Introduction to Process Modeling in Membrane Separation Process
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Lecture – 11
Osmotic Pressure Controlling Filtration (Contd.)

Good morning everyone. So, welcome to this class of modelling in membrane separation process. So, what do you we did in the last class we are looking into the modelling of cross flow filtration and this modelling is applicable for reverse osmosis ultra filtration nano filtration even for micro filtration.

Now, in the last class whatever we did we formulated the governing equations and the of the of the solved mass balanced in the mass transfer boundary layer and next then, we have found out that this is a problem of fully developed velocity profile and developing mass transfer boundary layer. So, we have written down the solved basic balance in a two dimensional form in the within the mass transfer of boundary layer.

Now in today's class will be formulating the boundary conditions associated with this problem and then will be looking into the solution and then will see how the solution can be utilized for the prediction of the system performance in order to calculate the permeate flux and permeate concentration. Therefore, the if you remember the governing equation was 3 u 0 y by h del c del x minus v w del c del y is equal to d del square c del y square.

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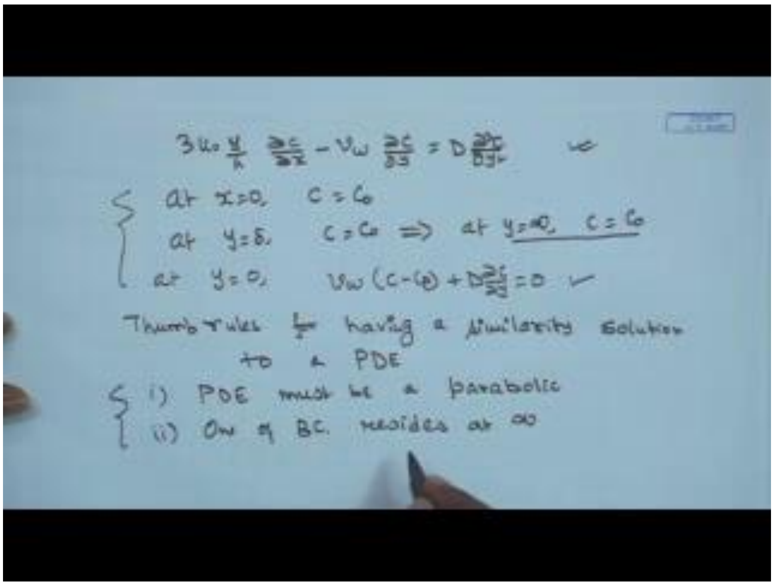

Now, will be for who writing down the respective boundary conditions at x is equal to 0 is a. So, called initial condition to his problem c is equal to bulk concentration at y is equal to delta c is equal to c not again in the bulk concentration or delta is the edge of the boundary layer mass transfer boundary layer. But as we have argued in the last class that, we are dealing with the system where the solute will be having very low diffusivity and hence very high mass transfer very high number and thickness of boundary layer is mass transfer boundary layer is inversely proportional to the number.

So, therefore, we are talking about a boundary layer thickness, mass transfer boundary layer thickness which will be around 1 present of the total chart channel dimension in that case that you can see that beyond the mass transfer boundary layer majority of the channel the concentration remains bulk concentration. So, we can safely replace this boundary condition as at y is equal to infinity c is equal to c naught. Now let us set up the boundary condition at. So, we need 1 more boundary condition on y because this equation is order two with respect to y. So, at y is equal to 0 we will be having at the steady state all the fluxes at towards the membrane will be all the solute fluxes towards the membrane will be equal to 0. So, that will be giving you the VwC minus c p plus d del c del y will be equal to 0.

So, the first term will be the convective solute flux towards the membrane surface second term is the convective solute flux away from the membrane surface and the third 1 is the diffusive solute flux away from the membrane surface. Now, these three conditions will be setting up the complete definition of the problem and as you have already known that we have already solved the 1 dimensional problems related to the mass transfer boundary layer we already solved this. Now if you can remember that the boundary condition of the two dimensional problem has now was the governing equation for the 1 dimensional problem. So, now, let us try to solve this in a set of equation if you look into the governing equation the governing equation is a parabolic partial differential equation and 1 of the boundary condition is residing at infinity. So, there are two thumb rules, thumb rules for having a similarity solution similarity solution to a partial differential equation.

