Computational Neuroscience
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Week – 04
Lecture – 17

Lecture 17 : Phase Plane Analysis - II

Welcome. We have been discussing the analysis of the Morris-Lecar equations using phase plane and we have been discussing primarily lately with the idea of the stability of equilibrium points. And so just a small recap, we said that in the phase plane for two-dimensional system of differential equations where this x dot is some function of the vector x. Then with this as two dimensions, we have the phase plane where we have one state variable x_1, the other state variable x_2. We have equilibrium points where this dx_1/dt equals 0 and dx_2/dt also equals 0. So essentially it is the intersection of the null planes.

That is in the Morris-Lecar case we have a v and w. So dv/dt equals 0 and dw/dt equals 0. And so if I say that this x_1 is v, this x_2 is w, then the v null client and the w null client which is essentially a function of w as a function of v, both of them can be represented as w as a function of v. And wherever they intersect are the equilibrium points.

And the property of the equilibrium points we said determine the property of the system in general and in order to understand how it influences the system, we talked about linearizing the equilibrium points around the equilibrium points in the phase plane. And through linearization we find that the eigenvalues of the Jacobian matrix, so if we have x that is now the deviation x is representing the deviation from the equilibrium point is some jx where again x is two dimensional vector and j is the Jacobian after the linearization. And the eigenvalues of j determine the properties of the equilibrium points. So we said that if the eigenvalues are λ_1 and λ_2, they are of two kinds that is stable and unstable. So if λ_1 and λ_2 both are negative, then they are stable and it is unstable when λ_1 and λ_2 both are greater than 0, this is unstable.

There is another case of stable and unstable where λ_1 and λ_2 are complex. So in the above two cases, the λ_1 and λ_2 are real but if they are complex, then the real part of λ_1 and the real part of λ_2 that would determine the stability. So if both of these are less than 0 or negative, then it is a stable spiral and when both of these are greater than 0, then it is an unstable spiral. And we said that the behavior of the system near the equilibrium point can be represented with these ideas of stable and unstable equilibrium points. So we are left with one more case which is the most interesting case maybe and also it is the basis of understanding threshold behavior in neurons.

So the last case is when this λ_1 and λ_2 are real and of opposite signs. So when they are complex, the real part of λ_1 and λ_2 cannot be of opposite signs because we have all real coefficients in the Jacobian or real values in the Jacobian matrix. So they have to be complex conjugates, the λ_1 and λ_2 and so their real part must be of the same sign. Both of them must have the same sign. So the only case that remains is when let us say λ_1 is greater than 0 and λ_2 is less than 0.

So we said that the solution to the system of differential equations near the equilibrium point is given by $Ae^{\lambda_1 t}$ and along the eigenvector e_1, the eigenvector corresponding to λ_1. So j the e_1 is equal to $\lambda_1 e_1$ and je_2 is $\lambda_2 e_2$. That is λ_1 and λ_2 are the eigenvalues and the corresponding eigenvectors are e_1 and e_2. And so this is $Be^{\lambda_2 t}e_2$. So this is the trajectory of the system near the equilibrium point.

So that is when we linearize it around the equilibrium point. And A and B can be determined from the initial values and we get the trajectory very close by. So essentially as we have discussed earlier, it shows that depending on the sign of λ_1 and λ_2, it either decays along e_1 and e_2 or it diverges along e_1 and e_2 based on the sign of λ_1 and λ_2 given the over time. So now when the case of λ_1 greater than 0 and λ_2 less than 0 is considered, then near the equilibrium point what is happening is it is diverging along one of the eigenvectors while it is converging towards the equilibrium point along the other eigenvector. So this gives an interesting behavior that is if let us say this is the equilibrium point whose linearization yields λ_1 and λ_2 such that one is they are of opposite signs and they are real.

So if near the equilibrium point this is the direction of e_1 and this is the direction of e_2, then if λ_1 is greater than 0, then over time if we start the system very near the equilibrium point, it will start to move away along e_1, whereas it will start to move towards the equilibrium point along e_2. What this does is that it creates trajectories that are of this nature that is it is coming down like this. So as you can see along e_2 the distance between them is decreasing. This distance to the equilibrium point is decreasing. So this is the distance along e_2 here, this is the distance along e_2, this is the distance along e_2, this is the distance along e_2, whereas the distance from the equilibrium point along e_1 keeps on increasing.

