Introduction to interfacial waves
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Lecture - 18
Lindstedt-Poincare technique

We were looking at the non-linear pendulum and we were trying to solve the non-linear pendulum using a regular perturbative technique. Using the technique we had found that the expression for the angle of the pendulum as a function of time has a term which is a secular term as I have indicated in the red bracket.

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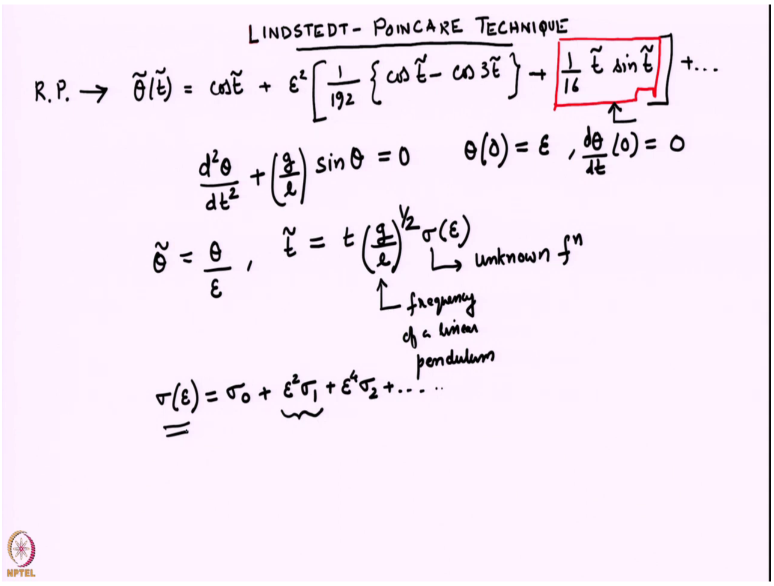

This particular term caused this expression to become invalid at large times. We had also said that there exists other techniques, using which we can rectify this behavior. In particular we had also looked at the origin of this kind of behavior and we had seen that this kind of behavior essentially arises, because the frequency of the pendulum depends on the perturbation amplitude, in this case theta naught the initial angle from which the pendulum is released.

So, now let us look at an alternative technique which will correct this kind of a behavior and will give us an expression which will not become arbitrarily large at large times. So, this technique is called the Lindstedt-Poincare method and it is named after the two people who were among the first to use it. So, let us understand the technique. So, recall that our expression for the so, let me write the dimensional expressions first.

So, our expression for the pendulum was and we were solving it under these initial conditions. In particular we had seen that our choice of scales was epsilon for theta and our choice for non-dimensionalizing time was the linear time period. We had non dimensionalized time using this g by l to the power half is has dimensions of inverse of time. Now, we know that this is not going to work so, the basic idea is that we have to modify introduce some modifications into this procedure and see how to eliminate these kind of terms.

We also have to remember that this is we have done this calculation up to order epsilon square here, using a regular perturbative technique, but if one proceeds using the same technique to higher orders one will get more and more such terms. So, our technique has to be general enough to eliminate such terms at every order. So, with that in mind I am going to introduce a modification to this procedure. So, what I will say is that the g by l is the frequency, g by l to the power half is the frequency of a linear pendulum.

I will introduce a function sigma of epsilon here. This is of course, a non dimensional quantity and so, this is an unknown function, but the purpose of this function is to take into account that the actual frequency of the pendulum depends on epsilon. Now, what can we infer about sigma of epsilon? I am going to write sigma of epsilon as an expansion in epsilon, just like I had already written theta tilde of epsilon.

So, sigma of epsilon is sigma naught plus epsilon square sigma 1 plus epsilon 4 sigma 2 plus dot dot dot. Like theta our expansion, our perturbative expansion starts at epsilon square and not epsilon. We can also infer that for sufficiently small epsilon, as epsilon tends to 0, this expression should reduce to unity that is, because the scale of the time scale or the frequency scale of a linear pendulum is just g by l to the power half.