These are if the p d e must be a parabolic 1 parabolic and second 1 of boundary condition resides at infinity, if these two thumb rules are met then these boundary these governing equation may admit a similarity solution. So, why the similarity solution is, important in case of boundary layer problems that has to be very clear to everyone. So, similarity solution is suppose we are talking about any boundary layer problem.

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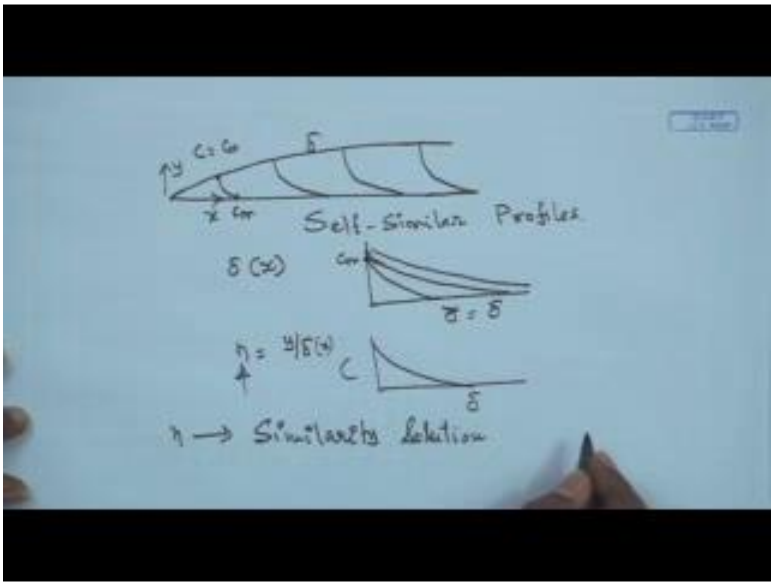

Let us say who are talking about a mass transfer boundary layer in this case. So, the boundary layer grows over the wall or over the membranes surface. Now if you look into the concentration profile the concentration profile is assuming the assume value is assuming c is equal to c naught at the edge of the concentration boundary layer of mass transfer boundary layer and the concentration is c at the membrane surface and this c m c n concentration of at the membrane surface will be a function of x. So, at every point you will be having a profile something like this and if you look into the profiles the profiles are similar in nature and these are these profiles are called self similar profiles.

So, whenever you will be having the self similar profiles will be having the self similar profile in the case of all sorts of boundary layer problems. So, whether it is a thermal boundary layer whether it is a mass transfer boundary layer whether it is a hydrodynamic boundary layer will be having the self similar profiles for velocity. If in case of hydrodynamic boundary layer temperature of in case of thermal boundary layer and concentration field in case of mass transfer boundary layer, this self similar profiles will be this therefore, they will be function of y and x the delta will be for and the boundary layer thickness delta will be essentially a function of x.

Now if you plot concentration profile as a function of x or function of y at y is equal to 0 this value is c m it will be very high value and y is equal to infinity or delta this is c naught. So, you will be having a value about profile something like this at a particular x location this will be at a this will be at a particular x, x location because c m will be varying and then this will be at a particular x location.

So, now will be defining a parameter eta which will be nothing, but y by delta then all these curves will be coincide and they will be super posing on a single curve, therefore, will be reducing the number of independent. So, delta will be function of x will be reducing the independent value will x and y into 1 single variable it is the similarity variable and this is also known as the combined variable this combine variable is eta, whenever will be plotting the concentration profile or temperature profile as a function of eta. So, all the curves will collapse in a single curve and will be having a variation with respect to 1 variable only that is known as the combine variable or similarity parameter. Therefore, what is the final advantage of this the advantage of this will be reducing the two parameter, two independent variable system into 1 independent variable system in other words will be converting a partial deferential equation into an ordinal deferential equation.

