fin and we will just quickly go through some of those other possibilities and the resulting expressions. I am not going to write all the expressions or will not talk about how to solve these, you should refer to your text where all these has been provided. What I am going to give us a glimpse of how depending on the boundary condition what are the realistic boundary conditions, in what can they tell us about the material of construction of the fin, the shape and size of the fin and so on.

So, let us look at a simple case in which the concentration, in which the cross sectional area does not vary with x. So, if the cross sectional area does not vary with x and I am talking about let us say a rectangular fin, a rectangular fin which is attached to the hot surface which is this one. So, if it is a rectangular fin of constant cross section, its cross sectional area is simply going to be perimeter times x, where x is the distance in the direction of, in the direction of heat flow or it could be a cylindrical fin, where the where the area, cross sectional area is also kept constant.

So, if we have the area, the cross sectional area to be a constant if we assume that to be a constant then let us see what is, what are the simplifications that we can make to the general equation which we have derived. So, if we go into look at this expression over here, what you have is this term should definitely remain in the equation. This term which tells us about the convective heat transfer, this should also remain, but as A c the cross sectional area is a constant with respect to x, I can safely drop this term. So, the governing equation for a constant area constant cross sectional area fin would consists of the first term of the, of the equation and the third term of the equation, the second equation second term can safely withdraw. So, let us use this expression for certain situations for example, for a rectangular fin or for a cylindrical fin of constant cross section and see how the temperature distribution would look like.

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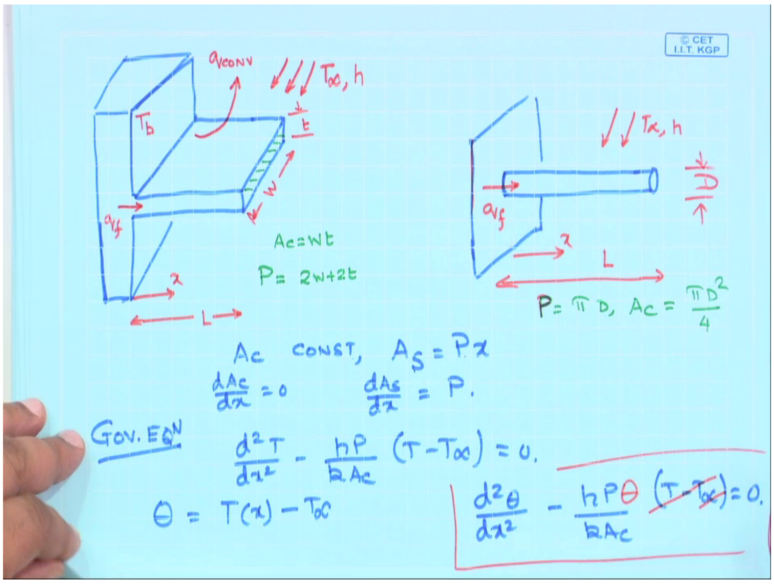

So, the first cases is rectangular fin, which looks something like this ok, in this is a direction of heat which is been picked up by the fin from the hot surface, which is at a temperature of T b, temperature of the base the conviction is from here. So, this is q convection and which is as a result of, let us see here flowing over the fin at a temperature of T infinity, with a convective heat transfer of h. This is my x direction and the entire length of the fin is L, the width of this is W, the thickness of the film let us say its t and so therefore, the A c, the cross sectional area which is available for conduction A c is simply going to be W times t.

So, A c is equal to W times t, the cross sectional area which is available for conduction and if I talk about the perimeter of this fin, which would simply be equal to 2w plus 2t. So, that is the perimeter of the fin and this is the cross sectional area of the fin, you can similarly have a situation where you have a cylindrical fin of constant cross section. So, here the heat goes, picked up by the fin we call it as q f the same over here, this is q f and we also have the same T infinity and h and let us see, assume that the diameter of this constant cross section pin fin is D. This is again x and the entire length of the fin is l. So, here the perimeter, the perimeter is going to be pi times D and the cross sectional area which is utilized for convict conductive heat transfer is going to be pi D square by 4.

Now, what we see is that A c is a constant, the surface area A s is simply going to P times x. So, if I take a slice of the area of length x, then the surface area available for conduction would simply be equal to P times x. So, what are the implications of that, then d A c by dx would be equal to 0 and d A s by d x is going to be equal to P. So, if you look at the governing equation which was there in the previous slide, which is this, this governing equation I cancel this term and I simply write instead of d A s by dx, I replace that with P, where P is the perimeter of the fin. So, the governing equation therefore, becomes d to T dx square minus h P by k A c times, T minus T infinity, which is equal to 0.

