Introduction to interfacial waves

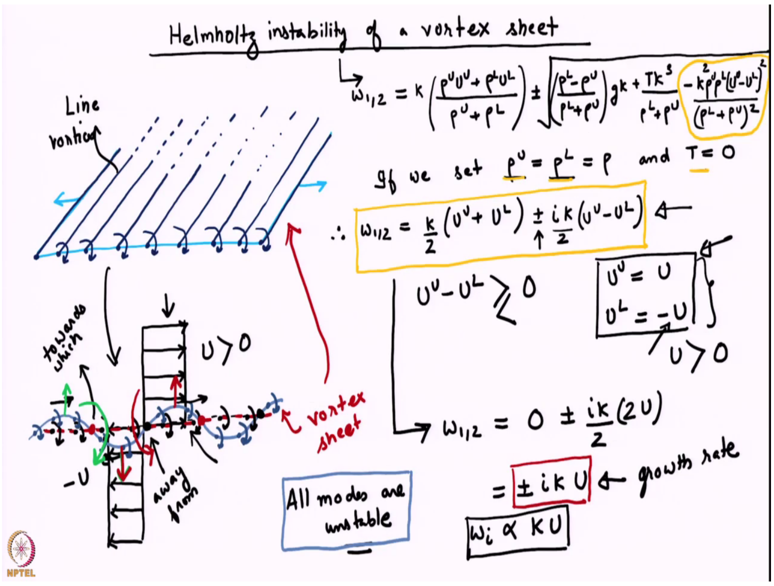
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 58
Helmholtz instability of a vortex sheet and summary

We were looking at waves on shear flows where we had analyzed the Kelvin Helmholtz instability as a model for wind wave generation. We had found that the model predicts a threshold wind speed where at least one wave number is on the threshold of instability and the threshold wind speed was about 650 centimeters per second.

We had mentioned that this is not a good enough model because experimental observation actually shows that both on the field as well as in the lab shows that the threshold wind speed is much lower and is about 110 centimeter per second. We now go on using the same dispersion relation we will now look at even simpler model which is the Helmholtz instability of a vortex sheet.

(Refer Slide Time: 01:00)



So, I have put here the full dispersion relation once again. And now we are going to do a number of simplifications, in particular we are going to say that we do not have 2 separate fluids. So, the density of the upper fluid and the density of the lower fluid are the same. So, rho U is equal to rho L is equal to rho. So, we just have a single fluid.

So, we consequently also do not have any surface tension. So, T is equal to 0. If your if we substitute these simplifications into the full dispersion relation, then the dispersion relation simplifies to that. Note that setting rho U is equal to rho L eliminates the gravity term there is no surface tension and consequently the what is inside the square root is always negative.

You can see that this term the first two terms inside the square root are 0 and this term is always negative. There is a negative factor outside and then the negative minus 1 multiplies a square. So, that is always positive and consequently this term that has encircled in yellow is always negative. Note that this is inside the square root and the first two terms are 0 because of these two approximations that we are making.

So, we are saying that there is only one fluid. So, there is a single density and there is no surface tension. So, that sets both the first two terms inside the square root to 0 and we are left only with the third term which is negative. So, consequently we find that there is always instability. So, the dispersion relation now simplifies to that expression.

The real part of the dispersion relation gives us the phase speed and it is telling us that if we put a perturbation k the real part of the phase speed tells us the speed with, which the disturbance propagates and it is the average of the two velocities. And you can see that there is an imaginary part also and the imaginary part actually depends on the difference between the two. So, as long as U U minus U L is greater than 0 or even less than 0 the difference is not equal to 0 you will always have instability.

You can see that if U U minus U L is greater than 0, then one of those plus minus there is a pair here will produce instability and if U U is less than U L, then the other will produce instability. So, in either cases as long as there is a difference in velocities between the two streams we are going to get an instability.

So, let us understand this instability a little bit more in particular because now we are dealing with a single fluid there is no interface as such. So, what does it mean? To have a difference between velocities between the two streams so, for that it is useful to take the; to take the to consider the special case where U U is equal to U and U L is equal to minus U, you can see that the difference. So, U let us say is greater than 0. So, this is what it would look like.

