Gasdynamics: Fundamentals and Applications
Prof. Srisha Rao M V
Aerospace Engineering
Indian Institute of Science – Bangalore

Lecture 07
Flow equations – Differential Form

In the previous class we have were looking at flow equations in a particular form which is known as the integral form of flow equations the control volume approach. And in this class we will look at another kind of analysis and the kind of equations of the same conservation equations now in differential form.

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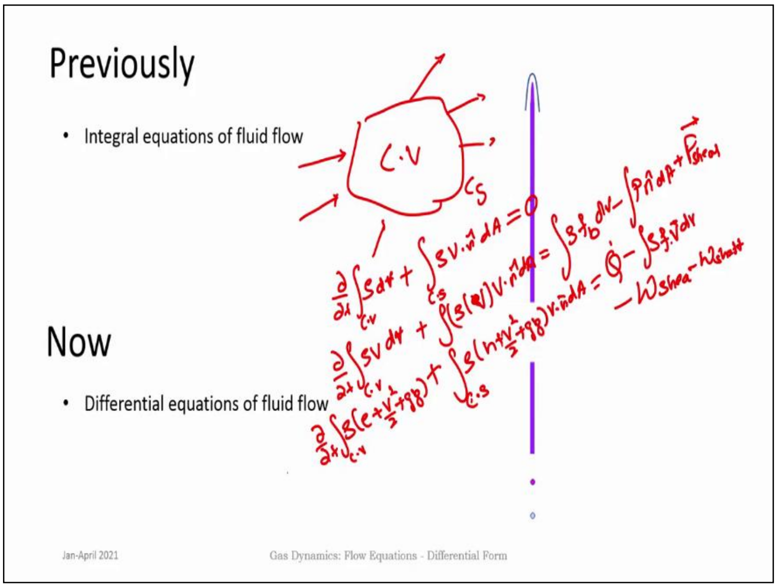

There are really two kinds of approaches looking at analyzing fluid flows. In the control volume approach or the integral forms we are really looking at analyzing the fluid flow through a certain system which is a control volume. And we look at how changes happen within the control volume due to fluxes which happen across the control surfaces. So you have control surfaces and control volume and we write the mass momentum and energy balance in an integral form.

The mass conservation written in for this control volume this control volume is ρdv which is this is the total mass of the control volume. If there are any changes to that mass it should have happened due to fluxes around the control surface which is $\rho\vec{V}\cdot\vec{A}$ so $\rho A\vec{V}\cdot\hat{n}$ where $\hat{n}$ is a normal. So this is really the flux of mass. And since the total mass it never changes this is equal to zero.

So this is the conservation of mass. Similarly if you consider the conservation of energy conservation of momentum $\rho\vec{V}$ is the momentum then momentum within the control volume if it has to change it is due to changes due to the $\rho\vec{V}$ which is the momentum and $\vec{V}(\vec{V}\cdot\hat{n})dA$ which is a momentum flux and this can be due to various forces they can be due to the body forces which is termed as just body force $\overrightarrow{f_b}$.

And this is integrated over the volume. The other is surface force which is here mainly the pressure forces and shear forces and $P\hat{n}dA$. So is just the statement of some shear force the statement of second law of Newton that change in momentum is equal to the sum of forces that come on to the body. And then the first law of thermodynamics which is $\frac{\partial U}{\partial t}$ changes to energy which is total energy is internal energy plus summation of kinetic energy and the potential energy.

That over a control volume is actually due to the fluxes of energy across the surface, $(\vec{V}\cdot\hat{n})dA$ over the control surface is can be due to some heat that is heat transfer and it can be also due to the work done. And here work done can be due to body forces if there are body forces then the work done on those body forces is $\overrightarrow{f_b}\cdot\vec{V}dv$ $\vec{V}$ is the velocity and some shaft work or shear work.

So this is the statement of the energy equation which is nothing but statement of first law of energy first law of thermodynamics. So this is the integral form of a equations and we will use them significantly in the course of the work. The other form is here we do not go into the details of the flow field what is happening at each point in the flow. But if you want to know that is important to know that to look at fluid flow phenomena in compressible flows then we should really approach every point. The way to go about doing that is through differential equations.