So, this is the governing equation where the A c is constant and of course, if A c is constant the area, the cross the surface area can be expressed as P times x. So, if I define theta, which is an x s temperature, which is T temperature at any location minus T infinity, then this governing equation should simply become just a more compact form d 2 theta by dx square minus h P by k A c times T minus T infinity is equal to 0. So, this is going to be the governing equations, same governing equation but in a in a slightly more compact form. In fact, I can use, instead of using this I can simply write this to be equal to theta as T minus T infinity is defined as theta.

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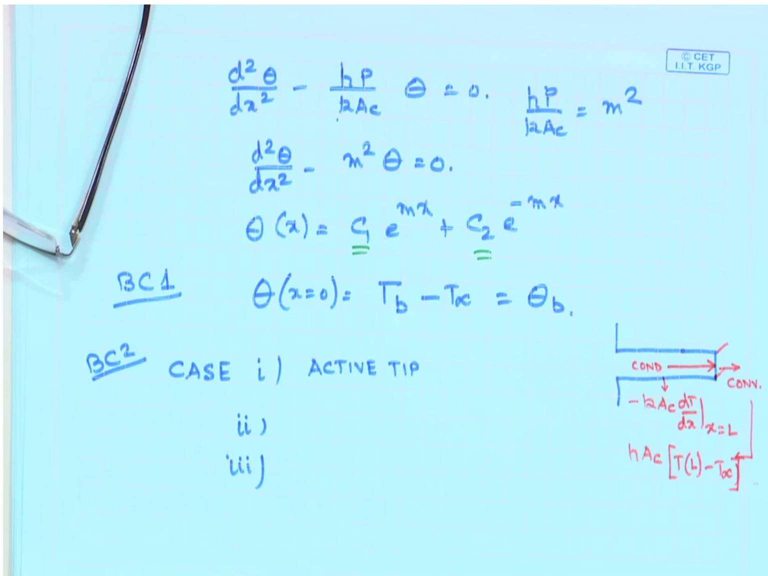

So, what is the governing equation once again, if you if you look at look at this one then it s d 2 theta by d x square minus h P by k A c times theta, is equal to 0. Let us define h P by k A c to be some other constant, because here h is call h P k and A c all are constant. So, if I define this h P by k A c to be another constant to be equal to m square, then the governing equation becomes d 2 theta dx square, minus m square theta is equal to 0. So, this is a linear homogenous second order differential coefficient, differential equation with constant coefficient and the solution of this is theta x c 1 e to the power m x plus c 2, e to the power minus m x, these c 1 and c 2 are integration calls, constants of integration.

So, I required 2 boundary conditions, boundary condition 1 is fixed, that is theta at x equals to 0, let us call it as, this should be equal to T at the base, minus T infinity and let us call is a, call it as theta b. So, that is the concept that is the temperature which is known at x equals to b and for boundary condition 2, you have you have several cases. So, I will discuss but not write what are the solutions for this. So, this is case 1, these then case 2 and 3, 3 cases we will talk about. So, the first case is known as the active tip, tip means at the other is of the, other is of the fin. So, what it is the other is of the fin, you have the heat coming in to the tip by conduction, the heat that goes out of it is by convection.

In order to maintain steady state the conductive flow of heat up to this point must be equal to the convective heat which is taken out from the tip of the fin. So, that is to maintain steady state, otherwise the temperature will change with, with time. So, what is the conductive heat which comes in? That must be equal to minus k A c dT dx at x equals to L so this is the conductive heat. What is the convective heat? This must be h, now the area available to the tip is A c. So, h A c times temperature at x equals to L minus T infinity.

So, this is going to be the new boundary condition of, this is going to be the boundary condition for the this case where the tip is active and therefore, the amount of heat which comes up to the point, up to the up to the tip by conduction must be taken out by convection. So, that is the standard way of looking at things and for these two boundary conditions in your text, you would see what would be the temperature distribution for these two boundary conditions.

So, I will not write that, you take a look at your text, the other condition what you can, you can do is let us assume that the fin is very very long, if the fin is very long. So by the time you will reach to the other end of the fin its temperature has decreased from its original starting value of T b and now it lies very close to the ambient air itself. So, if I, if this temperature based temperature is T b and I have a very large fin which is attached to it then as I progress the temperature is going, temperature of the fin is going to be higher than the temperature of the surrounding air, but it is going to be lower than the base temperature.