So, eta is known as the similarity parameter and similarity solution and the solution method is known as the similarity solution and the similarity solution will be obtained. If we have a partial deferential equation which is parabolic in nature and 1 of the boundary condition is residing at infinity. So, in this case also will be our problem satisfies that thumb rule and will be admitting a similarity solution in this case. So, therefore, let us look into the how to evaluate the similarity parameter.

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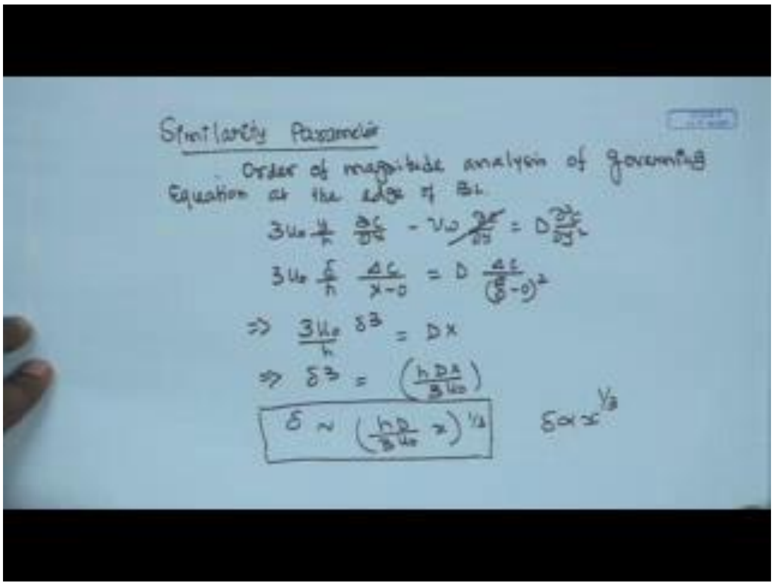

Now, similarity parameter can be obtained if y by let us do an order of magnitude analysis, analysis of governing equation, equation at the edge of the boundary layer. So, let us look into the governing equation once again three u 0 y by h del c del x minus v w del c del y is equal to d del square c del y square. So, we evaluated the edge of the boundary layer and any boundary layer will be satisfying two typical conditions or at the edge of the boundary layer whether it is a velocity boundary layer thermal boundary layer or mass transfer boundary layer.

What are this these conditions two conditions are known as the boundary layer conditions these are basically at the edge of the boundary layer the varied the dependent variable will be assuming the value of that of the stream. For example, at y is equal to delta c is equal to c naught if it is a mass transfer boundary layer and y is equal to delta t is equal to t infinity, if we are talking about a thermal boundary layer and at the edge of the boundary layer. So, v u is equal to u infinity if we talk about the hydrodynamic boundary layer.

Similarly, another condition will be satisfied at the edge of the boundary layer that is the gradient of the dependent parameter with respect to y will vanish; that means, at y is equal to delta del c del y will be equal to 0 at y equal to delta del t del y will be equal to 0 and del u del y equal to 0 delta in case of hydrodynamic boundary layer. So, therefore, will be at the edge of the boundary del c del y will be equal to 0. So, these term will be gone and let us have we evaluate the rest of the equation from the very beginning from at x is equal to 0 to a small value of x. So, 3 u 0 at the edge of the boundary layer y is equal to delta. So, this will be delta by h. So, del c will be delta c that is occurring between the in the boundary layer and delta x will be x minus 0. If we start from x equal to 0 and d del square c will be again a sort of delta. So, if you really do a numerical calculations in the finite difference method that you can you can you can see that del square c is nothing, but a kind of difference delta c. So, this will be order of magnitude delta c and delta del y square will be y delta minus 0 square of that. So, that will be giving you an estimate of how delta varies with x. So, this will be delta q is equal to d x. So, delta q will be nothing, but h d x divided by 3 u naught and delta will be h d by 3 u naught times x is to the power 1 upon 3.

So, therefore, these will be the variation of delta the mass transfer boundary layer in this case with respect to x. In fact, delta is varying as a as x to the power 1 upon 3. So, therefore, will be able to, now derive the similarity parameter and the similarity parameter is defined eta is equal to y by delta.