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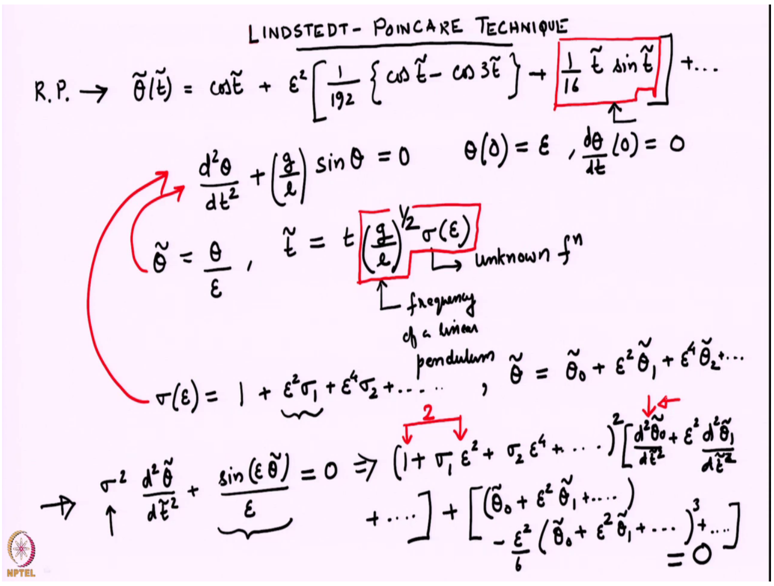

So, we already know that our expansion should start with 1 so, that at sufficiently small epsilon that the scale is so, this is the scale, the scale just reduces to g by l to the power half. So, with this two expansions so, I am now going to substitute these and this into this equation and now, again obtain a non dimensional equation. So, let us substitute this. So, first we have to non dimensionalize our equations.

So, if we do that then you will just get, you can see that this is a generalization of what we had done earlier. If you substitute sigma is equal to 1 in this equation that we have written here, then you will recover the previous non dimensionalization scheme that we had got. So, now, we have an additional sigma and we will do our usual expansion. So, let us do it. So, now, we have to remember that theta tilde has to be written as theta naught tilde plus epsilon square theta 1 tilde plus epsilon 4 plus dot dot dot.

So, I am just expanding sigma square here. So, this is, I am not writing the epsilon 4 term, because we are not going to go up to epsilon to the power 4. We just want to determine the order epsilon correction plus like before, we have to expand this in a Taylor series the first term will be just theta tilde and so, the first term will be theta naught tilde plus epsilon square.

The second term will be epsilon square theta tilde cube divided by 6 and we can write more such terms. So, this whole thing is equal to 0. So, now let us collect the terms at various orders.

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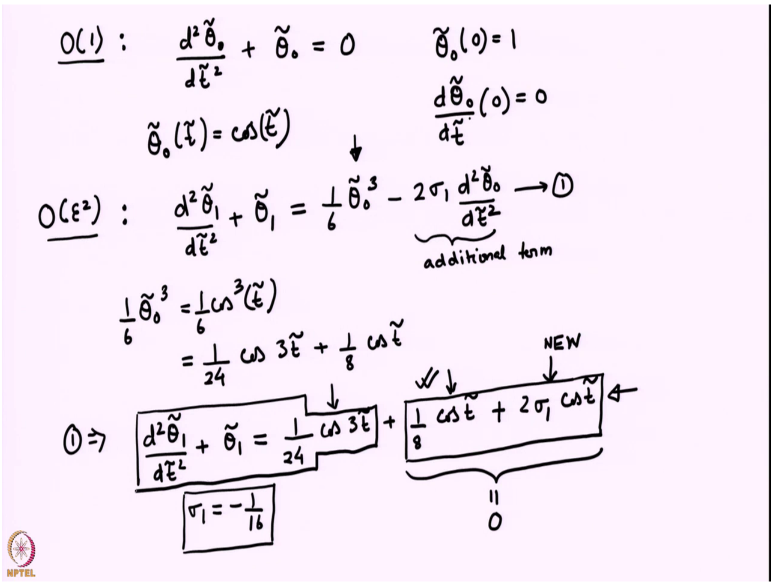

So, at order 1 we recover like before and the initial conditions, one can also do the same expansion on the initial conditions. We have already done that before. So, the unit displacement for theta in the initial condition is going to be absorbed at the lowest order and then all other initial higher order initial conditions are going to be 0.