So, if I provides sufficient length and knowing that the fins is, it sufficient length then what you would expect is that at the very end that is not going to be any difference between the temperature of the fin and the temperature of the surrounding air. So, that is the case as L tends to infinity, the temperature of the fin at that location tends to be equal to the temperature of the air. So, that is known as the long fin approximation.

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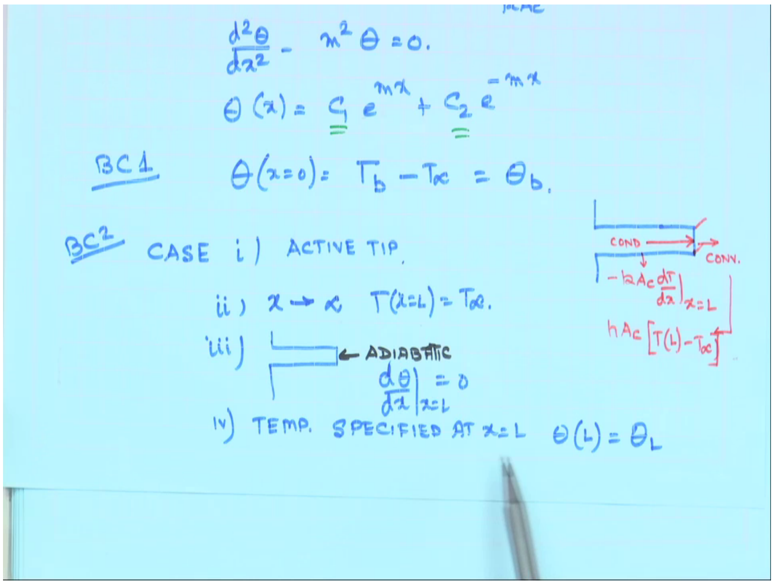

So, if I write it the expression for this is as, x as for a for a long fin as x tends to infinity that is a for very large values of a, value of the of the fin then you are going to get T at the T of x equals L is equal to T infinity. So, that is that is the second condition, similarly one can one can think that the tip of the fin is adiabatic, the third condition, the third condition is the tip of the fin is adiabatic. So, if the tip of the fin is adiabatic, then the corresponding boundary condition would be d theta dx at x equals L is 0. So, that is a standard condition, that is a standard condition what you would get in for any adiabatic surface and the fourth one, 4 possible boundary condition is temperature is specified at x equals L. That means, theta at L, x equals L is known, which is let us call it as theta L.

So, the first boundary condition is fixed; that means, the temperature of the temperature, at the base is known the second, boundary condition there can be four different possibilities. One is an activity that I have just described, the second one is for very long fins the temperature is going to be equal to T infinity that could be in adiabatic tip and the temperature specified at x equals to L which would give you theta to be equal to, theta to be equal to theta L. So, for all these 4 conditions, you would see in your texts that the solutions are provided. So, look at the solution and sometimes you may have to use it, the only thing which is remain to be discussed is; what is the total heat which is transferred by the fin?

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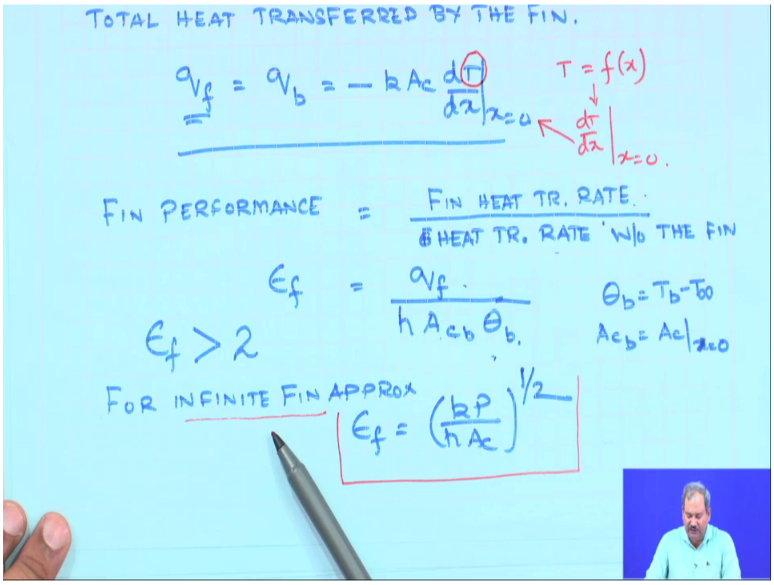

The total heat if you look at the figure which have drawn over here, the total heat which is transferred by the fin at steady state, must be equal to the heat that it picked up from the base at x equals to 0. So, which I have denoted as qf. So, I would like to know how much of heat that the fin can dissipate when it is attached to the base.

