Introduction to interfacial waves

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Lecture - 28
Introduction to waves on an interface

We have completed the first half of the course, so it is a good time to summarize what we have learnt so far. We started the course with looking at linear spring mass systems single degree and multiple degrees of freedom which are coupled to each other. It let us to coupled ordinary differential equations which were linear, we analyzed the, we also looked at the stability of the equilibrium or the base states we perturbed it about the base state and it gave us the coupled linear set of differential equations.

We analyze those equations looking for normal mode kind of solutions. It led us to an eigenvalue problem and we found that the eigenvalues are related to the allowable frequencies in which the system can vibrate and the eigenvectors contain information about the shape of the modes of oscillation.

We then looked at the n going to infinity. The number of degrees of freedom going to infinity or the continuum limit. It let us to a wave equation. We looked at the wave equation subject to fixed fixed boundary conditions in two kinds of coordinate systems, Cartesian and cylindrical axisymmetric.

In both cases we analyzed it using the method of normal modes and we once again found that it leads us to eigenvalues which contain information about the allowable frequencies of in which the system can vibrate and eigenfunctions here which contain information about the shape of the modes of oscillation. In this case there were a countable infinite number of such eigenvalues and eigenfunctions. We then moved on to a non-linear problem; the non-linear pendulum. We introduced ourselves to elliptic functions.

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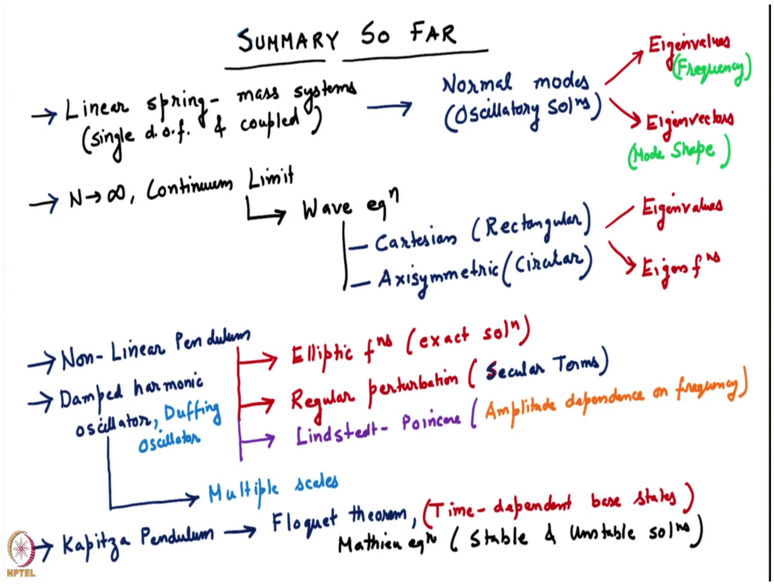

We found that the non-linear pendulum with just an initial angle perturbation and zero velocity can be solved in terms of Jacobian elliptic functions. We found the exact solution. We then also introduced perturbation techniques, in particular we tried to solve the pendulum; the non-linear pendulum using a regular perturbation technique and we found that it leads to secular terms.

We then learned that the using the force phase portrait, we learned that the qualitative behavior is that the time period of the oscillatory motion of the pendulum actually depends on the initial angle of a feature which is not captured in linear theory. So, for that we looked into an alternative perturbative technique, the Lindstedt Poincare technique where we eliminated the secular terms and in using this we found the how the frequency of the pendulum depends on its initial perturbation angle.

We then continued with more linear and non-linear oscillators. We looked at the damped harmonic oscillator, the duffing oscillator, we solved both using the method of multiple scales and we found the same behavior that the frequency depends on the initial perturbation amplitude.

Last, we studied the Kapitza pendulum this is an example different from the remaining topics that we had looked at so far. In this the equilibrium state or the base state is time dependent, because in the frame of reference in which the pendulum if you oscillate along with the pendulum then we find that the acceleration due to gravity becomes an oscillatory function of time.

We found that this leads to a, this equation still has the same two fixed points. We analyze the stability of the lower fixed-point by linearizing about the lower fixed point it led us to a Mathieu equation. The Mathieu equation is an equation with periodic coefficients. We learned Floquet analysis which allows us to analyze and make qualitative inferences about the nature of solutions to such equations with periodic coefficients.