So, that is the unit displacement unit angular displacement and we know that the solution to this set of equations with those initial conditions is just cos t tilde. You can write it as a linear combination of sin and cos and then determine the constants from the initial conditions ok.

So, now let us look at order epsilon square. So, at order epsilon square we expect the same left hand side, but now for theta 1 and on the right hand side we have already got the corrections. This term was already there before in our regular perturbation, but now, we will have a contribution from the extra term sigma square. So, you can see that we are going to get the product of this term, the product of this term and this term multiplied by this. So, when these two multiply each other, there will be a factor of 2.

So, the product of 2 times sigma 1 into epsilon square multiplied by d square theta 0 by d t tilde square. This is an order epsilon term. This is the only additional order epsilon square term which will come from this expansion for sigma. You can check this yourself. So, I am going to introduce one more term in addition to what I have written and that is minus twice sigma 1 t square theta naught by d t tilde square. So, now, this is the additional term.

So, this is the additional term compared to the regular perturbation series that we had written. In the regular perturbation series at order epsilon square this was the only term that we had obtained and there was no second term, because there we did not expand, we did not have a sigma and so, you can think of all the sigma's as being 0 ok. So, now, we will have to solve this equation. A natural question arises is that how are we going to determine sigma 1 sigma 2 and so on, let us see how.

So, we already know that theta naught cube is cos cube t tilde and this can be written as 1 by 24 cos 3 tilde. So, it is actually 1 by 6 that I am writing here. So, 1 by 6 cos cube tilde is 1 by 24 cos 3 tilde plus 1 by 8 cos t tilde. So, we obtains if I call this equation 1, then 1 implies we have d square theta 1 tilde by d t tilde square plus theta 1 tilde is equal to 1 by 24 cos 3 t tilde plus 1 by 8 cos tilde minus twice sigma 1 and then d square theta 0 by dt t square is just makes it a plus its just minus cos t tilde. So, this is just cos t tilde alright.

So, now we have this term and this term which were present in our regular perturbative calculation and now we have a new term. This is a new term compared to the regular perturbation expansion. You can notice one very interesting thing that the new term is proportional to cos t tilde. Remember that our secular terms arose from this term cos 3 t tilde was not really a problem.

It did not produce any unbounded term, but this cos t tilde has the same frequency as the frequency of the homogeneous equation. So, this would produce a resonant, this would act as a resonant forcing term which would produce a secular term in the particular integral.

However, now you can see the new term has added another quantity which is proportional to cos t tilde and I can immediately see that I can combine these two and say that I will determine sigma 1 in such a way that the sum of these two will be equal to 0. In particular if you see, if you set sigma 1 is equal to minus 1 by 16, then this entire term vanishes.

So, this is the procedure that we will follow that at every order you will find that the additional expansion for sigma that we have done will introduce an additional degree of freedom in our system. It is a degree of freedom, because it will introduce a coefficient whose value will be unknown.

However, it will be the term itself will be proportional precisely to the resonant forcing term at every order and so by choosing the value of that coefficient that unknown quantity, that extra degree of freedom that we have such that all the resonant forcing terms at that order are cancelled, one can eliminate systematically the resonant forcing terms at every order.