You know. So, look at the picture on the left. So, what I have put here are a series of line vortices. So, these are line vortices. So, this is my wave state velocity. So, in 2 D it would look like this. So, I have a 0 line z is equal to 0 and then the fluid at the top is moving with a positive U velocity capital U, which is greater than 0.

The fluid at the bottom is moving with negative U velocity. So, it is equal and opposite velocity. So, this is U and this is minus U. So, you can see that this is it in this particular limit, it is easy to understand the physical mechanism of this instability and what I have indicated at the figure at the top are basically line vortices. These are point sources of vorticity and you can see that I have driven I have drawn them in a way such that they induce a vorticity field which is they induce a vorticity which is clockwise.

So, you can see that this will induce a velocity field which will be exactly this. So, the fluid at the top will be moving with a positive velocity and the fluid at the bottom below the z is equal to 0 line will be moving with a negative velocity and there will be a jump in the derivative of velocity.

The derivative of velocity the z derivative velocity is related to the vorticity. So, you can see that there is vorticity at z is equal to 0, but nowhere else and so this is this line, which I am indicating here in red is basically what is known as a vortex sheet. Note that, so this is a vortex sheet and I have drawn this sheet in 3 D here, it is not inconsistent to assume vorticity although our analysis has been inviscid irrotational.

Note that the vorticity is confined only to the sheet in the base state. So, only the vorticity is at z is equal to 0 there is no vorticity anywhere else. So, in this case the vorticity is like a delta function which is present only at z is equal to 0. So, it is a very singular distribution of vorticity and we are interested in asking the question that if we perturb this sheet of vorticity using a Fourier mode of wave number k then what happens to the sheet of vorticity ok.

Now, this analysis: So, if you look at this dispersion relation and if you substitute U U is equal to U and U L is equal to minus U in this dispersion relation, then we obtain omega 1 comma 2 the first term just goes to 0 and then we have i k by 2 U U minus U L is just 2 U and so this is purely imaginary and this is just i k U.

So, this is telling us something very interesting this is telling us that any wave number any k basically grows. So, all modes are unstable all Fourier modes are unstable and this is basically related to the growth rate. The imaginary part of omega will give me the growth rate and that is proportional to K into U. The imaginary part of omega is actually equal to plus minus K into U. So, one of them will actually grow.

So, you can see that this is what is predicted by this model and this is telling us that if we put a sheet. So, let me draw if you put a vortex sheet and if you perturb it. So, let me put a Fourier mode here and this is telling that if you put a Fourier mode here, then independent of the wave number of the Fourier mode every Fourier mode grows in time and all Fourier modes are unstable.

So, all modes are unstable; is it possible to understand the physical mechanism of this instability? Yes. So, for that recall that we have chosen our line vortices to have a clockwise sense in the base state. So, I am going to draw the line vortices in 2 D they appear as points. So, that is the sense of vorticity. Now when I take this vortex sheet deform it in the shape of a cosine wave or a sine wave and that is indicated in blue here, then let me draw the line vortices in the perturb state and so on.

Now, you can see that in the perturb state. So, you can think of this as a vortex sheet, which is seeing a base state induced by itself. So, this is the base state induced by itself. So, in a linear approximation you can think of this as this base flow this vortex sheet will be carried by the base flow.

So, there will be a tendency of the base flow to advect the vorticity towards these kind of points. You can see that from the picture that the vorticity will be advected towards the points which I have highlighted in red. Note that I have highlighted the alternate zeros of the Fourier mode I have not highlighted consecutive zeros.

In the consecutive zeros. So, this is the other 0 which I have left out. So, this is a point towards which vorticity is advected and this is a point away from which vorticity is advected this is an effect of the base state velocity profile on the vortex sheet. And the effect is that, that the vorticity is advected towards the points in red and away from the points in black.

This causes an excess of vorticity at the points in red and a deficit of vorticity at the points in black. Note that the way I have drawn it this clockwise sense of vorticity by the right hand rule is a negative vorticity. So, this causes a negative excess of vorticity at the points in red and a deficit in vorticity at the points in black.