We found that the Mathieu equation contains in the in the space of alpha and beta, which are the two parameters which appear in the equation. There are tongues which appear, there are tongue shaped curves which appear and if you are inside the those tongues one has exponential growth, but it grows in an oscillatory manner.

And the frequency with which the solution oscillates, it depends on the tongue in which one is in. In general there are also stable oscillatory solutions, but they may or they may not be periodic. So, this is what we have covered. So, far we have looked at a variety of mathematical techniques and now we are ready to start with the main topic which is the focus of this course.

Introduction to Interfacial Waves. We will see that many of the things, almost everything that we have learnt so far will be utilized in interfacial waves. So, let us start with Introduction to

Interfacial Waves. As the name suggests, the first thing that comes to our mind when we talk of waves on an interface is probably the open ocean.

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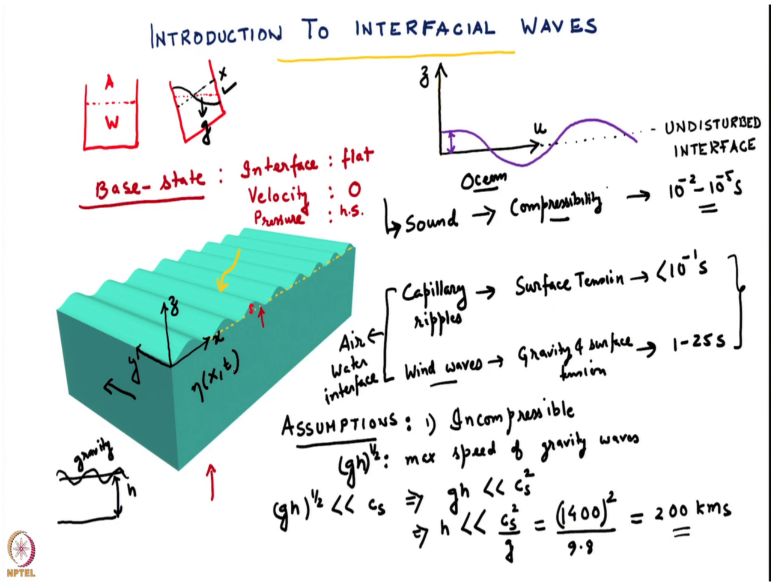

So, this is a cartoon of an ocean and we have made a very regular kind of a perturbations. This is actually a Fourier mode, a sine or a cosine wave which has been imposed. Now, we will in this course we will learn how to analyze these kind of waves both in the linear and the non-linear regime.

As I have told you earlier, you will find that here the equations of motion are all partial differential equations. If we look for small amplitude perturbations then we can linearize them. We will have to identify the base state clearly in every problem that we solve and then we have to linearize about the base state, that will lead us to a coupled set of linear partial differential equations with linearized boundary conditions and initial conditions.

We can once again use the method of normal modes to analyze those systems and it will lead us to a frequency relation or a dispersion relation. So, let us learn how to look at interfacial waves. So, as you will see as you will notice from experience, if you take a glass of water. So, let us say this is the glass and this is the interface between air and water. So, this is air and this is just water.

If I very gently tilt the glass, so as to be careful not to impart as far as possible not to impart any kinetic energy to the water, then we know from experience that the interface always maintains its orientation. The interface for example will not look like this. The interface is not going to look like this it is going to continue to look like horizontal.

So, we know that the natural or the equilibrium tendency of the interface is to remain perpendicular to the local direction of the acceleration due to gravity vector. This you can understand from the fact that this mass has a certain gravitational potential energy. If you disturbed the mass.

So, if I impose a perturbation, so if I for example, impose a perturbation of this form on the meniscus or the on the interface, one is led to oscillations if you these oscillations will typically damp out in time and after sometime the in interface will once again become flat.

It is easy to understand why these oscillations occur. Any perturbation on the surface actually increases the gravitational potential energy of the system and so the system tries to oscillate in an attempt to bring down its gravitational potential energy. You can think of it as being analogous to a spring mass system. So now, this is the reason why under the influence of a gravitational field if you have an interface and if one introduces perturbations to the interface it sets up oscillations.

There are many reasons why such oscillations can occur. You can have wind blowing, you can have the surrounding fluid which imposes shape perturbations on a finite amount of mass. We will look at various such examples of perturbations and how do they lead to oscillations or do they lead to instability as we go along.