In particular this will also determine the expansion for sigma. So, now, we have already found out that sigma 1 is equal to minus 1 by 16. So, later when we dimensionalize our expressions we will see that these corrections to the expansion for sigma these coefficients sigma 1 is equal to minus 1 by 16 and then we can find sigma 2 sigma 3 and so on, these coefficients are nothing, but the non-linear corrections to the frequency of the pendulum.

So, let us proceed having eliminated the resonant forcing terms at second order at order epsilon square I am left with only this portion, the rest is 0, the rest of the right hand side is 0. So, I only have to solve this equation. Once again, it has the structure of a complementary function plus particular integral. The particular integral we have already worked it out and so we can find the particular integral in this case to be minus 1 by 192 cos 3 tilde.

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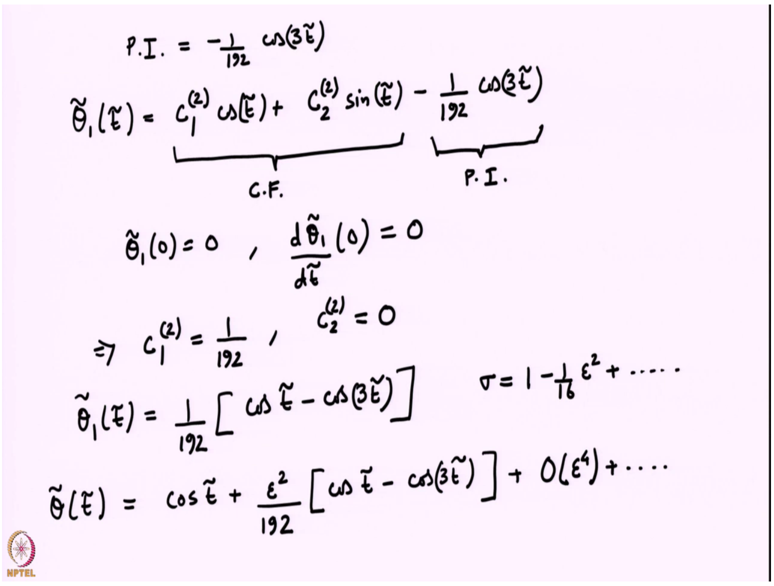

So, the general solution may be written as shall follow the same structure. This is a second order solution so all the constants of integration will have a 2 as a superscript. So, this is the complementary function and this is the particular integral. once again we have to determine the unknown constants using the initial conditions. The initial conditions are as we had discussed earlier. If you use these two then you will find.

So, our solution looks like; so, we have determined our first non-linear correction to the angle of the pendulum. Notice, that this solution is free of secular terms. This is a sum or a difference between two cosines and so, it always stays bounded at all times. So, we do not have the problem of having a term like t sin t or t cos t at this order. One can carry this procedure to higher orders, I encourage you to try, you will find that the algebra becomes more and more complicated as we become as we go to later orders higher orders.

So, you will find typically that you are solving the same left hand side, but the right hand side will keep accumulating more and more terms as you go to higher and higher orders. In this case we had three terms on the right hand side. We eliminated two of those by saying that we will choose sigma 1 such that all the resonant forcing terms on the right hand side gets eliminated. At higher orders you will get more resonant forcing terms, you will have to choose sigma 2 in such a way that at the next order again all the resonant forcing terms gets eliminated.

So, now let us see what is the physical meaning of this solution. So, we have found that theta tilde is written as so theta tilde is cos t tilde plus epsilon square by 192 into cos t tilde minus cos 3 t tilde plus you will have higher order corrections and so on. Now, you can let us dimensionalize these expressions and understand what does it mean in terms of thing. So, we also have found that sigma is equal to 1 minus 1 by 16 epsilon square plus dot dot dot. With this let us dimensionalize the corresponding expressions.

So, if we dimensionalize the expressions so I am just going back to unscaled quantities ok. So, theta was in any case dimensional. So, remember recall that theta tilde was defined as theta by theta naught. So, I want to go back to theta from theta tilde and I want to go back to dimensional time from t tilde which was a non dimensional type. If we do that we are just substituting it in this expression that I have written at the bottom of this slide.

