Gasdynamics: Fundamentals and Applications
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Lecture 54
Method of Characteristics - Applications

So, we are looking at Method of Characteristics approach to solve supersonic flow field problems in case of inviscid irrotational isentropic flows. In the previous classes we discussed what these Characteristic lines are and how do you get the Characteristic equations and Compatibility conditions.

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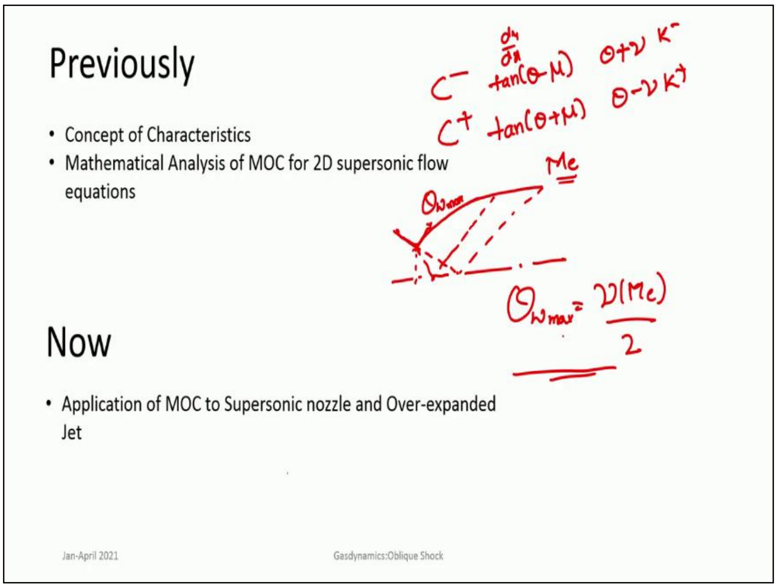

For a 2D supersonic flow the characteristic lines are C- and C+. The slopes of the characteristic lines dy/dx = $\tan(\theta - \mu)$, where μ is the Mach angle. For C- characteristic and t$\tan(\theta + \mu)$ for C+ characteristic along the characteristic lines '$\theta - \nu$', ν is the Prandtl-Meyer angle, C- it is $\theta + \nu$, that is constant for C+. So, these are given values K- and K+ these are constant.

We also had an introduction to what is meant by contoured Nozzle, where the Nozzle is designed in such a way that the flow at the exit, right the flow at the exit is uniform. Otherwise, if the Nozzle wall is not properly controlled then these characteristics keep reflecting off the walls and at the end you get a non-uniform flow. To get a uniform flow, you must design the wall in such a way that it cancels the waves, these Mach waves, once they impinge on the wall that can be done by making the wall angle equal to the flow angle at each Mach wave.

So, that is what is the principle that we are going to use to look at how to design such Nozzles. We had a relationship for what should be the flow turn. So, now I know what my exit Mach number, M_e, should be. I have to give a certain flow turn. So, that the flow expands here that is the θ_{wall} maximum, that I have to give in order to achieve a certain Mach exit and this relationship was $\theta_{wall,max} = {\nu_{exit}}/{2}$. This also we proved by using Method of Characteristic itself.

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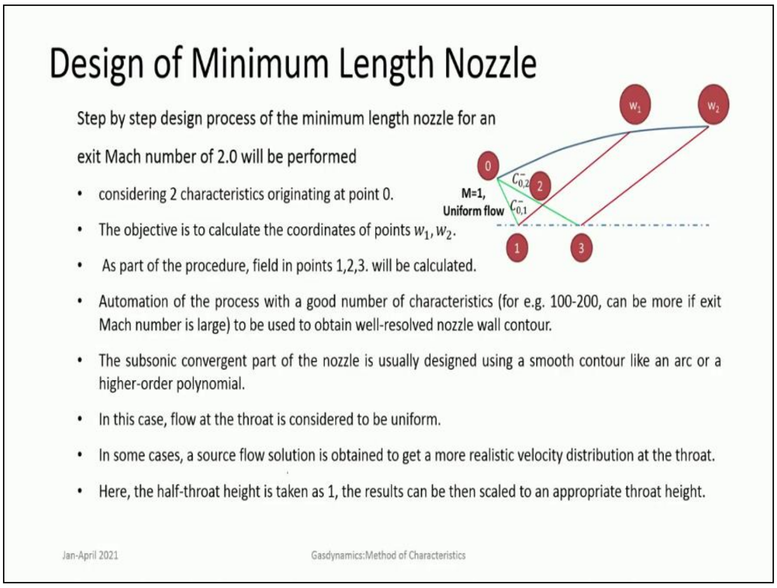

So, now let us apply this process for design of a minimum length Nozzle for Mach 2. We will just consider 2 points because you can see just for these 2 points, we have 1, 2, 3, 4, 5 points need to be evaluated also property is at 1 at 0 which is the originating point. We have uniform Mach 1 flow at section 0. Sometimes this is not always true you may have certain non-uniform flow also may be present.