So, let us now look at some examples. So, we are mostly going to be concerned with waves on this interface, but it is a just to get a sense of how the time period of these waves compares to for example, the time period of another wave that we are all familiar with, sound waves, let us write down some representative numbers.

So, for example, in the ocean. So, I am comparing sound waves with capillary waves. So, it is called capillary ripples. So, capillary ripples are basically waves which occur because there is surface tension between air and water. So just as gravity acts as a restoring force surface tension can also act as a restoring force. If you have a spherical mass of liquid, surface tension tries to keep it spherical.

Any deformation increases the surface energy of the system and produces a restoring force which tries to bring it back to being spherical. In the process if you inject a finite amount of potential energy one can get oscillations. We will look at such oscillations later. So, these are capillary ripples, then one can also have waves generated by wind and so on.

So, just to get some numbers, so you can have capillary waves capillary gravity waves. So, typically the restoring force here is the compressibility of the medium. Capillary ripples as the name suggests, the restoring force is surface tension. Wind waves is a combination of gravity and surface tension.

We will be mostly be looking at these kind of waves where the restoring force is surface tension or a combination of gravity and surface tension. We will start with gravity driven waves first gravity driven surface waves. Just to give get an idea of how does the time period of these waves vary.

So, these are typical numbers in the ocean. So, we are looking at sound propagation for example, in the ocean. So, in the ocean the typical period of sound waves in the interior of the ocean is about 10 to the power minus 2 to 10 to the power minus 5 second. For capillary ripples it is less than 10 to the power minus 1 second. This is about 1 to 25 seconds ok, but in turn these two are much bigger than typically this the sound waves ok. So, as you can see this is typically visible on the air water interface.

But, in many industrial situations as I had mentioned in the introduction to this course, you can have a situation where you have flow being taken through a pipeline, heavy viscous oils, water is can be used as a lubricant and so you can have a thin layer of water which adheres to the wall and then the inside part is a thick viscous oil and so you have an interface between oil and water..

Whatever we study here is also going to apply to those kind of situations. So, we will think about air water oil water and these kind of situations. So, now before we proceed with our mathematical analysis let us make a number of simplifying assumptions. As you will see the equations of motion with where there is a unknown interface. You can see that in these kind of problems the shape of the interface is an unknown, we have to determine it as a part of the solution.

So, there is an additional unknown in the problem, and so this typically leads to initial boundary value problems which are mathematically quite difficult. So, one has to make a number of simplifying assumptions. You will see that although these assumptions appear very dramatic, but still there is enough physics left in the resultant equations which are obtained after making these assumptions.

So, let us make the assumptions. And we will start with some of these assumptions and as we proceed along in the course, we will relax some of these one by one to understand what happens when each of them are relaxed. So, our first approximation is going to be that we are going to assume that the medium is incompressible.

So we are going to eliminate sound waves. Because sound waves as I said, comes from the compressibility of the medium. So, we are going to eliminate sound waves from a system ok, we are only interested in waves on the interface. When is this justified? So, you can see that we will see later that g into h.

So, if you have a layer liquid layer of depth h then and we are looking at gravity waves, where gravity acts as the restoring force, then g h is the typical upper limit or speed that these waves can take, g into h g is the acceleration due to gravity. So, this incompressibility assumption is, so g h is typically, so g h to the power half is the maximum wave speed.

We will see this later. And, you can see that if this is the maximum wave speed then the incompressibility assumption is going to be good if my maximum wave speed g h to the power half is much much less than my sound speed. Remember, that if we set the compressibility of the medium to 0 then the sound speed goes to infinity. So, if my maximum speed is much much less than sound speed then on that scale, I can treat the sound speed as infinite.

So, this implies that this is going to be a good assumption if g h is less than c s square, c s is the sound speed we know the typical values of sound propagation speed in air and water and this implies that this imposes and a limit on the depth h we can calculate that limit from this relation this is c s square by g.

So, if the thickness of the layer on which waves are propagating, which we are considering is much much less than c s square by g then we are justified in neglecting compressibility of the medium and ignoring sound waves. Let us calculate how much is this. So, in water I will approximately take the speed of sound to be about 1400, slightly more than that and g is 9.8, if you work this out this approximately evaluates to 200 kilometers.