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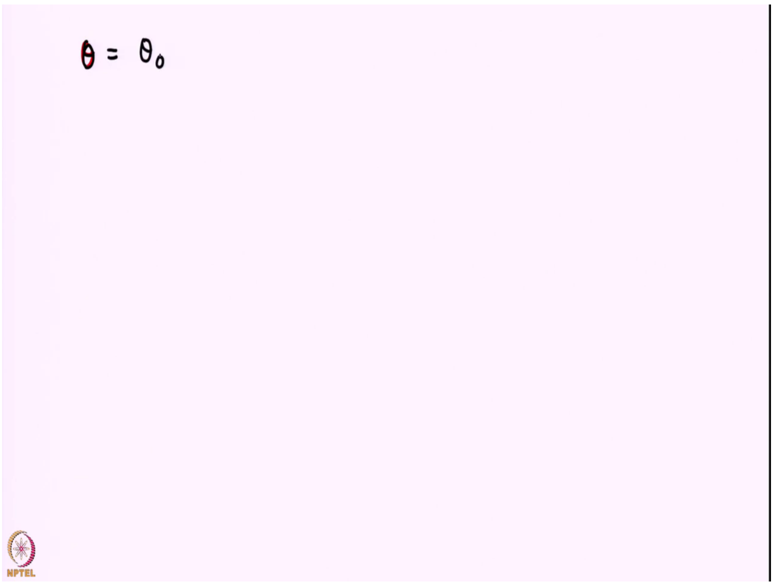

If we do that then we recover theta is equal to theta naught or rather let me call it epsilon cos square root g by l into t into sigma of epsilon and sigma of epsilon we have found it to be 1 minus epsilon square by 16 plus dot dot dot.

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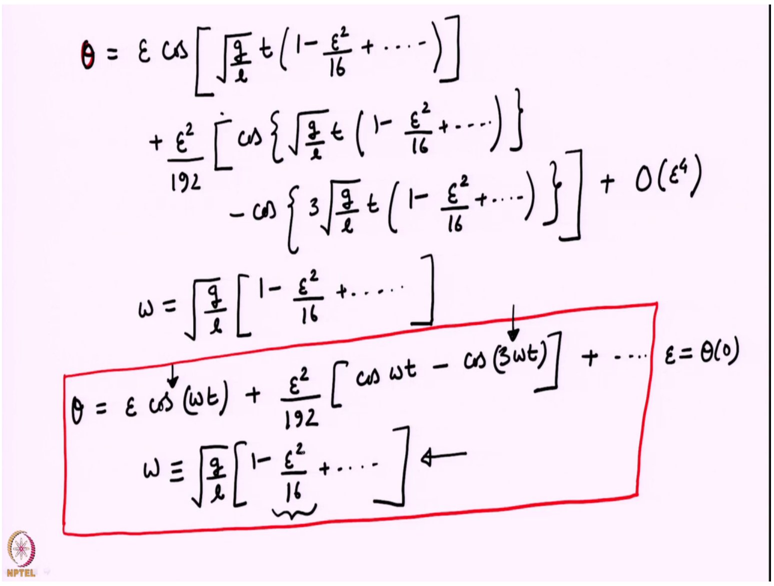

This is the linear solution plus a non-linear correction. Note, that the linear solution has its frequency where there is a non-linear correction plus the non-linear part which is just this cos and the inside part of the argument always takes the same. It is minus plus order epsilon 4.

If we define omega is equal to square root g by l into 1 minus epsilon square by 16 plus dot dot dot then we can write this above expression in compact form as; so, with omega defined as, this is our Lindstedt-Poincare solution to the non-linear pendulum up to order epsilon square.

You can notice a couple of things, the first thing is the process of elimination of resonant forcing terms has automatically produced the non-linear corrections to the frequency. The linear frequency was just square root g by l. We now have a non-linear correction to the frequency in terms of epsilon square.