And if we look at the other opposite directions of it, we will find the trajectories near the equilibrium point behaving in this manner. So that is here and here. So anything that is close by would move away along the positive eigen, the eigen vector corresponding to the positive eigen value. So we also along with this we bring in the idea of what we call manifolds or the two essentially trajectories associated with any equilibrium point like this which is called a saddle node. So this kind of as we had said earlier this kind of an equilibrium point is called a saddle node.

So what are the manifolds? That is now if we start the system right near the equilibrium point just along e_1 with an infinitesimally small deviation from the equilibrium point along e_1 and now no component along e_2. Then what would happen is that if I draw this here let us say if this is our e_1 and this is our e_2 and e_1 has the λ_1 is greater than 0. Then if we start along this it will go straight along this line along e_1 as long as the linearization is valid. That is the e_1 is a correct representation of the trajectory. Remember we have linearized it to be this is valid only very close to the equilibrium point.

So if we move along e_1 slightly further then that e_1 is not valid anymore that is it will not continue only along e_1 away from the equilibrium point. So just as it goes out of the range of the linear behavior then it will start its own direction in the sense of whatever is the dx_1/dt and dx_2/dt or dv/dt and dw/dt and accordingly the derivatives will appear and it will keep on moving accordingly and will end up somewhere. I mean it usually ends up on another equilibrium point or goes outside of the range of allowed values. So or usually on another equilibrium point. So what this kind of or this trajectory is called one of them is the unstable manifold.

Similarly along the negative e_1 there will be the other part of the unstable manifold that is the direction along which the system will go if you start it just outside of the equilibrium point along e_1 with no component along e_2. The other is if we have e_2 if we start slightly away from the equilibrium point along e_2 or along minus e_2 so let us say this is our e_2 and our λ_2 is less than 0 is less than 0 then if I start very close by the equilibrium point and leave it there the system will go directly into this saddle node because that is the linear behavior near the equilibrium point that and that is the trajectory it is supposed to follow. Similarly along minus e_2 if we start it very nearby it will again converge on to the equilibrium point or saddle node. So the question is that somewhere around the saddle node the range over which it is linear if I start the system there it will continue into the saddle node. So and if I start anywhere outside of that e_2 then it is not going to go into the saddle node anymore because we will have a component along e_1 and so it will diverge.

So if we start very close by exactly along e_2 then it has to go into the saddle node whereas any other point or starting away from that line infinitesimally away from that line it will never go into the saddle node. So this brings the point that wherever it is that this trajectory is pulling in the system into the saddle node there is only one point outside and so there must be a trajectory outside that ends up being at that point which will direct into the saddle node along the stable manifold. So this trajectory along which the solution is ending up along e_2 into the saddle node is what we call the stable manifold. So similarly thus extending the same idea along negative e_2 then we have another trajectory that ends up into the saddle node which is the another stable manifold. So now as you can see these manifolds are actually trajectories.

So that means that these manifolds if a saddle node is present in the phase plane these manifolds are separating out the phase plane into different regions because if you remember we said that two trajectories cannot cross each other since the manifolds are trajectories if I start the system here it is going to be bounded by this manifold around within this region. Similarly if I start the trajectory here or the system here it is also going to remain within this manifold region within this region similarly on the other part here and here. Of course let us say that this ends up being on to a stable node or this ends up being on to a stable node then the trajectory can go around and go there. So it is not crossing the manifold. So the manifold is ending at this stable node or it is ending at this stable node and then the trajectory from here may go into this region but that may not necessarily be the case.

So you must appreciate here a few things that is the idea of the saddle node and the associated manifolds the stable and unstable manifolds and how they behave as separators of the phase plane or called the separatrix. And you must also remember that if we are infinitesimally to the right in this case or to the left in this case we will have very different behavior in the system. So if we have a manifold like this a stable manifold going into a saddle node I start the system infinitesimally to the other side or infinitesimally to the opposite side. So they are extremely close initial condition very close by yet their behavior will be entirely different because they will diverge away along plus along the positive eigenvector with the along the positive direction of the eigenvector whose eigen corresponding eigen value is positive and the other will diverge along the negative direction of that same eigenvector. So if this is the other two directions this is where this is how it is going to go.