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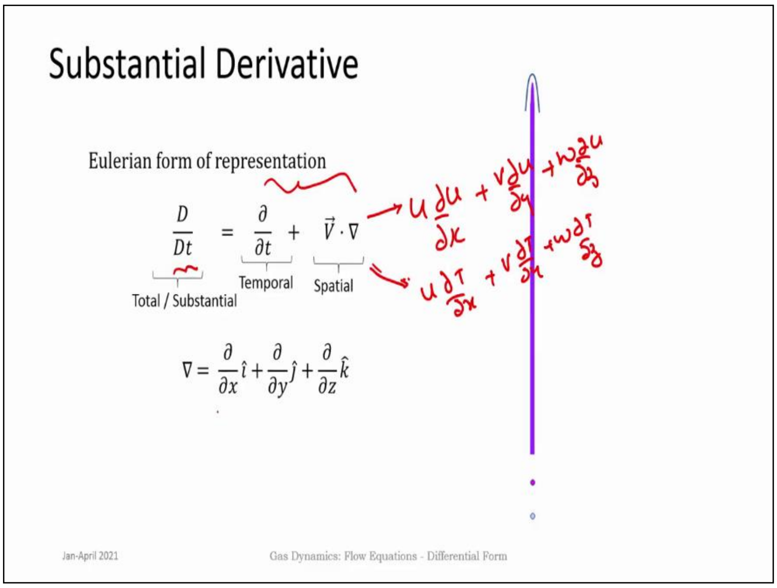

So we will just look at the differential forms the same three laws the three conservation laws we will express in differential form. When expressing them in differential form we have to understand the differences in approaching them. Usually when you look at differential forms they are looked at you are looking at a particle and you follow the particle and then you say how is the velocity changing the changes in velocity is related to flow to the forces as you follow the particle that kind of an approach is Lagrangian approach in terms it is called a total derivative or a material derivative.

But if you look at fluid flows there are so many particles there are so many such particles would you follow all of them individually? So in fluid flows the other approach is usually look at a particular point in the flow field and then see how velocities vary within that point and so on that is known as the Eulerian frame of looking at things. So these two frames can be represented or these two frames can be related to each other through the definition of the total or material derivative consisting of a spatial a temporal part and a spatial part.

So this is the Eulerian representation while $\frac{D}{Dt}$ which is on the left hand side is a Lagrangian representation, $\frac{D}{Dt}$ total derivative or material derivative is equal to this form where $\vec{V}\cdot\vec{\nabla}$ is actually for a Cartesian frame it is $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ this term comes out to be like this. So, if it is say you are talking about a particular velocity u component of velocity it will become $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$.

Or if it is temperature it becomes say $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$, so it comes so on. So this term gets expanded into such a form. So where del operator, $\vec{\nabla}$ is the usual gradient operator. So this form is important so lot of vector calculus and so on are important over here.

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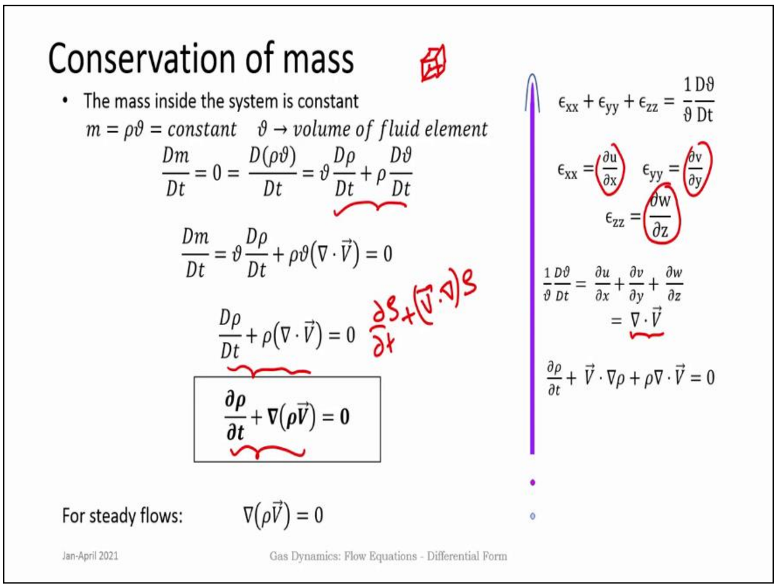

So you should just revise them if you haven't if you need to again so if you consider a very, very small fluid parcel a very small fluid element. And we are going to do analysis on such a small fluid element which is infinitesimally small. The mass inside that particular element is nothing but ρv. So what we should say is that mass is constant, mass does not change so $\frac{Dm}{Dt} = 0$.

So as you follow that particular fluid particle or fluid parcel so $\frac{Dm}{Dt} = 0$. So $\frac{D(\rho v)}{Dt} = 0$ which is you can then use the rules of differentiation to find this out. Which consists of changes to density and changes to volume and the changes to volume can be related to how individual these particle this fluid volume changes due to velocity gradients. It is so in x-y-z directions the changes are related to $\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial v}{\partial y}\right), \left(\frac{\partial w}{\partial z}\right)$.