So, this tells me that as long as the layer on which the interfacial waves propagates. So, I am looking at waves on the interface. So, as long as the layer thickness is less than 200 kilometers we are absolutely fine in justifying the speed of sound, saying that the speed of sound is infinite compared to all other speeds in my all other wave speeds in our system.

You can see that this is much much greater than the average depth of the ocean. So, even when we are looking at ocean waves, it is perfectly justified to take the medium as incompressible when we want to analyze surface waves on the ocean ok. This is of importance when we look at wind waves ok. So, this justifies our first assumption.

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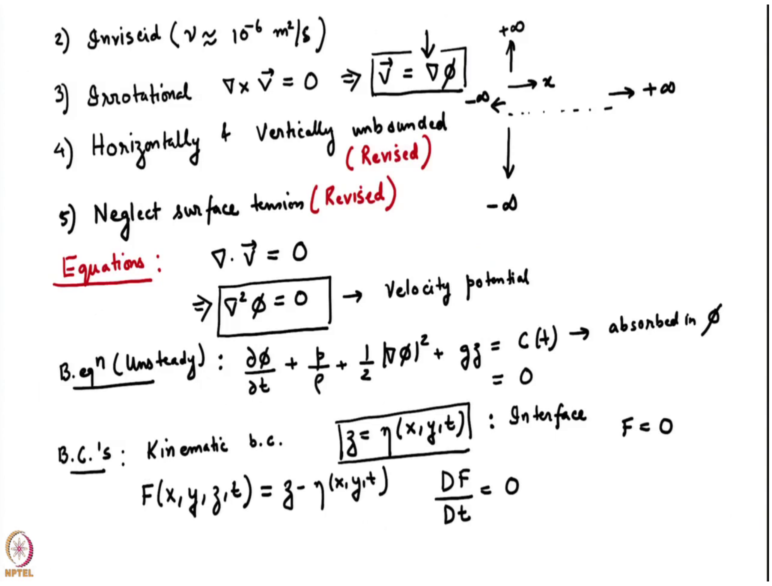

Let us go to the next we are also going to assume that the medium is inviscid. In a typical air water kind of a situation this is going to be not a bad approximation. The kinematic viscosity of water is approximately 10 to the power minus 6 meter square per second. This is a small value and we will see that in most regions of interest it is saying that the fluid is inviscid is a good approximation.

Of course, using such kind of theories one cannot estimate damping if one is interested in damping of waves then one cannot estimate damping. So that will be a limitation and we will discuss it towards the end of the course. Then the 3rd approximation that we are going to make is irrotational. From your basic fluid mechanics you will recall that a flow is called irrotational if the curl of velocity is 0.

The curl of velocity is defined as the vorticity in the flow, it quantifies the local rate of rotation. So, we are going to assume that the flow is irrotational. This automatically implies that we are not going to analyze boundary layers which might be created in the flow.

Such boundary layers are typically thin they arise either near the interface itself or near walls. So, we will be making some errors near the interface and near the walls, but in the bulk of the fluid our theory even when it is irrotational will get the approximation correct.

Now, this also implies that v can be written as gradient of some potential. You can see that curl of gradient of some potential is 0 for regular function phi. We are going to say as of now that our domain is horizontally and vertically unbounded. So, if this is our interface, I will say that this is the x direction, so it goes from plus infinity on the right to minus infinity on the left and it will go to minus infinity below and plus infinity above.

So, I am not going to consider as of now any confining boundaries. We will revise this approximation once we have finished doing our first derivation. We will revise this and introduce boundaries and see what effect do boundaries have. Once again, we will we are going to initially neglect surface tension.

This also will be revised and then in later analysis we will look at the effect of surface tension alone as well as together with gravity. But the first example that we are going to do we will keep it simple and we will ignore surface tension and just keep gravity as the restoring force.

So, with those approximations what are the equations which govern our system. Let us first quantify the base state, say as I showed in the last slide. So, in our base state the interface is flat. So, this figure in blue represents a perturb state. The base state will be indicated by that blue by the dotted yellow line ok.

The interface would be perfectly flat, that is what it would do if there was no perturbation. We are going to introduce some perturbations, because of those perturbations the interface is going to move up and down as a result there will be some velocities induced.