So, that must be equal to the conductive flow of heat into the fin from the base which, which is denoted by q f for these 2 cases. So, what is q f? Since, I have mentioned that q f is nothing, but the amount of heat taken up by the fin, is the amount of heat taken up by the fin at the base at steady state by conduction. So, this must be equal to k A c dT dx at x equals 0.

So, knowing the profile of T, if I know the profile of T, that is T as a function of x, then I would be able to obtain what is dT dx in what is the value of d T dx at x equals 0. Put it back in here and what you would get is the what is get is, what is known as what is the total heat to be taken up by the fin under different conditions. and I have also mentioned that this, this temperature distribution will; obviously, vary depending on what is the what is the expression for the, what is the second boundary condition that you have taken in this case. Depending on the second boundary condition that is 2 a b c or d you will get different expressions for the final temperature distribution.

But no matter what, once you have the temperature distribution you can find out its gradient and once you have the gradient you can put x equals to 0 in order to obtain what is the value of the gradient at the beginning, at the point where the fin is joint to the solid surface. And thereby you should be able to calculate: what is the total amount of heat to be taken up by the fin? Which has a, which has some direct bearing on fin performance. So, this fin performance as we have discussed before, the fin performance is defined as the fin heat transfer rate, this is the amount of heat taken up by the fin and the heat transfer rate without the fin. So, what is heat transfer rate, when you have the fin is q f which is this one, divided by what is the heat transfer rate without the fin it should be h, area would be A c, area cross sectional area at the base divided by theta at the base.

So, theta b is nothing, but T b minus T infinity, A c b is A c b is nothing, but A c at x, x equals to 0 and h is the heat transfer coefficient. So, obviously, in order to justify the use of the fin, the amount of heat taken up by the fin must be more then, when the fin was not there. So, this fin performance must be greater, must be greater than one and it is denoted as epsilon f. So, unless epsilon f, unless you get take out more heat then you could take out when the fin is not there, the use of a fin is not justified. And the as I said the rule of thumb is epsilon f must be greater than 2 the fin should at least be able to dissipate twice the amount of heat which is, which is going to be dissipated when the fin is not there.

So, this q f must be twice, at least twice of the heat transfer rate without the fin. Now, when you go you just give me an example for infinite fin approximation, approximation this the expression for epsilon f turns out to be k P by h A c to the power half. So, this apparently simple result, this is only for infinite fin approximation and you understand the there are different different, you would, your expected to get different values of different expressions for epsilon. Depending on whether you have a an infinite fin approximation and active fin, active tip approximation and so on.

But this tells us something about the utility of fin performance, how does the expression of fin performance tell us something about the shape of the fin, the material of construction of the fin, what should be the, what is the, what is the usual situation in which use of a fin is justified. So, you have k in the numerator, since k the thermal conductivity is in the numerator in order to enhance the effectiveness of the fin k should be large. So, the value of the so the material of construction of a fin should always be of pi thermal conductivity, you have P by A c should be large, that is p the perimeter a c is the cross sectional area.

So, thin fins are always preferred and you have h is, h in the denominator. So, you use the fin is justified only for those cases where the value of h is low, h is small and when h is small the most likely condition of h being small is when you are going to have heat transfer from the solid to air, which typically have very low heat transfer coefficient. So, this more or less completes my discussion on fins, their performance, their efficiencies, the governing equations, the modified form of the governing equation, if you have a constant cross section fin. And what are the different types of boundary condition one can have, one can one can use to obtain the expression of the temperature distribution in the fin, in the fin and how that expression can be used to obtained the total heat of take. That means, the total heat dissipated at steady state by the fin.

So, these are interesting applications. So, I think it, if you take a look at your text right now it should be more clear. So, very quickly I will give you a problem, practice problem with answers and you should try to solve it and if there are any queries are any questions then we are, we are will get back to you can come, you can post the question to us will try to answer it.