So on an average the change of volume $\frac{1}{v}\frac{Dv}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ this is known as the dilatation of that fluid parcel. So $\vec{\nabla}\cdot\vec{V}$ and therefore we can say that the change in mass should be equal to change in density $\frac{D\rho}{Dt}$ or it can be due to dilatation $\vec{\nabla}\cdot\vec{V}$. So, it the total change in mass is anyway zero so we get this particular equation $\frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{V}) = 0$.

And $\frac{D\rho}{Dt}$ can be expanded $\frac{\partial\rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho$. So the taken together you can also write it in this form $\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{V}) = 0$. For steady flow $\nabla(\rho\vec{V}) = 0$. Now this is a compressible fluid flow so density is a variable it is not a constant anymore. So this is conservation of mass.

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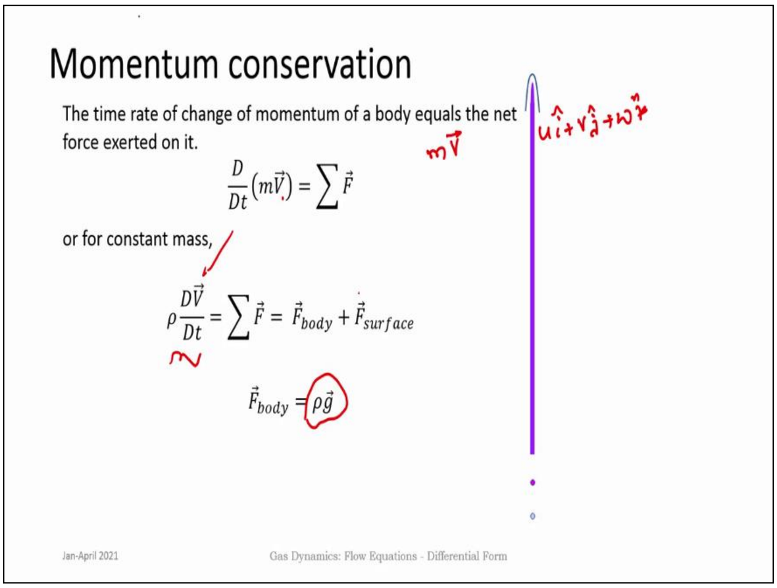

Now let us go and look at momentum conservation. So here now momentum is mass multiplied by the velocity. Velocity is a vector momentum is a vector so we are dealing with vector equations. If you are looking at Cartesian coordinates this involves three components $u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ so three different velocity components. And similarly forces also in three different directions. These forces can in general be body forces or surface forces.

The body force is usually due to gravity $\rho\vec{g}$. Now if you consider left hand side you have $\frac{D}{Dt}$ of momentum which is mass multiplied by velocity, $m\vec{V}$ but already we know there is a conservation of mass applied here implied here. Therefore you get this term it comes out to be $\rho\frac{D\vec{V}}{Dt}$. So now what are the surface forces? so left hand side is nothing but the acceleration terms $\rho\frac{D\vec{V}}{Dt}$.

While the right hand side is due to surface forces and body forces. So what are these surface forces?

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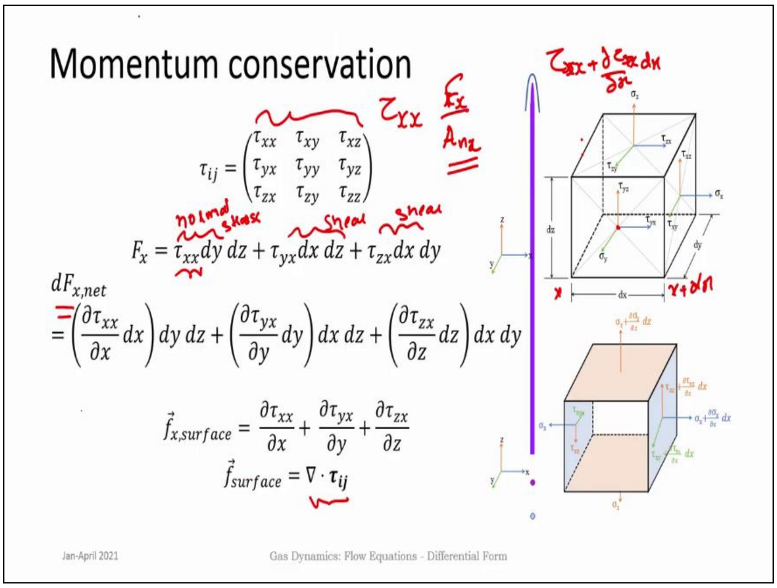