So, we have now I am going to use another color. So, at the place in red we have a net excess clockwise vorticity at the points in black we have a net deficit negative vorticity ok. So, at the points in red we have a net excess negative vorticity.

At the points in black we have a net deficit negative vorticity a net deficit of negative vorticity is equivalent to a positive vorticity. So, you can see that this thing in green that I have drawn will actually lead to a velocity profile which will give which will cause the amplitude of the vortex sheet to increase.

Whereas at the points in black there is a net deficit of negative vorticity that is equivalent to having a there is a net deficit of negative vorticity that is equivalent to having a positive vorticity. So, it will be like that and this is also going to cause the amplitude of the vortex sheet to grow.

So, at alternative points we have this mechanism, where alternative points have an excess of vorticity and again alternative points have deficit of vorticity and both of this lead to growth of the vortex sheet and consequently the instability, which is predicted here.

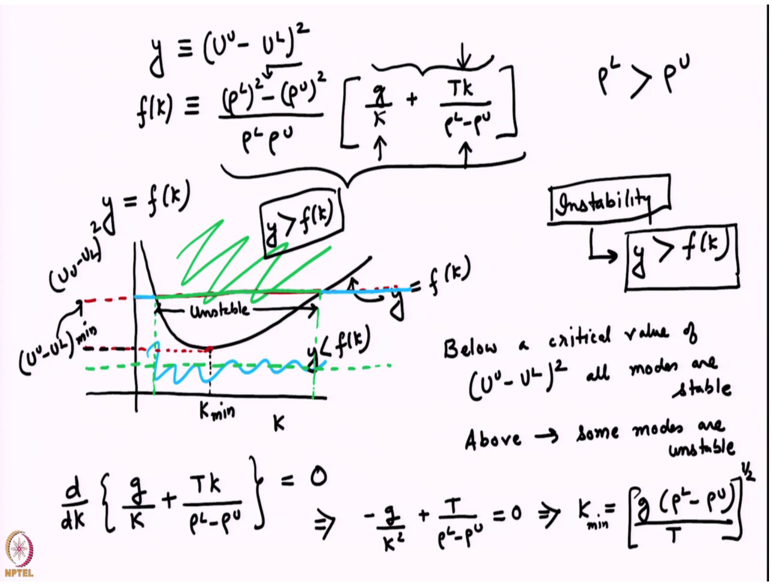
You can see readily that this mechanism is not specific to any Fourier mode, this will operate for any Fourier mode although the growth rates depend on the wave number of the Fourier mode. Consequently, this mechanism operates for all Fourier modes and thus all Fourier modes are unstable.

With this background it is now also possible to understand why did we get only a band of unstable wave numbers in the Kelvin Helmholtz model when we had two different fluids.

This particular analysis is the Helmholtz instability of a vortex sheet where there is only a single fluid.

So, there is no contribution from gravity no contribution from surface tension, but suppose we applied the same model that we had done earlier for two different fluids and we had found that there is a band of wave numbers indicated here in green which are unstable.

(Refer Slide Time: 13:40)



So, now, it is possible to understand why we have only the specific band of wave numbers which are unstable. You can see from the vortex sheet instability model when we have only one fluid that all Fourier modes are unstable. Now, suppose we have two different fluids the vortex sheet would also predict the same thing that all Fourier modes are unstable.

However, now we have surface tension and gravity also. Because of surface tension there is a penalty to be paid in terms of surface energy because of gravity there is a potential there is also a penalty to be paid in increasing the potential energy of the system. Recall that we are looking at when we are looking at the Kelvin Helmholtz model, we had found instability when the heavier fluid was below and the lighter fluid was above.

So, we are looking at a statically stable configuration. So, start from the Helmholtz vortex sheet model where all Fourier modes are unstable, if you put in gravity to them then it puts a penalty on gravitational potential energy. Remember that wave numbers or wavelengths which are sufficiently long are strongly affected by gravity.

So, consequently at very large at very large lambda or very small k we expect gravity to have a stabilizing effect ok. So, this is shown up here in this figure. Similarly, at very small wavelengths where surface tension is becoming important there is a surface tension penalty to be paid on the Helmholtz vortex sheet model. So, consequently not all wave numbers are unstable when we put in surface tension and gravity.