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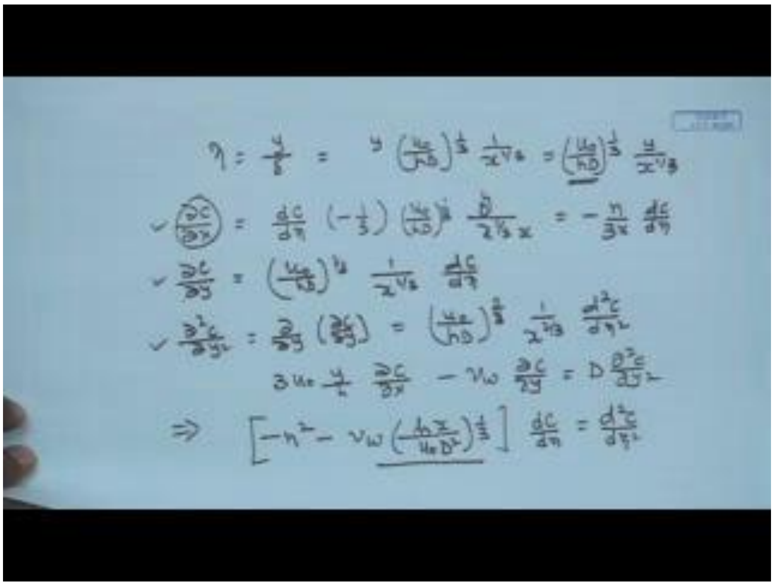

So, y and delta will be writing as u 0 over h d raise to the power 1 upon 3 1 over x to the power 1 upon 3. So, this will be nothing, but u 0 h d raise to the power 1 upon 3 y divided by x to the power 1 upon 3. So, we if you have put this term this term will be will be there because, if you include. So, u naught is a constant h is that geometry the channel d is the solute diffusivity which will be acting as which will be assumed to be constant. So, the combinations of all these parameter you can see that eta will be a dimension list number.

So, eta does not have a dimension the similarity parameter and now what will be doing next will be expressing all the derivatives if the governing equation del c del x del c del y del square c del y square in terms of the similarity parameter eta, in terms of the combine variable parameter eta. So, therefore, we can reduce. So, there will be no independent variable x and y in the governing equation there will be only 1 parameter eta into the governing equation.

So, partial differential equation will be boiled down into an ordinary differential equation. So, next what will be doing will be deriving will be will be will be expressing all the partial derivatives in terms of eta or the similarity parameter. So, let us do that. So, del c del x will be nothing, but d c d eta because since now c will be it is sole function of eta the partial derivative will become a complete derivative or total derivative d c d eta minus 1 upon 3 u 0 h d raise to the power 1 upon 3 y x to the power 1 upon 3 times x and this will be nothing, but minus eta divided by 3 x d c d eta and similarly you can derives del c del y as u 0 h d raise to the power 1 upon 3 1 x to the power 1 upon 3 d c d eta and second derivative del square c del y square will be nothing, but 1 more derivative of del c del y. So, if you really do that the expression becomes h 0, u 0 by h d raise to the power 2 by 3 1 over x to the power 2 by 3 d square c d eta square.

Now, if we substitute these you know partial derivative in the governing equation and we will do a and if you remember the governing equation is 3 u 0 y by h del c del x minus v w del c del y is equal to d del square c del y square. So, I am replacing del c del x from here into this equation del c del y here and del square c del y square into this and then after doing a simplification the final version of the equation becomes eta square minus v w h x divided by u 0 d square raise to the power 1 upon 3 d c d eta is equal to d square c d eta square. So, we will next what will be seeing that the partial differential equation as now boiled down into an ordinary differential equation. So, now, let us tackle with this part of the equation. So, as we have discussed earlier that as the boundary layer grows in thickness it offers more resistance against the solvent flux.