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$F_x = \tau_{xx} dy\, dz + \tau_{yx} dx\, dz + \tau_{zx} dx\, dy$$

$$dF_{x,net} = \left(\frac{\partial \tau_{xx}}{\partial x} dx\right) dy\, dz + \left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dx\, dz + \left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dx\, dy$$

$$\vec{f}_{x,surface} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\vec{f}_{surface} = \nabla \cdot \boldsymbol{\tau_{ij}}$$

If you take any small volume it has let us take this Cuboidal volume it has different surfaces. And at each surface you will have forces that occur both in the same directions. For example if this is the x direction then you have forces along x direction and you have forces along y direction as well as this direction at a particular point. So if you take any particular point and have different surfaces you have forces along different directions.

Therefore this particular if you divide by the area, force divided by area, is stress. So at a particular point you can have a state of stress and that is defined. The stress is actually a tensor because at every surface you can have forces in all other in every direction. So τ_{xx} represents if you consider a stress τ_{xx} is actually a force along x-direction F_x over an area, which is an area and the normal of this area is also along x-direction.

So normal is also along x-direction. So, you can look at the coordinate system here and understand them. So if you consider a force along x-direction it is composed of several different stress components. One is of course the stress it is due to the area with having a normal x and force along x. But it can also be due to a stress where it has a normal is at y but force is along x. So this component is the normal component normal stress is a normal stress it is acting perpendicular to the area.

While this τ_{yx} and τ_{zx} which are all x components of velocity where the areas are perpendicular to y and z these are tangential components so these are shear stresses. So, shear stresses. Now these components of forces in general can vary across different areas so if you take the effective force which is from one section to the other section there is a change which is so $\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx$ is the change as you go from x to $x + dx$.

Similarly y to $y + dy$ net force will be the difference between these two forces which is $\frac{\partial \tau_{yx}}{\partial y} dy$ similarly for every other force. So, just the F_x that force along x direction is a summation of different components of the stress tensor. So $\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$. So similarly in the vector equation which is the momentum equation which has both x all x, y and z components in general the surface force is actually composed of F_x, F_y and F_z.

And they in turn are composed of these derivatives. And so divergence of the stress tensor which is represented over here is the surface force in general.

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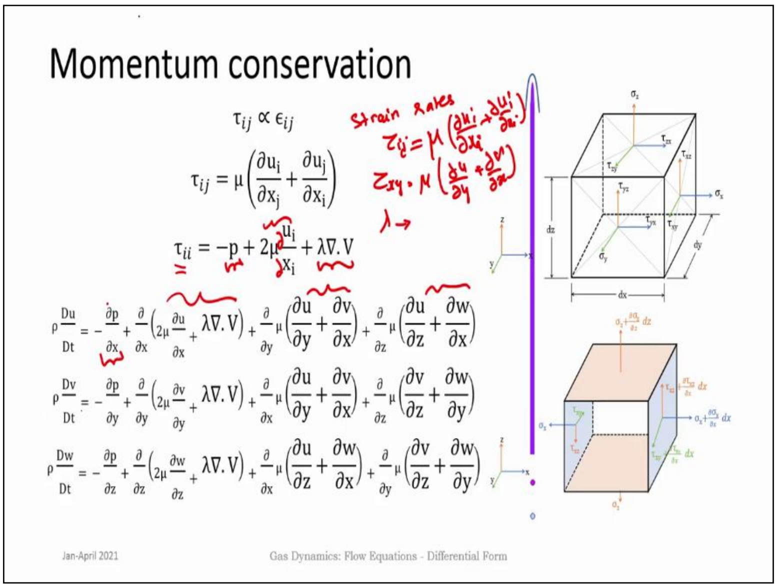

Now what is the particular stress? How is it related to the velocity gradients is what we have to see. And we are considering a Newtonian fluid and in that fluid the stress is directly proportional to the strain rate shear strain rate or strain rate and the strain rates are given by the velocity derivatives. So this is the Newtonian flow or Newtonian fluid. So therefore you know that τ_{ij} or this stress is the viscosity times this gradient of velocities $\frac{\partial u_j}{\partial x_i}$.

So if you take for example τ_{xy} this will be $\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$ so it is a general representation where you can replace i j by x y and then the corresponding coordinates you can get it. But if you look at the normal stress it consists of a pressure term pressure is a normal force and it consists of another part which is related to dilatation and where λ is known as bulk viscosity and you have the normal stress term $2\mu\frac{\partial u}{\partial x}$.