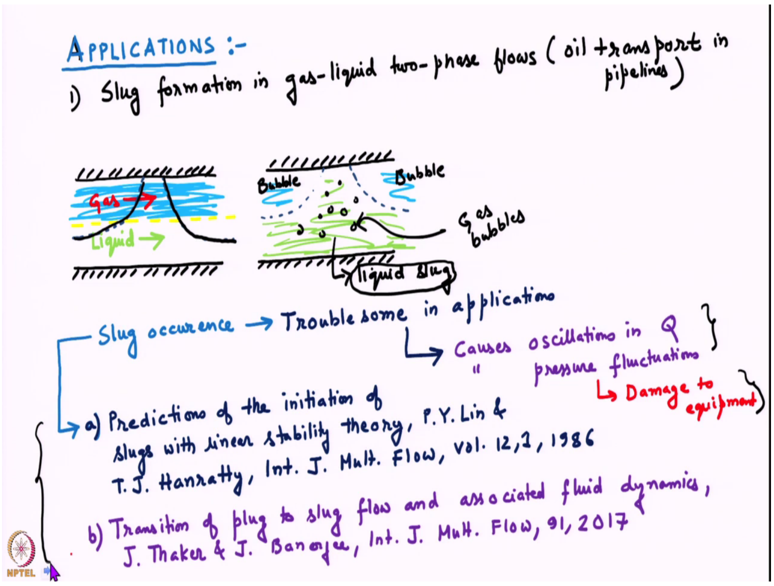
Sufficiently long waves are stabilized by the presence of gravity in the model sufficiently short waves are again stabilized by the presence of surface tension in a model. It is only in this intermediate regime of wave numbers where gravity and surface tension are roughly of the same strength, which cannot overcome the instability of the vortex sheet model where this instability of the or the Kelvin Helmholtz instability is to be found.

And so, consequently this is a rough physical explanation for why only these band of wave numbers are unstable this band of wave numbers are basically coming from the Helmholtz vortex sheet model whereas, the ones which have got stabilized are stabilized either because of surface tension or because of gravity.

So, this is what we wanted to discuss about the Helmholtz instability of a vortex sheet, its physical understanding of why all wave numbers are unstable and then what happens when we put surface tension and when we put gravity into the model and why does it cause some wave numbers to stabilize while others continue to remain unstable.

There are let us ask what are the engineering applications of this? There are many engineering applications, I will mention only two of them. Firstly, the Kelvin Helmholtz model is of great relevance in understanding slug formation in gas liquid to phase flows. I have mentioned this in my introduction to the course that this is relevant to oil transport in pipelines where water can be used as a lubricant.

(Refer Slide Time: 16:38)



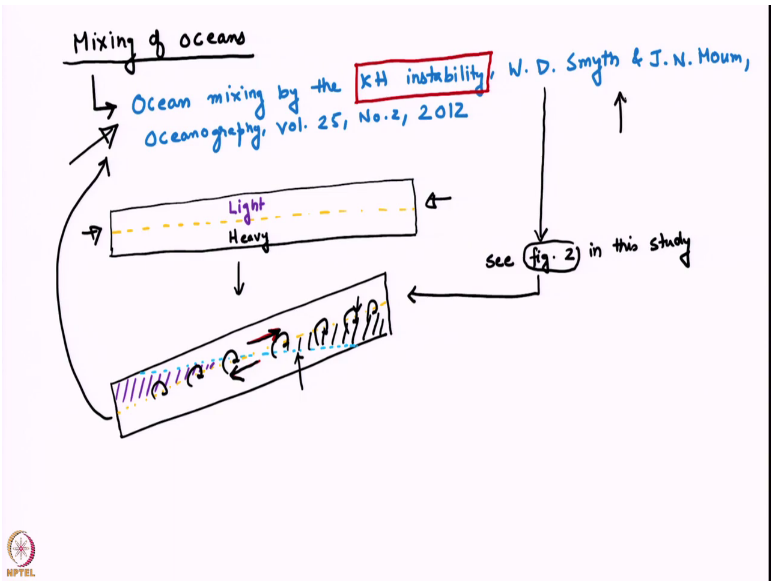
So, I have plotted here for example, a situation where we have a co-flowing situation where we have gas and liquid flowing in a channel, the gas is at the top the liquid is below. The line in yellow indicates the interface between the two in the base state. One can do an analysis of this and this also becomes unstable and waves are formed and these waves can become large amplitude waves and the liquid can actually go and occupy the entire cross section of the channel.