So, in our base state the velocity field is 0 in the base state. So, quiescent fluid interface is flat, no motion inside. The only quantity which is going to be non trivial in the base state is the pressure field the pressure is going to be hydrostatic. So the pressure is going to be hydrostatic. There will be a linear pressure profile as we go deeper into the pool.

So, with those approximations, this is our base state and we are going to introduce perturbations about our base state. We are going to consider those perturbations as small amplitude perturbations and then we will proceed to find what is the oscillatory behavior about the base state.

So, let us proceed. So, what are the equations? So, because we have assumed that velocity is gradient of some scalar function phi, we know that the medium is incompressible that was our first approximation. So, we know that when density is a constant, if the medium is incompressible density is a constant.

So, the continuity equation reduces to this. Note that this is an unsteady flow, but density is treated as a constant, so this is still the continuity equation now this immediately tells you that if you put v is equal to grad phi in this equation it leads you to the Laplace equation.

So, a lot of things that we will do in this course will involve solutions to the Laplace equation, in various kinds of geometries. We will start with the Cartesian geometry first. So, this is our Laplace equation which will govern the velocity potential. As I said before, in our base state the flow is quiescent, so the velocity potential is 0 in the base state.

However, when we introduce perturbations at the interface it will cause the interface to move up and down that will in turn cause velocities to develop in the fluid inside. This velocity potential will govern those velocities. In addition, we also have pressure. So, we are going to use the Bernoulli equation.

The unsteady Bernoulli equation which will tell us how does the pressure fluctuate in time how does the pressure field fluctuate in time. Here again, because of the perturbations there will be a perturbation pressure field and we will see that the perturbation pressure field does not vary linearly with depth unlike a hydrostatic variation.

So, the unsteady Bernoulli equation is just del phi by del t plus p by rho plus half grad phi square plus g z is equal to some constant of time which can be typically taken to be 0, because it can be absorbed the c of t can be absorbed in phi. If it if you can write c of t as d by d t of some function f which is a function of time then you can redefine your velocity potential as phi minus f ok, that in general will not alter the velocity field.

So, in other words we are going to take this constant to be 0 because this we will assume has been absorbed in the velocity potential. I urge you to go back and look up the derivation of this equation. You start from the Euler equations and then there is a vector identity which has to be used on the non-linear term in the Euler equation that will lead you to two terms. When you say that the flow is irrotational one of the terms will go to 0 and the other term will be half grad phi square and then that will lead you to gradient of a set of whole quantities.

So, it will actually be the gradient of whatever I have written on the left of the Bernoulli equation and then when you integrate that you will get a unknown function of time which is c of t ok. So, this is the equation. So, if we solve for the velocity potential then this is the equation that will govern the evolution of pressure.

We will see later that in the way we will solve things we will not worry about the pressure field initially. There will be the Laplace equation and there will be some boundary conditions. One of the boundary conditions will arise by taking the Bernoulli equation and applying it at the interface.

What are the boundary conditions? So, the first boundary condition is the Kinematic boundary condition. The Kinematic boundary condition basically says that the interface is a material surface fluid particles which remain on the interface always remain on the interface, they do not go into the bulk. So, the interface basically remain is basically a material surface.

So, we need to so we need in some sense an additional equation of mass conservation which says that there is a surface which separates the two fluids and there which is a material surface. So, let us say, so you can refer to this picture. So, let us say that this is the.

So, in this picture you can see that the interface is given by this wavy surface. Let us say that the equation of this interface is given in implicit is given in explicit form to us. So, the equation of the interface is given as z is equal to eta of x comma y comma t. What is eta? Eta is just this perturbation and.

So, you can see that eta is how much the interface deviates from the flat state; obviously, that deviation depends on where one is in the on the horizontal plane and so eta is a function of x y. And because the interface will in general fluctuate in time it is also a function of t note that by definition eta is not a function of z. So z, so when we come to z is equal to eta then we are on the surface.

So, this is the surface, this is the interface. So, because this is a material surface, one can always construct. So, this is a z is equal to eta is the interface. So, let us construct a function f, which is a function of x y z and t, such that F has a constant value everywhere on the interface. You can see that if I define F as z minus eta x comma y comma t then F is equal to 0 everywhere at the interface, because the interface is given by z is equal to eta.

So, when z is equal to eta z minus eta is 0 and so everywhere on that blue surface which I showed you earlier capital F is 0 no matter where you are on that surface as long as one is on the surface exactly at any point on the surface the value of F is 0, f has been defined in such a manner.