So this is as we will see is the going to be the basis of threshold behavior and why we call action potentials all or none or that if it is on below a threshold point it behaves in one particular way just above the threshold point it will behave in an entirely different way in fact in our case it will produce an action potential. So extending the this concept of saddle node let us consider the Morris-Lecar equations as we have and if we recollect it is Cdv/dt and we have terms with

$M_\infty(v)$ and $w(v)g_k - -v - e_{calcium} g_{calcium} bar$ and this is $v - v_k - g_{leak} v - e_{leak}$ and $i_{external}$ and we of course have the dw/dt is equal to $w_\infty(v) - w/\tau_w(v)$. We have the functional forms of w_∞ and τ_w as a function of voltage. We have all the values of all these parameters and we have discussed how this system behaves and shows near threshold behavior and not exact threshold behavior and the reason was that the fast dynamics required to model the threshold behavior which is given by this M_∞ we are excluding its changes that is its dynamics that is we are not including dM/dt in these equations and we are setting M to be instantaneously changing to M_∞ because it is very fast. So we said that the initial changes cannot be modeled very correctly and hence we did not get true threshold behavior as we have discussed and it is not possible to get to threshold behavior in that sense but if the parameters of this system can be changed are changed then this system can actually show true threshold behavior.

So what I mean by that the set of parameters it will be provided to you and the code also the set of parameters for which we can see threshold behavior is not a real neurons behavior it is just mathematically it will show a threshold behavior that is why we are going to study this. So there are a second set of parameters for the Morris-Lecar equations with this g_k, g_l, e_l, e_k and all the values the capacitance and so on for which we get an equilibrium point a stable equilibrium point along around minus 42 milli volts. So as we have said that us in that case let us draw the phase plane so this being V and this being W. Earlier when we studied the Morris-Lecar equation we always had one equilibrium point the case where we showed near threshold behavior and the case where we talked about how it spirals out into a limit cycle or oscillations or periodic behavior. In this case with these set of parameters the null clients become different so if we look at the dv/dt null client it turns out to be something like so and the dw/dt null client so this is the dv/dt null client and the dw/dt null client turns out to be something like this.

So there are three equilibrium points in this case this is around minus 42 milli volts this is around minus 20 milli volts and this is around 0 milli volts. So if you linearize the system around these equilibrium points let us say this is the first equilibrium point which we will call R or the resting membrane potential this R is a stable equilibrium point. This particular one is an unstable equilibrium point and the middle one turns out to be a saddle node. Now if we are to construct the manifolds of this particular saddle node we will see so if we go to the actually with a different color this is an unstable equilibrium point so stable manifold comes in from here another stable manifold comes in from here. So remember all these you will be provided to code to simulate it you will see this for yourself and that is the only way to understand this fully.

Here we can only describe how it would behave but we cannot quantitatively show it in this lecture you have to numerically simulate them. So these are the stable manifolds of that saddle node and if we look at the unstable manifolds what happens is that one of the unstable manifolds goes into this stable node. The other one that is along the negative side of it goes along this along this and down and into this stable node. So these are the unstable manifolds and this is these are the stable manifolds. So now if you recollect that let us say the system is at rest at this particular resting membrane potential that is minus 42 millivolt or to resting membrane potential it is not really neuron behavior.

And now if I start the system with the current injection as we had described earlier somewhere to the right of this then remember without changing w we can start anywhere along this line and this line. This already should give you the clue of where the threshold is. So trajectories to the left of this stable manifold will end up going like this and like this into this stable node. Anything that is to the left of this stable manifold will go back to the resting membrane potential that is minus 42 millivolt. And infinitesimally to the right here it will go diverge and move along this unstable manifold and go back and into the resting membrane potential.

And this is what is going to be the action potential because now if you plot V as a function of time for the two sides of the manifold what you will see is. So this is time and let us say this is where the stable manifold voltage is along this particular line. Let us say along this line this particular voltage is this value here. And so trajectories that start below or the their voltage component and will go back down to the resting membrane potential. Let us say this is V_r this is $V(t)$ and so it will go back to the resting membrane.

Anything like this will go back into the resting membrane potential. Anything like this will go back into the resting membrane. As soon as we start the trajectory above the threshold it will follow the other unstable manifold and will show this kind of a behavior. I am sorry it will go down and go below the resting membrane potential and then go to V_{rest}. So this is essentially the action potential and the stable manifold that separates the two regions is what is providing the threshold behavior.

So this same idea so we are talking about toy system here by changing the set of parameters where we can see the two threshold behavior. So now if we extend this same idea to the Hodgkin-Huxley system then we have to consider a reduced system where the sodium channel kinetics or the activation gates of the sodium channel need to be included. So now if I if we remember the Hodgkin-Huxley equations. So we are now shifting back to the Hodgkin-Huxley equations. This

CdV/dt equals $I_{external} - G_{sodium} m^3 h, V - E_{Na} - G_K, n^4, V - E_K, -G_L V - E_L$.