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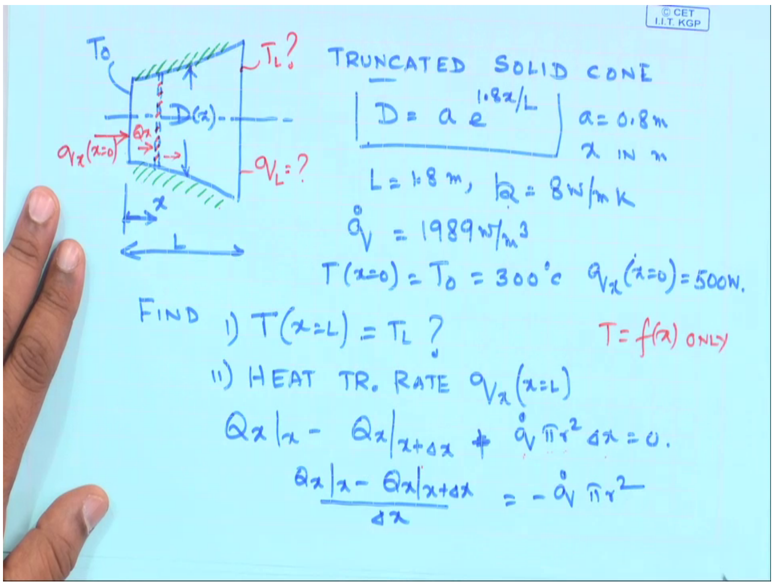

So, the problem that I have I am going to give you which is the practice problem, there is a fin of this shape ok, where the diameter of the fin is a function of x and the temperature at x equals to 0. So, this is x and the entire length is L. So, T 0 is the temperature at this end. So, this is nothing, but a truncated solid cone, whose a diameter is a changing with x and D is a which is a constant, e to the power 1.8 x by L. So, this is the, this is the functional form of how diameter changes with x and a is a constant, the value of a is 0.8 meter, x in meters and L is the length of the fin as I have, I have shown as I have shown you. The length of the fin is 1.8 meter, it has a thermal conductivity of 8 watt per meter Kelvin.

So, this is the thermal conductivity of the material; however, there is a uniform heat generation in the film in the cone as 1989 watt per meter cube. So, the amount of heat generation is generated in the truncated solid cone is 1989 watt, watt per meter cube, the lateral surface of the fin, these surfaces of the fin are insulated ok. So, heat can enter or leave only from these 2 at x equal to 0 and x equal to x equal to L, the temperature at x equal to 0 which I call as T 0 is known as 300 degree centigrade and the heat rate that is the amount of heat which comes into, comes into the cone this is q x at x equals 0 this q x at x equals 0 is 500 watt.

Find, 1 what is the temperature at x equals L that is what is T L, what is the value of T L? And second the heat transfer rate there is q x at x equals L at the right hand surface. So, the thing that you have to find out is what is the value of T L? And what is the value of q L; that means, q at x equals L? So, at one end the temperature, the temperature and the heat rate are known, some amount of heat is generated in the solid cone all the sides are perfectly insulated.

So, whatever heat that is generated has to, has to be has to travel in axial direction only there is no radial flow of heat and will assume the T is a function only of x T is not a function of R. So, T is a function of x only, in the temperature at this end is given as 300 and the heat enters at x equals 0, the amount of heat that enters at x equals 0 is 500 watt, what you have to find out is what is the temperature at x equals L and the and what is the heat rate.

I will quickly give you some point rates on this, the total heat which enters at x, minus the total heat that enters that leaves at x plus del x, must be equal to the amount sorry, the plus q dot the amount of heat which is generated, multiplied by pi r square delta x should be equal to 0. So, the conservation equation, if I take a thin strip of this, as my as my control volume, some amount of heat is coming in which I call it as Qx, the amount of heat which goes out of this is Qx plus dx and the amount of heat generated in here, this is heat generation per unit volume.

So, I multiply it, multiply q dot with the volume which is pi r square times delta x. So, that is going to be my conservation equation. In once you divide both sides by delta x; that means, Qx at x minus Qx at x plus delta x divided by delta x should be equal to minus q dot pi r square. In a this, when in the limit when I take delta x tends to 0; that means, I am using the formula for the first derivative, what I would should get out of here is d d x minus d dx of Q Q x is equal to q dot times pi r square and these two minus is will cancel out.