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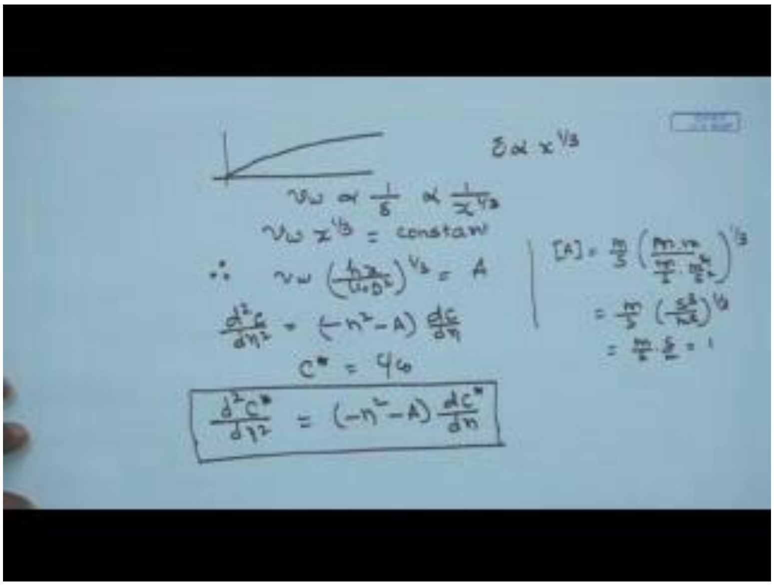

So, therefore, the thickness of the boundary layer in initial part of the boundary layer; that means, initial x location of the channel the resistance against the solvent flux will be very, very less and 1 will be getting a very high solvent flux as the boundary layer thickness grows in grow grows in thickness. Then it will be offering more resistance and the solvent flux will be decreasing. So, therefore, permeate flux or the solvent flux of permeate flux is inversely proportional to the thickness of the mass transfer boundary layer.

So, v w is inversely proportional to delta and we have already observed a delta is directly proportional to the x to the power 1 upon 3. So, v w inversely proportional to the x to the power 1 upon 3 that simply indicates that v w x to the power 1 by 3 is a constant. So, we define a constant in this term v w x to the power 1 upon 3 including the other terms. So, this is constant h is constant u 0 is constant d is constant. So, you define the whole term becomes a constant and we define that constant as a. So, we define v w h x divided by u 0 d square raise to the power 1 upon 3 is a constant a.

So, once we do that then, the governing equation becomes now, very simple d square c d eta square is equal to minus eta square minus a d c d eta and now if you see that we have since we have already observed the eta is a eta is a non dimensional parameter and a is also non dimensional parameter. So, if you look into the dimension of a let us look into the dimension of a what is the dimension of v w dimension of v w is meter per second and h is in meter x is in meter u 0 is in meter per second and d square is meter square per second.

So, it will be meter to the power 4 seconds square raise to the power 1 upon 3. So, this becomes meter per second and this will be second cube by meter cube to the power 1 upon 3. So, this will be meter by second into second by meter. So, it will be dimensional s. So, they will be cancelling out. So, therefore, let us make the numerator and numerators the variable c to non dimensional. So, we write c star is equal to c y c naught. So, therefore, d square c star d eta square is equal to minus eta square minus a d c star d eta. So, these become the governing equation of concentration profile within the mass transfer boundary layer.

Now what will be doing next will be in order to solve this equation we need to have two conditions on eta. Because it is of second order with respect to eta and will be formulating those two boundary conditions in terms of similarity parameter eta and if you remembered we had two conditions on y. So, will be converting those two boundary conditions into non dimensional parameter combined per similarity parameter eta and then will be solving these equation.

So, let us do that. So, at y is equal to 0.