But the flow is supersonic if the flow is supersonic then that can be considered, and you can draw an initial characteristic net starting at this point. So that there are methods detailed methods for that on how to consider such situations which is not in the scope of what we are discussing. We are just discussing that we have a uniform flow and then we turn it by a Prandtl-Meyer expansion at as a corner over here.

Then look at designing a wall such that there will be no wave reflections at the wall therefore at the end you will get a uniform flow. If this needs to be achieved correctly and a smooth profile needs to be got then we need many waves you may need to take hundreds of waves to get a smooth profile. But here for a classroom case we will just take 2 characteristics.

We will see the procedure. So, that you become familiar with it and if somebody is applying this in their work then they need to write a MOC code a simple code to do this calculation for many, many waves.

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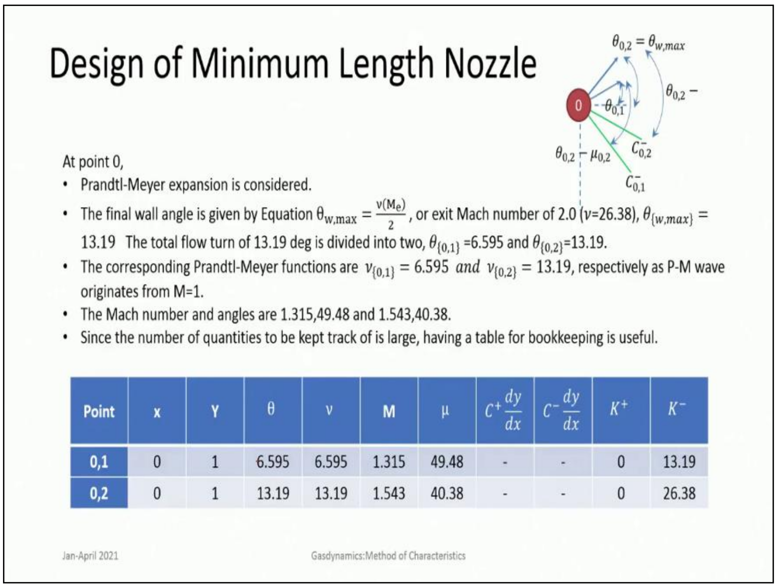

So, we let us begin from we look at the point 0. So, at the point 0 you can see. So, this is the point 0 and here this flow here is uniform Mach number equal to 1 and the flow turns at point 0 by an angle. So, Mach number equal to one, $\theta = 0$ and it turns by an $\theta_{wall,max}$ which is equal to ${}^{\nu_{exit}}/_{2}$, Mach exit by 2. Now this total turn can be accomplished by several set of waves.

In this case we are considering only 2 waves, 1 the final flow turns, and the 1 at a half of that. So, the flow turns for say Mach number 2 the $\theta_{wall,max}$ needed to be achieved is 13.19^{o} degrees and that is divided into 2 angles, 1 at 6.595^{o} and the other 1 at 13.19^{o}. Corresponding Prandtl-Meyer functions are also 6.595^{o} and 13.19^{o}.

$\Delta\theta = \nu_2 - \nu_1$ but Mach number, M_1, is 1. So, ν_1 is 0 therefore, $\Delta\theta = \nu_2$ or in this case, since θ is also 0 initially, $\theta_2 = \nu_2$. So, the point is there are 2 waves essentially, considering that the half angle or the half height of the Nozzle at the throat is 1. So, you can appropriately scale these coordinates once you get the results.

Now when we do this kind of solution, we are solving both for x and y as well as the flow properties at each point. It becomes very difficult to keep track of these various quantities, therefore it is useful to always have a table of the salt when you are doing it by hand. Otherwise, generally people write a code to do this automatically. Now to understand the procedure we can do for a few points.

One can use this procedure. So, at point at the wave which is 1 we know θ value, we know ν value, corresponding Mach number is 1.315 and corresponding, ν that is Mach angle is 49.48 the value of K+ and K- is also given here. K+ is $\theta - \nu$ that is 0 but K- is $\theta + \nu$, which is 13.19 and 26.28 for the 2 different waves.

Now we go to point 1. Now point 1 is located here, it is the first wave that is produced at point 0 it is going and coming to the point of line of symmetry at point 1. So, we apply the symmetry boundary condition here that is $\theta_1 = 0$, this is a C- characteristic. So, $\theta_1 + \nu_1$ =13.19.

Therefore, corresponding Mach number, you can get is 1.543 and corresponding Mach angle also you can get 40.38°. Now how to get the value of x and y here this is the x-y. So, this is along x-coordinate, this is y-coordinate, now this is a point on line of symmetry which is at x-axis. So, y = 0. But we need to find the value of x for that you write the equation for this C-characteristic,

$$\frac{y_1 - y_0}{x_1 - 0} = \tan\left(\frac{(\theta_1 + \theta_0) + (\mu_1 + \mu_0)}{2}\right)$$

you can get x_1 as 1.125. So, we write the value for 1 here, x is known y = 0, θ is 0. Now ν we got it is 13.19, corresponding Mach number and μ are also written, and the value for the value of the slope that we used. So, that we got the values for x and y is also given over here.