So, I have indicated this here and this is what is known as a slug. So, you can see that the liquid has gone and occupied part of the entire cross section of the channel. There can be wave breaking also which leads to aeration and formation of gas bubbles in the channel. So, this is a liquid slug with gas bubbles in it.

This is very important this is a troublesome feature in the industry and consequently one needs to study, there have been a lot of studies the formation of slugs can cause oscillations in the volume flow rate and which in turn can also lead to pressure fluctuations as well as cause damage to equipment.

So, it is of industrial relevance to understand the origin of slug formation and its relation to waves and instability in gas liquid to phase flows. I have outlined two papers here those of you who are interested in understanding more about this can go and read these two papers ok one of them is a very recent study.

(Refer Slide Time: 18:08)



The second area where many of these things find applications the Kelvin Helmholtz instability in particular finds applications is mixing in the ocean ok. So, if you are interested you can go and read this paper which is a nice introduction to how Kelvin Helmholtz instability leads to mixing in the ocean you can go and read this paper.

Also look at figure 2 in the study, where it is there is a nice demonstration of the Kelvin Helmholtz instability there are many such demonstrations on the internet. So, one typically takes such a tube of finite cross section and which are closed at the two ends. And then you have a light and a heavy fluid which is in static equilibrium.

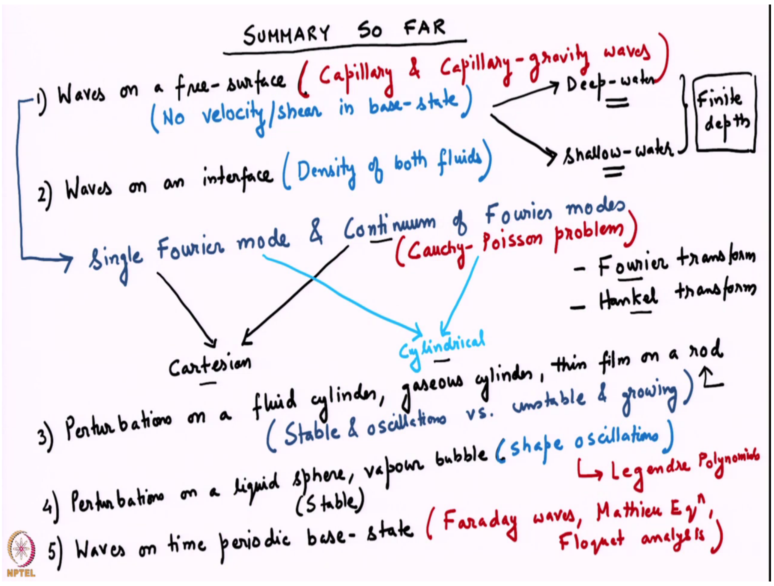
And then if you suddenly turn or tilt the tube or the channel, then you can see that those regions in which I have indicated in hash in black and in purple, you can see that the black region was originally occupied by the heavy fluid, the purple region was originally occupied by the lighter fluid in the base state.

However, now because the interface has to remain horizontal. So, the interface was earlier the yellow interface here and after the tube is suddenly tilted the interface will become like that in blue. So, you can see that those hash regions will have to be exchanged. So, there will be heavy fluid which will occupy the region in purple and there will be light fluid which will occupy the region in the hash region in black. This effectively causes a net flow in this direction.

So, the lighter fluid moves from left to right the heavier fluid moves from right to left and this causes instabilities to develop on the interface. There are nice pictures of visualization of these instabilities and how they lead to mixing in that thin layer between the two fluids. This is of great relevance in of mixing in oceans you can read more about it in this paper.