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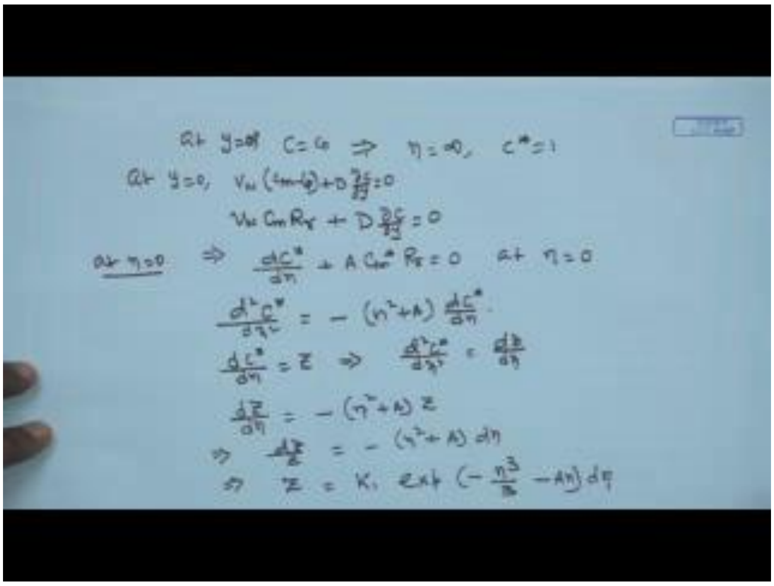

If you remembered c was equal to c naught what y equal to infinity c is equal to c naught at y is equal to infinity means eta is equal to infinity c star is equal to 1 at y is equal to 0 we had VwCm minus c p plus d del c del y was equal to 0 c it was in. In fact, c minus c p, but c at y is equal to 0 is nothing, but concentration of solute and membrane surface. So, in terms of similarity parameter if you really. So, if you if it do that it will becomes c m r r we can we can remove c p we replace c p in terms of r r definition of real retention as we have discussed in the last class d del c del y is equal to 0. So, therefore, in terms of similarity parameter, if you substitute the expression del c del y in terms of the similarity parameter after simplification this will turn out to be d c star s eta plus a c m star r r equal to 0 at eta equal to 0 because at y equal to 0 means at eta equal to 0.

So, we have formulated the two boundary conditions and we have the governing equation now the governing equation let us solve. So, d square c star d eta square is equal to minus eta square plus a d c star d eta let us say d c star d eta is equal to z. So, d square c star d eta square is nothing, but d z d eta. So, d z d eta will be is equal to minus eta square plus a z. So, d z by z is equal to minus eta square plus a d eta and one 1 integration will give you l n z. So, it will be z plus k 1 is equal to k 1 exponential minus eta cube by 3 minus a eta d eta and what is z z is nothing, but d c star d eta z is nothing, but d c star s eta. So, will be having 1 more integration in terms of with respect to eta.

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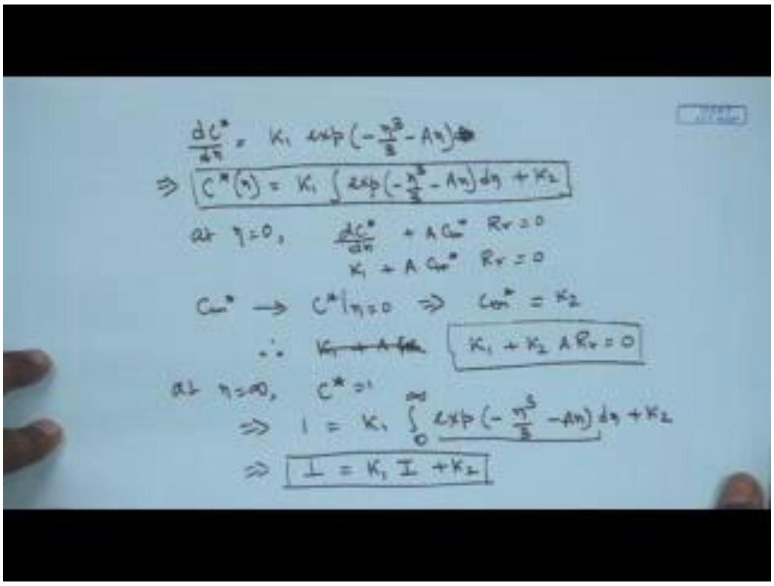

So, d c star d eta is equal to k 1 exponential minus eta cube by 3 minus a eta d eta. So, 1 more integration give you the expression of c star as k 1 exponential minus eta cube by 3 minus a eta d eta plus k two. So, these will be giving you the complete profile of concentration within the mass transfer boundary layer and now we evaluate the integration constant k 1, k 2 from the boundary condition at eta equal to 0 we have at eta equal to 0 we have d c star d eta plus a c m star r r equal to 0. So, what is d c star d eta d c star d eta is equal to these expression. So, these will be d eta there. So, you evaluate d c star d eta at eta equal to 0 at equal to 0 is exponential 0 1. So, it will be value of k one. So, this will be k 1 plus a c m star r r equal to 0.