Recall that epsilon is basically the initial angle. Epsilon is the value of theta at 0. So, omega is equal to square root g by l into 1 minus epsilon square by 16. You can rewrite this in terms of time period. If you do that you will have to take a minus 1 and then you can use the binomial theorem and you will recover the result that we had obtained earlier.

We had obtained an expansion of for the elliptic function capital K which is a function of small k, you should go back there and check that this and though that expression are actually the same. I am here I am writing it in terms of frequency, there I have written it in terms of time period this frequency is equal to 2 pi times 1 by time period, alright.

So, now you can see two things the exact solution to this problem for those initial conditions that we are focusing on was a complicated looking function. It had a sin inverse and then there was a elliptic sn inside and things like that. The Lindstedt-Poincare technique has produced a term, has produced an expansion which can be systematically taken to larger and larger orders and is a relatively simpler looking expression.

The at the lowest order it is just a pendulum which oscillates harmonically with the frequency cos omega t where omega is just square root g by l. If you allow for non-linear corrections then you the first non-linear correction to theta is order epsilon square and it has two terms, one of which is again proportional to cos omega t, but then now we also have a cos 3 times omega t.

This 3 is coming from the cubic non-linearity that we encountered in the original governing equations. Recall that sin theta was theta minus theta cube and so this produced theta naught cube a cos cube of the lowest order solution and we expressed the cos cube in terms of cos 3 theta, the cos cube theta was expressed in terms of cos 3 theta. So, this is cos 3 omega t is coming from there.

And we also have a expression for the frequency and it is telling us that the frequency of a non-linear pendulum is slightly less than the frequency. At least at this order of calculation you can see that it is minus and so 1 minus epsilon square by 16. So, it is actually, slightly less than square root g by l which is what we would conclude if we did a linear analysis.

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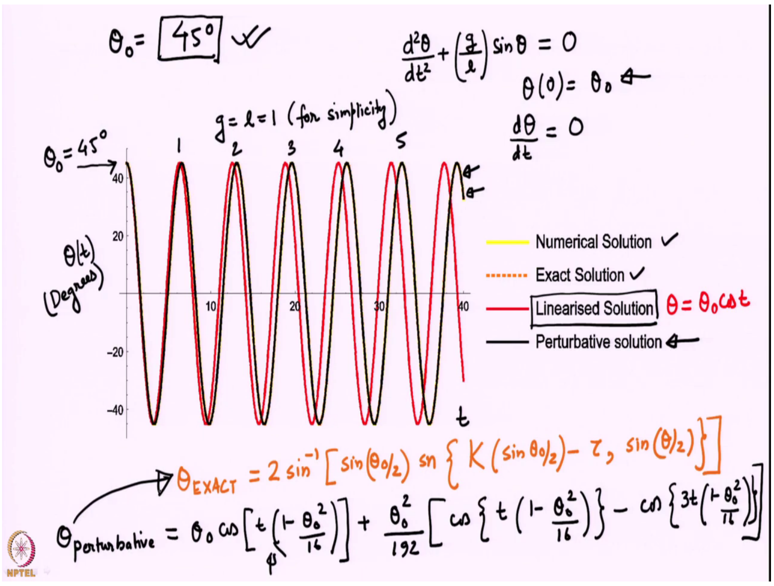

Let us compare the solutions with what we had done earlier. In an earlier exercise; so, here I have just gone back to theta naught using instead of epsilon. So, the initial angle is theta naught and this plot is for theta naught equal to 45 degrees pi by 4. I am only going to show you comparisons for large angles; compared to 1 degree, 2 degrees and so on, because we have already seen that at such small angles the linear approximation is a very good approximation.

So, let us go to large angles like 45, 90 and so on and compare the four ways in which we have got solutions. So, now, here in this plot you can really see only two curves. You can see a linearised solution and you can see a perturbative solution ok. The other two are actually hidden here. You cannot, you can see a little bit of yellow color there, the which is the numerical solution.