I am sorry this is these are not vectors this is simply bar G_{Na} bar and G_K bar. So we have now dm/dt if you remember and dn/dt and dh/dt along with the first differential equation. Now if we have the dn/dt if we if we want to have a true threshold behavior as we said we need to include the fastest of the state variables in this case the dm/dt and that opening of sodium channels or the activation gates of sodium channels that is required. If you remember we qualitatively said how the activation gate keep on opening and sodium keeps on coming in which causes the action potential.

So m becomes crucial. So we include the dm/dt in the phase plane analysis with $m_\infty(V) - m/\tau_m(V)$ and our dn/dt is removed dh/dt is removed and we set n equals to $n_\infty(v_{rest})$ or which is in the Hodgkin-Huxley case around minus 60 milli watts. Similarly h also we set to $h_\infty(v_{rest})$ why because n and h are far slower compared to m more than order of magnitude h and an order of magnitude slower for n. So what we are doing is we are keeping the n and h whatever they were at the resting state or the resting membrane potential that is v_r and we are allowing m. So without the changes in n because n is fixed at v_{rest} and h is also fixed at v_{rest} we cannot get the full action potential behavior. Remember we said that the validity of the reduced systems in this case we are talking of the v, m reduced system of the Hodgkin-Huxley equations.

The validity of this reduced system is there only when the assumptions are valid that is n is not changing h is not changing. So it is only in the initial few milliseconds one or two milliseconds that we can mimic the behavior with the reduced system. In order to truly model the system over time longer than that we need to include n and h. But the threshold behavior is early on through the is in the very beginning when we do a current injection and see if we can move the system beyond the threshold or not. And that can be seen with just this reduced system.

So if we have the reduced system like this if we draw the phase plane V and we have m in this case the threshold of the system or the V_{rest} of the system is around minus 60 millivolt. And we will see that we have the m or the m-nullplane to be like this. And if we use the same equations the V-nullplane would behave in this manner. So this region needs to be blown up here and what we essentially have is two equilibrium points one that is the intersection of the V-nullplane and the w-nullplane creating two equilibrium points.

So one is the stable one at minus 60 millivolt. So if you linearize the system around this particular point or equilibrium point you will see that the Eigenvalues are all negative and it is a stable equilibrium point. And this again turns out to be a saddle node. And the last one at the top is this is when m has actually reached 1 and this is voltage of around 53 millivolt which is like the sodium reversal potential. And this one this equilibrium point the voltage here is around minus 56 millivolt or something near that. So you will have codes and you will you can run it for yourself and see.

So now again this saddle node has the stable manifold which very very much like the Morris Lecher second parameter system has stable manifolds that are coming in to this particular point this saddle node. And the other one actually comes in from outside the phase plane. The other unstable manifolds which we were drawing in cyan will end up on to the stable node and the one negative to that will end up actually will end up on this stable node. So it will go out along this direction. So this particular unstable manifold is when when extended it goes up to the other stable node at the right end of the right top corner.

So as we can see here if we start the system anywhere along this yellow line which is at the resting n. So this is n and this is what it was at rest and this is our V_{rest}. Then by shifting it through a current injection and initial value of voltage is somewhere near this stable manifold but on the left then the behavior is like this and like this. And in the other case the behavior is like this where it ends up being on this particular saddle node in the particular stable node as shown here. So this is extending on to the other saddle node which is at the top.

So now you will ask so where is the action potential? Well as we had said we would not be able to show the action potential because it needs n and h to be changing. So what we because we do not have them changing we have the point to which the sodium will take it to which is the stable node where m equals 1 which is the saturation of opening of the activation gates. And now h and n are not changing so this is only the left part or the initial part of the action potential that will be shown here. So with this we conclude about how the action potential has a true threshold behavior and how even in the Hodgkin-Huxley we can produce the and see the true threshold behavior and this is why we will consider action potential as all or none. So if it is on the below threshold that is on the left hand side of the stable manifold it goes back to the resting membrane potential if it is to the right or above the threshold that is or to the right of the stable manifold it will produce an action potential.

So with this we conclude the ideas of threshold and next we will be talking about oscillations and mainly about limit cycle behavior. Thank you.