And, this is true at all times. So even when the interface deforms at every point, F remains 0 ok. So, the kinematic boundary condition basically says, that because the interface is a material surface and F has a unique value associated with it which is in this case is 0, so DF the material derivative of F is 0. Let us work out the consequences of this equation.

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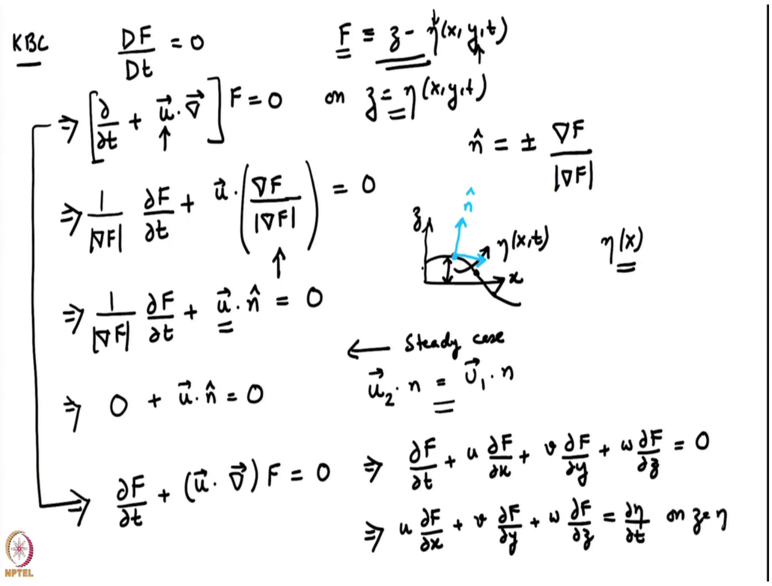

So, we will have DF by Dt is equal to 0. Remember, F is defined as z minus eta of x y t. So, if I apply the d by dt operator the d by dt operator is del by del t plus u dot gradient of F is equal to 0, and this is true only on z is equal to eta. Why do we have to mention this?

Because, this velocity in general is dependent on where you are in the fluid, and so, in this equation this velocity has to be evaluated at z is equal to eta. And so we are going to use one form, but let me show you that this this equation is consistent with what we would know if the flow was steady. So you can see, that I can divide throughout by mod grad F and this becomes one by mod grad F del F by del t plus and this just becomes u dot grad F divided by mod grad F.

Why do we did we do this? We know that the unit normal to the interface is given by grad F by mod grad F. This is the instantaneous unit normal on the interface. So at any point on the interface, if I want to calculate the unit normal which is normal to the local which is a local normal to the interface this is the formula. The plus minus indicates that it can either be from the pointing outwards or pointing inwards, they are just 180 degree in the opposite direction.

So, this implies, I can write this as mod grad F del F by del t plus u dot this is the local unit normal to the interface. Suppose the flow was steady, so my interface, so let us say, I am representing the interface as a 2D which we are going to do shortly. So, the interface is not a function of y.

So, in the y direction there is no variation recall that the y direction is this direction is this direction. So, I am just saying that my eta in general is a function of x and t, there is no y direction variation as you can see here, we have done that approximation while plotting this figure.

So, if I just make that approximation, you can immediately see that I can draw a 2-dimensional diagram instead of three dimensional and at any x, so this is x this is z and and this is eta as a function of x t. So, at different at every x you have a certain eta. Now, suppose the interface was steady. So, eta would just be a function of x, it is not oscillating in time or fluctuating in time. So, in particular F del F by del t would be 0, because the only t dependence of F comes through eta, so this is 0.

So, for steady case, this is 0 and we would just have u dot n is equal to 0 or in other words this u I recall I told you is the u at z is equal to eta or the u on the interface, the u at every point of the interface. So, this is just telling us that the local, so for a steady interface this is just telling us.

So, suppose I am at this point and this is the unit normal, this is the n direction the local unit normal, this is just telling us that u is tangent to the unit normal. This is what it should be because if the interface is a is does not fluctuate in time then the interface becomes a stream line.

So, at every point the local velocity the local direction of the velocity vector is tangent to the stream line. So, this is just telling us that the local velocity vector becomes a stream line. You can also see that if we have two fluids, this this would be in general given by, so if I have two fluid there would be one velocity coming in from the above fluid and another velocity coming in from the below fluid and in order for the fluids to remain in contact their normal velocities have to be continuous.