So, now, it is a good time to summarize what we have covered so far in the second half of this course on interfacial waves. So, we started with waves on a free surface. By free surface I mean those analysis where the density of the upper fluid was neglected if we neglect surface tension then we had set pressure to be equal to 0 at the free surface. So, that is why it was a free surface we looked at capillary waves we looked at capillary gravity waves.

(Refer Slide Time: 20:35)



The important feature here was that there was no velocity in the base state we looked at two approximations the deep water and the shallow water approximation. We had also found that the dispersion relation in both the approximations and in the general case of finite depth we had also found the dispersion relation for both capillary as well as capillary gravity waves.

We have looked at what simplifications happen to the dispersion relation in the limit of deep water as well as shallow water. Recall that the deep water limit is one where the depth of the base state pool is much much greater than the wavelength of the perturbations which are placed on the free surface.

The shallow water limit is the opposite limit or it is also known as the long wave limit. We then went on to waves on an interface where we model the density of both the fluids and we found corrections to the dispersion relation both in finite depth as well as in deep water, when we took into account density of both the fluids.

We looked at initial conditions, which involved a single Fourier mode we also looked at a continuum of Fourier modes which we called as the Cauchy Poisson problem. We have discussed the Cauchy Poisson problem in some amount of detail we have in particular looked at the Cauchy Poisson problem in Cartesian as well as in cylindrical base state geometries.

In Cartesian based state geometries when we looked at the continuum of Fourier modes, we required use of Fourier transforms and eventually our answers were written as inverse Fourier integrals with the integrand containing the Fourier transform of the initial condition.

We also looked at some asymptotic features of those integrals using the method of stationary phase. In cylindrical geometry we used the counter part of the Fourier transform namely the Hankel transform. Here also we use the method of stationary phase recall that using this method led us to the concept of group velocity or the velocity of the amplitude of the envelope.

Now, subsequently we looked at perturbations on a fluid cylinder we went on to another base state geometry which was cylindrical, we looked at waves on a fluid cylinder which we modeled as being infinitely long in the axial direction. We replaced it with a gaseous cylinder where there was a gas inside and liquid outside and we ignored the we considered treated it as a free surface problem and we only solved for the liquid outside and assuming the gas inside to be exerting constant pressure.

We also looked at thin films on a solid rod in all the three cases of this problem, we found instabilities as well as stable base states, which are stable to certain perturbations as well as unstable to certain other perturbations.

Recall that the criteria for which wave numbers are unstable remained the same for all the three. In this particular case we ignored the in this all the three cases fluid cylinder, gas cylinder as well as thin film on a rod we ignored gravity and these were pure capillary waves or pure capillary perturbations.

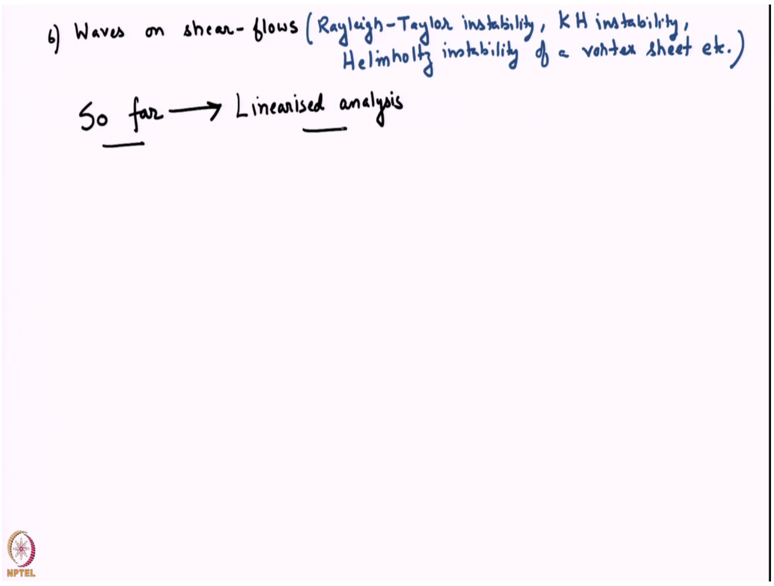
We then moved on to studying perturbations on spherical geometries in particular we looked at perturbations on a liquid sphere here too the restoring force was only that of surface tension and we looked at liquid spheres we also looked at vapor bubbles the 2 limits and we in particular we looked only at shape oscillations. We did not allow the volume of the sphere to change this was done using Legendre polynomials. We have looked at Legendre equation as a part of this analysis.