So, let us find out what is the value of c m star at what is c m star c m star is c star evaluated at eta equal to 0. So, therefore, c m star is equal to at eta equal to 0 means we we evaluate these c m star at eta equal to 0 these limit 0 to eta. So, this will be from 0 to 0. So, this integration will be off. So, c m star will be nothing, but k two. So, therefore, k 1 plus a c c m star is k two. So, I will write it clearly k 1 plus k two a r r will be equal to 0. So, now, let us look into the other equation let us look into the other boundary at eta equal to infinity c star equal to 1.

So, therefore, 1 is equal to k 1 0 to infinity exponential minus eta cube by 3 minus a eta d eta plus k two. So, this will be 1 is equal to and these will be a definite integral varying from 0 to infinity if we know the value of a. So, if know the value of a this integral will be returning as some value. So, let us write down this integral as k 1 I plus k 2. So, I have two governing equation for evaluating k 1 and k 2 and you can get the expression of k 1 and k 2.

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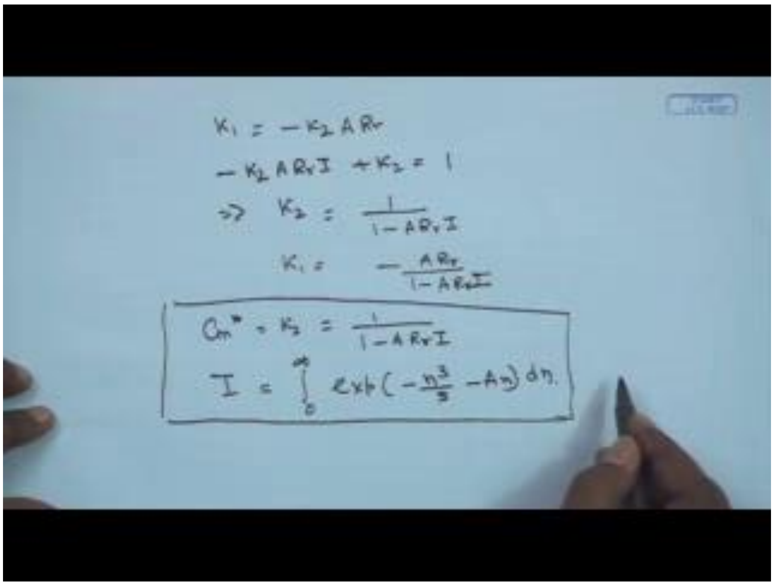

So, let us get that k 1 from the first 1 will be getting k 1 is equal to minus k 2 a r r and from the second equation you will be getting I write k I replace k k 1 as k 2 A R r I plus k 2 is equal to 1. So, k 2 becomes 1 over 1 minus A R r I and k 1 becomes minus k two. So, minus a r r divided by 1 minus A R r I. So, what we now required is that c m star c m star is equal to nothing, but k 2 and this k 2 is equal to 1 divided by 1 minus A R r I. Where the integral I is from 0 to infinity exponential minus eta cube by 3 minus a eta d eta. So, these will be. So, what we have done. So, these two relations are extremely important. So, whatever we have done is that we evaluated the mass transfer concentration profile in the mass transfer boundary layer in as c star is a function of eta, which will be basically combined variable y in terms of y index and eta is basically varying as y divided by x to the power 1 upon 3. So, we evaluated the concentration profile at y equal to 0 or eta equal to 0, in order to get the c m is as a function of k two r r etcetera I etcetera.

In the next class what will be seeing that how this expression can be hooked up with the transport loss to the porous membrane and then you can give get the system prediction perfectly.

Thank you very much.