In my next plot I will increase theta naught to even larger value of 90 degree and then you will see that the perturbative solution is a slightly different from the exact and the numerical solution. The exact and the numerical solution are always with each other, on top of each other ok. As we increase theta naught to 90 degree so at 45 degree you can see that the perturbative solution is a very good solution ok.

You, it exactly matches the numerical solution and the exact solution whereas, the linearised solution is you can see that after a few oscillations so this is 1 oscillation, 2, 3, 4, 5. So, by the time its five oscillations have happened you can see that there is a substantial phase lag between the linearised pendulum and the non-linear pendulum ok. So, now, make let us make this angle bigger and let us try to understand how good or bad is the approximation.

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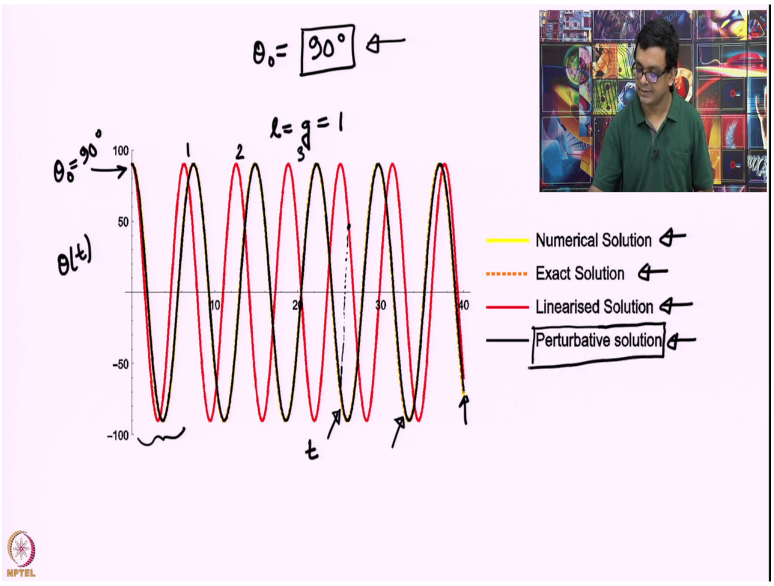

So, as I said before on the black curve here you can see that there is an orange curve and there is also an yellow curve ok. So, you can see it here and you can see that there is a mild difference between the black curve and the orange. So, the black curve is a perturbative solution that we have derived up to order epsilon square, in this case I have written it in terms of theta naught. So, up to order theta naught square.

So, this is theta naught equal to 90 degree and the exact solution and the numerical solution these are all on lying nearly on top of each other whereas, the linear linearised solution which is just cos t or cos square root g by l in into t is just completely out of phase. So, you can see that after 1, 2, 3 so, you can see if you take a certain time 30, let us say and if you compare how many oscillations has the exact pendulum done compared to the linearised pendulum, you will find that the answer is different ok.

So, around this time for example, you know. So, around this time the linearised pendulum is showing a negative angle whereas, the exact pendulum is showing a negative angle, the linearised pendulum is showing almost equal and opposite positive angle. So, you can see that at angles as large as 90 degree the linearised approximation is a poor approximation even within 1 oscillation. At higher oscillations it will become even worse.

Whereas, the Lindstedt-Poincare technique which is our perturbative solution which is a simple looking expression, it is just a sum of the linear solution here and a non-linear correction which has two terms one of which is cos t and other another is cos 3 t plus the frequency correction in all of the terms. So, there is a theta is 0 square by 16 in all of the terms. So, this relatively simple looking formula is doing very good job and is able to approximate this very complicated looking formula up to angles of 90 degree.

I leave it to you to try for higher angles. So, you can take this formula and use a package like mathematic or MATLAB and compared these two expressions at even larger angles. So, you can go up to 135 degree for example, and see how good or bad is a two term expansion compared to the exact formula.