So now we can plug these different forms of the stress tensors into the equations that we had just formed earlier and we get terms related to pressure, pressure gradient $\frac{\partial P}{\partial x}$ in
this is the normal stress terms in u directions and shear stress term in y and z direction. This is the general form of equations for momentum equation similarly in v and in w coordinates.

This is nothing but Navier Stokes equations but in this equation. Now density is a variable ρ is a variable not only is ρ a variable μ which is the viscosity is also a variable it is a function of temperature.

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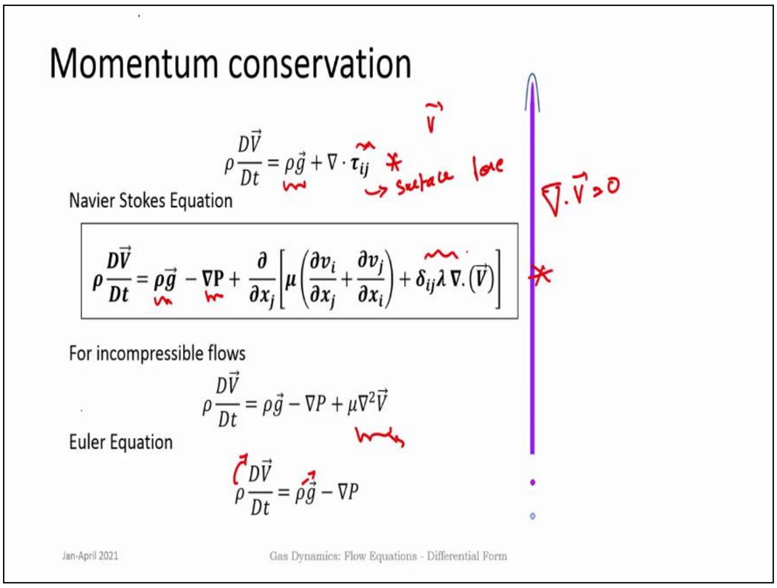

So in general you can write so read $\rho\frac{D\vec{V}}{Dt}$ where $\vec{V}$ is a vector is the body force $\rho\vec{g}$ $+\nabla\cdot\tau_{ij}$, $\nabla\cdot\tau_{ij}$ in general the divergence of the stress tensor. This is the surface force, surface force and with the definitions of the stress tensor related to the velocity gradients we can write them in general here that $\rho\frac{D\vec{V}}{Dt}$ is $\rho\vec{g}$ and this is gradient of pressure and then the other stress terms.

So for incompressible flow when you consider incompressible flow in general $\vec{\nabla}\cdot\vec{V}=0$. So all these terms related to $\vec{\nabla}\cdot\vec{V}$ drop off so you get a simplification with this $\mu\nabla^2\vec{V}$ where μ is also taken to be a constant if you consider in general the inviscid equation where you do not consider viscous forces. So, this completely drops off viscous forces all viscous forces drops off.

This is now $\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P$, here ρ is still a variable. So this is Euler equation where ρ is a variable this is an incompressible flow equation where ρ is taken as a constant. So these are various forms of the momentum conservation equation the most general form is represented here and expanded over in this equation.

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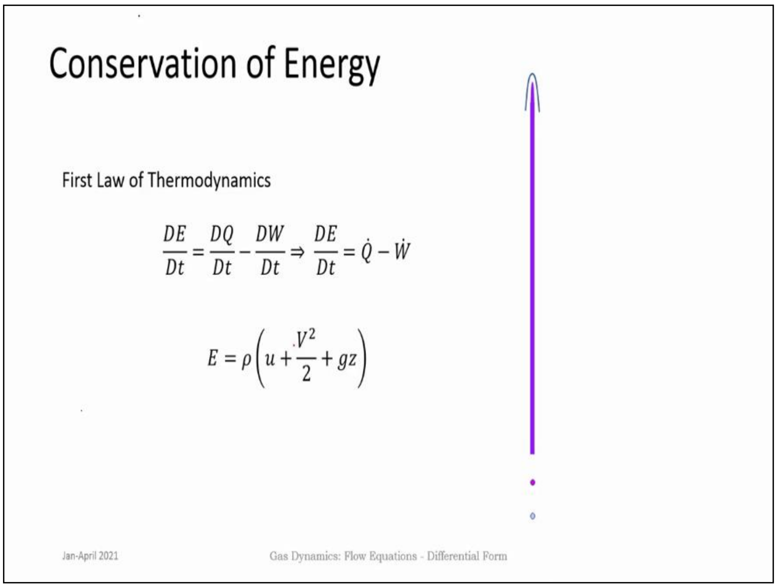

Now if you consider the energy conservation it comes back to first law of thermodynamics change in energy is related to heat added and work done on the system. So here it is given negative sign so it is a work done by the system so $\frac{DQ}{Dt} - \frac{DW}{Dt}$ energy is composed of internal energy, kinetic energy and potential energy. So total energy is ρ multiplied by this particular thing.