So, u 2 dot n would should be equal to u 1 dot n at the interface the u 2s and the u 1s have to be representative of the values of the two fluids at the interface. So, this is just continuity of normal velocities. So, you can see the kinematic boundary condition, so we are discussing the kinematic boundary condition I will call it KBC.

So, the kinematic boundary condition has various forms, but basically it is some kind of a mass conservation equation which basically ensures that the interface is a material surface. And unless one has phase transitions or mass transfer happening there is no mass exchange between the two fluids ok.

Now the form of the kinematic boundary condition that we will use is obtained from here ok. So, we will write it as del F by del t plus u dot grad of F is equal to 0, and one has to remember that this is true only at z is equal to eta. Now, with that in mind we will now decompose it into a coordinate system or x y z coordinate system.

So, we will have del F by del t plus u del F by del x plus v del F by del y plus w del F by del z is equal to 0. From this definition of F is equal to z minus eta you can see, that del F by del t is minus del eta by del t, I can shift that del eta by del t on the right-hand side and make it positive. So, this means that this is u del F by del x plus v del F by del y plus w del F by del z is equal to del eta by del t on z is equal to eta. Let us work on this further.

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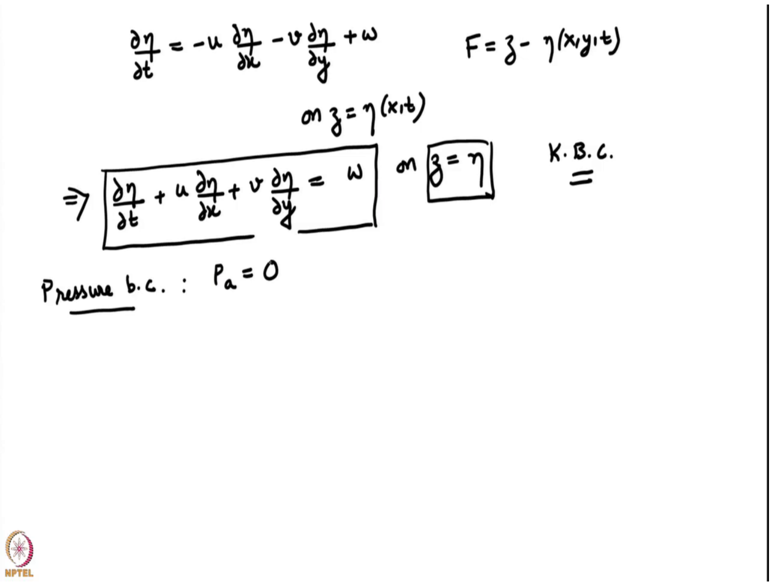

So, we have del eta by del t is equal to u del F by del x F is z minus eta x. So, let me write F here. So, F is z minus eta x y t. So, you can see that del F by del x is going to have a negative sign. So, this is going to be minus u del eta by del x minus v del eta by del y and then w del F by del z.

So, this will be plus w. And again you have to remember that this is at z is equal to eta of x comma t. If I shift all the negative terms to the left-hand side, then I get u, let me write the del eta by del t first plus u del eta by del x plus v del eta by del y is equal to w on z is equal to eta. This is the form of the Kinematic boundary condition that we are going to use in our derivation.

So, this is the Kinematic boundary condition this is true only at the interface this is not true anywhere inside. In any case eta is not defined in the bulk of the fluid eta is a surface quantity and is defined only at the surface. It tells us how much the interface deviates from flatness at that point where we are.

So, this is one boundary condition. The next boundary condition is just pressure. So, we had one boundary condition which was Kinematic boundary condition, we will also have a pressure boundary condition. Now, we are going to as a first step we are going to ignore the dynamics of we are looking at an air water kind of a situation and to simplify matters even further, let us ignore the motion of air that will happen ok.

So, we are assuming that the motion in the air is negligible; hence, the disturbances in the air do not set up any substantial motion. And so the pressure in the air is just a constant. I can set that pressure to 0 we will have to modify this when we want to include the dynamics of the second fluid.

We will do that shortly. We will also have to modify this when we want to include surface tension we will also do that, but right now let us assume that the pressure imposed by air is just a constant, we can take that constant to be 0.