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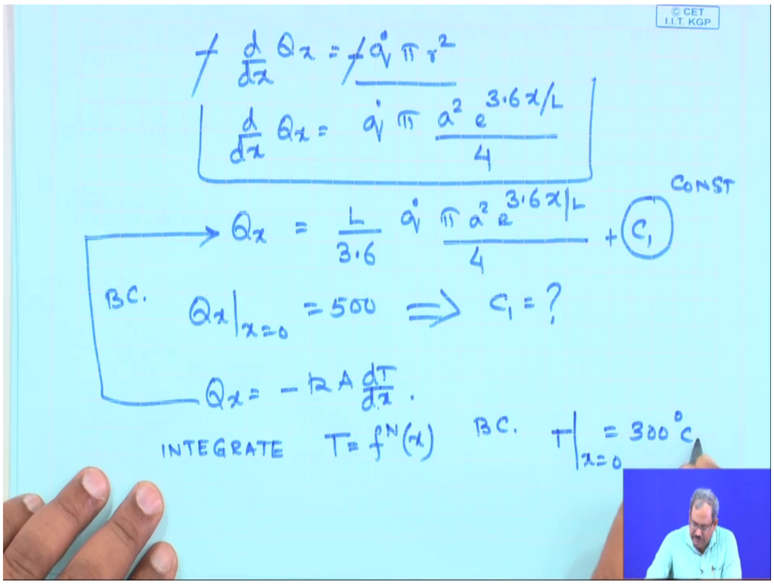

So, your final form of the equation, governing equation Qx is q dot pi instead of r, I am going to use this expression for diameter, I converted to r in, what you get over here is pi a square e to the power 3.6 x by L times 4 by this square by this square by 4. So, this is my governing equation, now this governing equation can now be integrated, Qx is going to be L by 3.6 q dot pi a square e to the power 3.6 x by L and I am going to have here as 4 plus c 1. C 1 is a constant of integration, what is the boundary condition? I know what is the value of Qx at x equals 0 to be equals 500 find out what is the x value for c 1 and secondly, you can also this is a total rate of heat.

So, Qx is equal to minus k A dT dx from Fourier’s law. So, if you substitute that in here what you should, what you should get is another expression, another expression in terms of temperature. Integrate that, integrate that to obtain the temperature as a function of x, but you need another boundary condition because there would be one more, one more integration constant which would come and that is T at x equals 0 is 300 degree centigrade.

So, substituting Fourier's law into my governing equation, after I evaluate c 1, I should be able to obtain a differential equation for T in terms of x integrate that expression to obtain T as a function of x and the integration constant can be evaluated with a known temperature at x equals 0.

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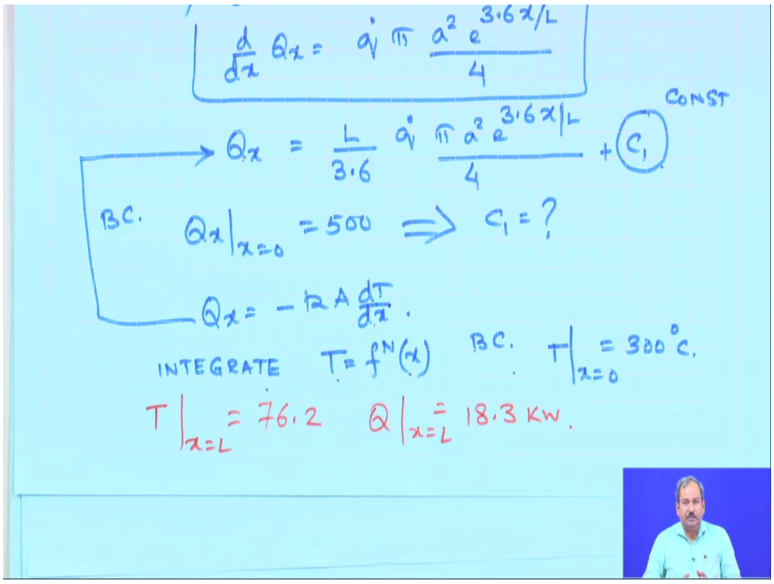

So, the answer to the problem that you should get that T at x equals L to be equals 76.2 and Q at x equals L to be 18.3 kilowatt. So, this is a nice example to show that the, it is always better to start from first principles. Write the conservation equation derive what would be the temperature distribution or heat flux distribution, identify the proper appropriate boundary conditions, solve for the integration constants in arrived at the desired value of either the temperature or the or the heat flux.

So, this concludes our study of conductive heat transfer in extended surfaces and from next class onwards we should start one of the very important, one of the very important chapter in heat transfer which is convective heat transfer. But conduction heat transfer is going to give us the base, based on which we are going to venture into convection and I would show you that it cannot have convection without conduction. So, a prerequisite for any study of convection is that you have a fair idea of how conduction works, since we have the background right now from next class we will start about convective heat transfer.