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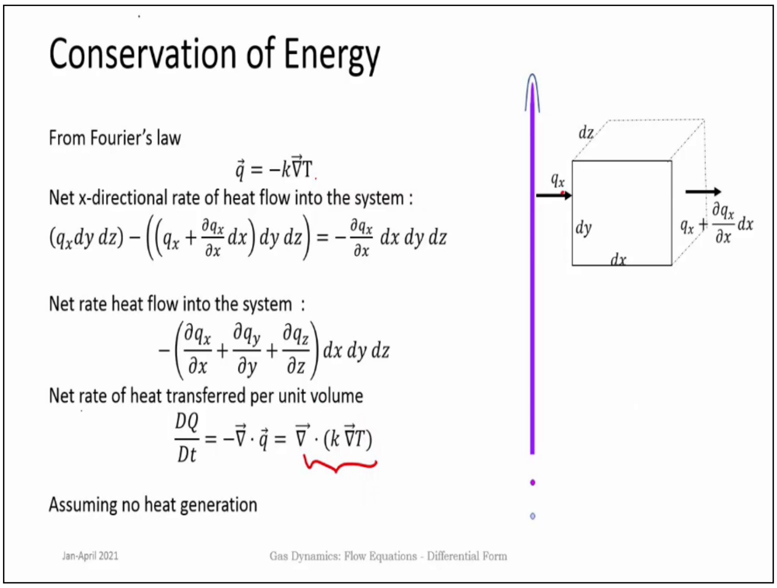

Now we have to relate how heat gets into the system how work is getting done on the system. So heat coming into the system is getting transferred by conduction is Fourier's law of conduction $-k\vec{\nabla}T$ and you have to consider different surfaces how heat gets transferred the net rate of flow into the system is nothing but gradient of this $k\vec{\nabla}T$ which is $\vec{\nabla}\cdot k\vec{\nabla}T$.

So this is the formulation for the net heat transfer that happens per unit volume. So $\frac{DQ}{Dt} = \vec{\nabla}\cdot k\vec{\nabla}T$ and here we are assuming there is no heat being generated.

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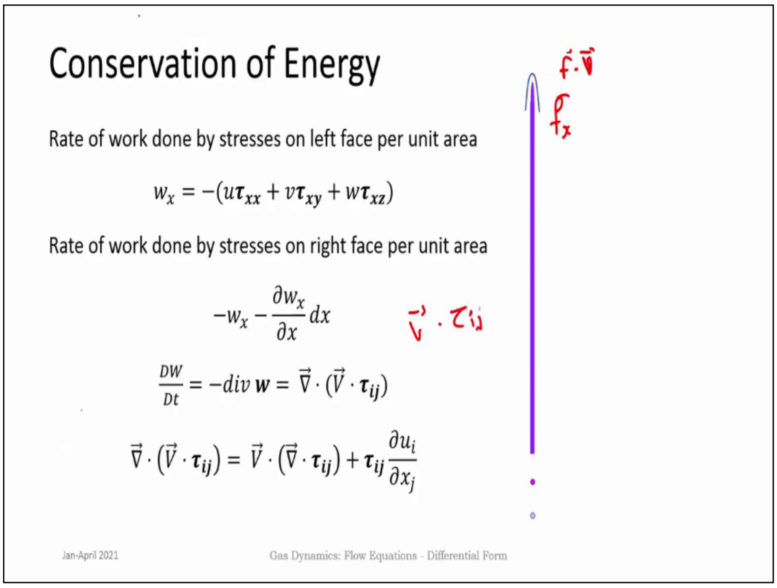

Now; what about work done in a similar manner as before we will be considered heat transfer we are checking the work done. And work done you knows the force multiplied by velocity or if $\vec{f}\cdot\vec{V}$ is the work done. So if you take f_x the force, force in x direction it has components

τ_{xx}, τ_{xy} and τ_{xz} and you have to multiply it by the corresponding velocities u, v *and* w you get W_x work done on the back, work done by the stresses on the x on one particular face.

Consider the other face, net rate of work done $\frac{DW}{Dt} = -div\ W = \vec{\nabla} \cdot (\vec{V} \cdot \tau_{ij})$. Now this can be expanded by vector identities and you can get two terms one having a $\vec{V} \cdot (\vec{\nabla} \cdot \tau_{ij})$ the other one $\tau_{ij} \frac{\partial u_i}{\partial x_j}$.