Next, we went on to the next level of complication where our base state did not contain any velocity, but was time dependent. In this case we had a pool of liquid, which was being shaken up and down with a certain amplitude and a certain frequency. Here also the base state in the base state the interface is flat the pressure field is hydrostatic, but the effective value of the acceleration due to gravity becomes a periodic function of time.

A linearized analysis led us to the Mathieu equation, we looked at Floquet analysis we understood the Floquet theorem and using the Floquet theorem derived earlier in the course, we analyze the Mathieu equation and we found that there are tons of instability where the response of the free surface to perturbations can be oscillatory and it can grow in an exponential manner in particular we found that there are this harmonic tons and sub harmonic tons.

Harmonic was when the oscillatory the frequency of the oscillatory response was the same as that of the forcing frequency. Sub harmonic was where the oscillatory the frequency of the oscillatory response was one half the forcing frequency. So, this was this led us to Mathieu equation Faraday waves and Floquet analysis.

(Refer Slide Time: 25:24)



Next, we looked at waves on shear flows. This is the last topic that we covered here we looked at a dispersion relation which was rather complicated, we looked at various limits of the dispersion relation in particular we found that if we one limit of the dispersion relation reduces to all the problems that we have studied earlier. Or in other words the problem number 1 waves on a free surface or waves on an interface.

Then we also looked at Rayleigh - Taylor instability where we exchange the heavier fluid with the lighter fluid. So, the heavier fluid being above and the lighter fluid being below we found is expectedly unstable; however, the interesting feature here was that that some of the modes can be stabilized by the presence of surface tension.

We then looked at waves on shear flows where the base state actually had a velocity and there was a velocity discontinuity at the interface. So, this is what we mean by shear. So, we looked at the Kelvin Helmholtz instability and we found that the Kelvin Helmholtz instability even when you have a heavier fluid underlying a lighter fluid, it stick still predicts an instability.

There is a band of wave numbers which are unstable when the difference between the speeds of the two streams exceeds a certain threshold. We use that for analyzing the minimum speed at which wind can form waves on a calm body of water and we found that it gives a prediction which does not agree very well with experiments.

Next, we try to understand the Kelvin Helmholtz model a little bit better by looking at Helmholtz instability of a vortex sheet. Here we said that we do not have two fluids we have a single fluid phase and then there is only a single density and consequently there is no surface tension and so the first two terms inside the square root of the discriminant drop off and we are left with just the third term which turns out to be negative.

So, here we found that all Fourier modes are unstable, we looked at the physical mechanism of this instability and that physical mechanism when we went back to the Kelvin Helmholtz model using this understanding of the physical mechanism, we could also understand why only a band of wave numbers are unstable in the Kelvin Helmholtz model.

In each of these problems that I have listed and that we have covered so far, we have also highlighted specific engineering applications. There have been a large number of varied engineering applications of everything that we have studied so far. I have also given you pointers to the literature for those of you who are more interested in studying making a deeper study of specific topics.

So, so far we have covered all of this and the common theme that runs across all the 6 points that I have just summarized is that that each of them have been linearised analysis. So, far we have not covered any interfacial wave problem or any free surface wave problem which is non-linear all our problems. So, that we have studied so far have been linear problem.

We now go on to the, we will now go on to the last part of this course where we will do a non-linear problem. We will look at something called a stokes wave and we will find what does non linearity, what are the new features that non-linearity brings on to a free surface wave.

We will use the Lindstedt Poincare technique that we had studied early on in this course for understanding or for analyzing the nonlinear pendulum, we will find application of the Lindstedt Poincare in analyzing the stokes wave and finding out the amplitude correction to the dispersion relation. We will do this in the next video.