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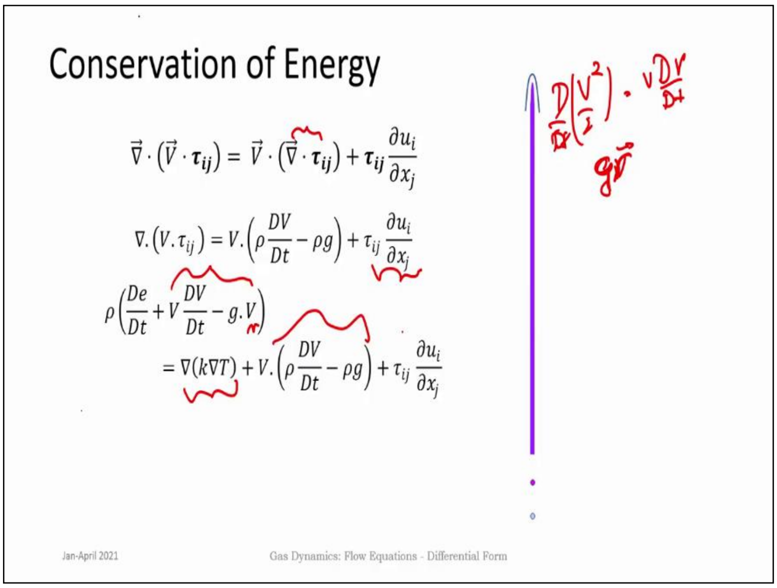

So now if you get these terms and this is related this can be extracted from the momentum equation $\rho \frac{D\vec{V}}{Dt} - \rho \vec{g}$ while this term still remains. So now you put all of them together $\rho \left(\frac{De}{Dt} + V \frac{DV}{Dt} - \vec{g} \cdot \vec{V} \right)$ because $\left(\frac{V^2}{2}\right)$ is differentiated $\frac{D}{Dt}$ is $V \frac{DV}{Dt}$ while you had the term $\vec{g} \cdot \vec{r}$ where $\vec{r}$ was the the displacement vector. So $\frac{d\vec{r}}{dt}$ is $\vec{V}$ so now you find these terms here $V \frac{DV}{Dt}$ and this term they are common and they get cancelled off.

And you are left with only the heat transfer terms and that is conduction heat transfer term and this particular term $\tau_{ij} \frac{\partial u_i}{\partial x_j}$.

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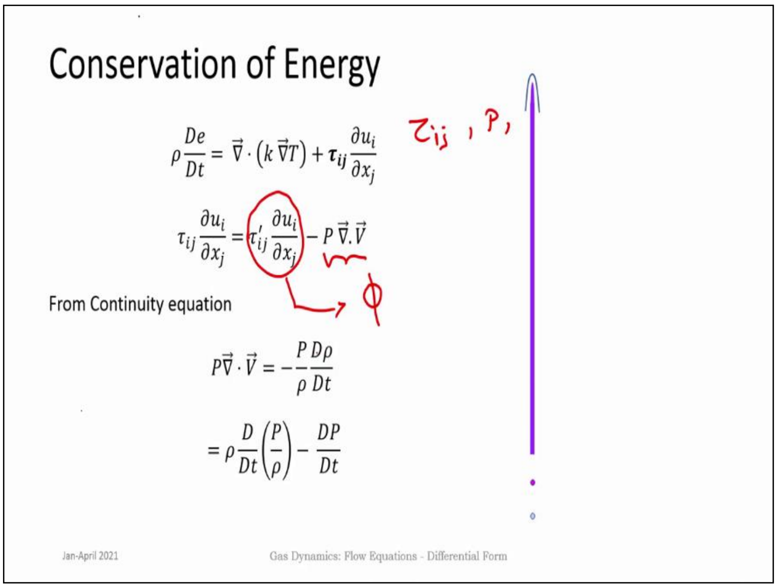

This τ_{ij} consists of pressure term and the terms due to the velocity gradients. So they can be separated the pressure term comes out as $P\vec{\nabla} \cdot \vec{V}$ while you are left with all velocity gradients and this term is known as the viscous dissipation term. This is all the dissipation that happens due to viscosity and velocity gradients. And you can simplify the pressure term use the continuity equation where $\vec{\nabla} \cdot \vec{V}$ can be represented in terms of $\frac{D\rho}{Dt}$.

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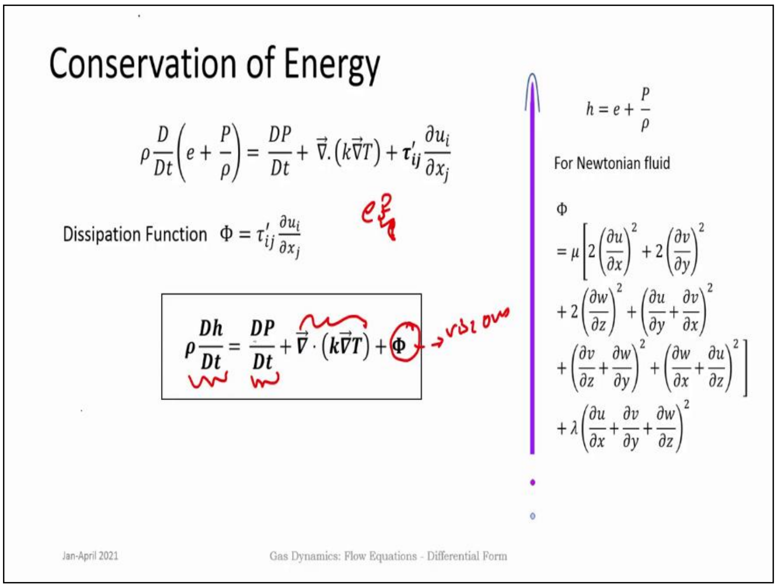

And therefore this pressure term comes within the brackets $\left(e + \frac{P}{\rho}\right)$ is nothing but enthalpy, h. So finally from the conservation energy conservation term we get the equation for change in enthalpy $\rho\left(\frac{Dh}{Dt}\right)$ is equal to changes in pressure $\frac{DP}{Dt} + \vec{\nabla} \cdot \left(K\vec{\nabla}T\right) + \Phi$ this is the conduction heat transfer and viscous dissipation term. So, Φ , this is viscous dissipation. This is first law of energy, energy conservation this is enthalpy pressure and heat that is getting transferred and viscous dissipation.

So this now forms the three forms of equations which is conservation of mass momentum and energy.

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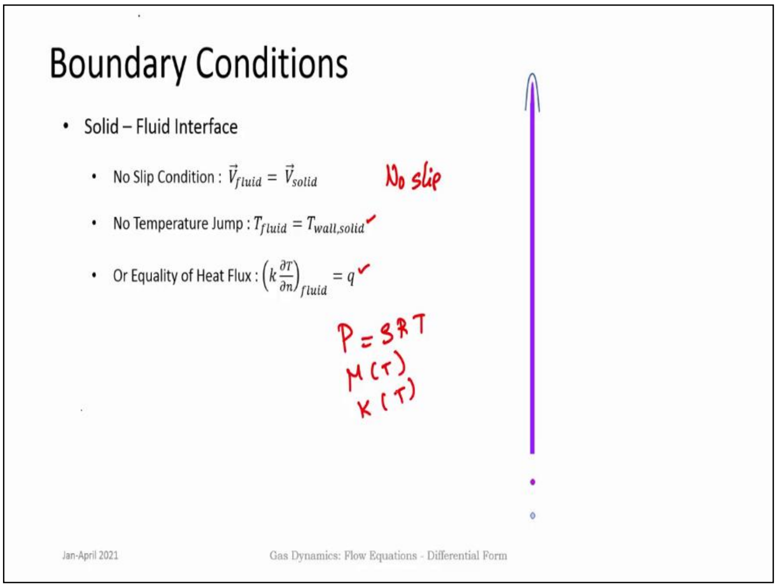

Along with you have to apply relevant boundary conditions the appropriate boundary condition is that at the fluid solid interface the velocities of fluid and solid are the same which is no slip condition. And temperature you can when looking at the temperature boundary condition you can either be that the temperature at the fluid and solid are the same no temperature jump across them.

Or if you are solving you can apply either of the two cases one is no temperature jump or equality of heat flux at the boundary. They are two different kind of problems either or the of them have to be applied so if you apply. The appropriate boundary conditions with the equations and along with this you need of course the equation of state $P = \rho R T$ and some function of for the viscosity and thermal conductivity.

If you get them then you can solve these equations but they are extremely complex and non-linear so there are no analytical forms they are usually solved in a numerical form. So this is another kind of analyzing this fluid flows and this is done using differential equations. So it is called the differential form. So we have integral forms and differential forms of equation. The first kind of analysis that we will do is always using the integral forms.

So we can get to know what is happening within a certain control volume due to fluxes across them and we can get simple relations between what is happening to the fluid flow. So we will proceed with a specific kind of analysis with certain assumptions known as Quasi-1D assumptions in the coming classes, thank you.