Now K+ and K- can be determined again here, K+ now is $\theta - \nu$ which is -13.19. So, θ is 0 and ν is 13.19. So, now we go from point 1 to point 2. Now point 2 is an interior point.

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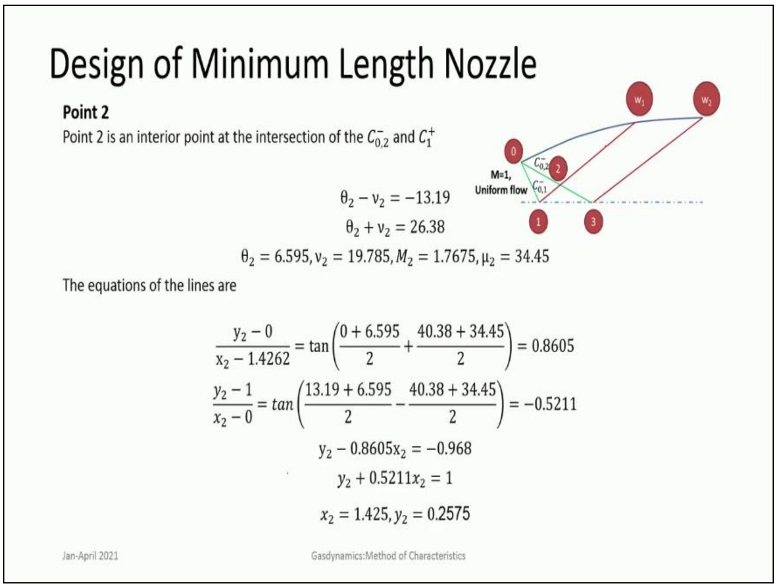

So, this problem is also a description of various unit processes that we had discussed. So, at point 2 we have a C+ characteristic coming from 1, and a C- characteristic coming from 0 this is the second wave from 0.

So, we know these values. So, just we must solve them algebraically you have the equation for that from here we can calculate what is θ_2, ν_2 which is the Prandtl-Meyer angle. So, θ_2 is 6.595. So, you find that this flow is not uniform because here the angle is 0, but here at point 2 it has an angle 6.595. So, the flow is deflected by 6.595^o with respect to x-axis. Mach number has increased, it has come to 1.7675 and similarly μ_2 has changed. Now we need to find x_2 and y_2 we write the equations of straight lines between these 2 points.

So, 0, 2 and 1, 2 we know the values for 1 that is already known, and we know the value for 0 we substitute those values, and we get the numbers now x_2 and y_2. So, now this can be again put into the table.

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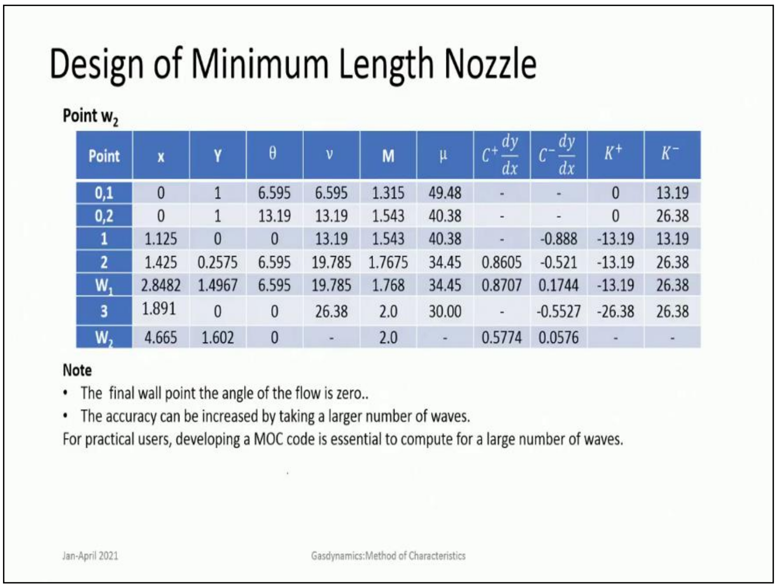

Point	x	Y	θ	ν	M	μ	$C^+ \frac{dy}{dx}$	$C^- \frac{dy}{dx}$	K^+	K^-
0,1	0	1	6.595	6.595	1.315	49.48	-	-	0	13.19
0,2	0	1	13.19	13.19	1.543	40.38	-	-	0	26.38
1	1.125	0	0	13.19	1.543	40.38	-	-0.888	-13.19	13.19
2	1.425	0.2575	6.595	19.785	1.7675	34.45	0.8605	-0.521	-13.19	26.38
W_1	2.8482	1.4967	6.595	19.785	1.768	34.45	0.8707	0.1744	-13.19	26.38
3	1.891	0	0	26.38	2.0	30.00	-	-0.5527	-26.38	26.38
W_2	4.665	1.602	0	-	2.0	-	0.5774	0.0576	-	-

So, we can keep expanding the table now point 2 is available. So, it is slightly downstream of 0.1 and slightly above the line of symmetry and here the flow is having an angle of 6.595°. So, now we look at the first we look at the first wall angle.

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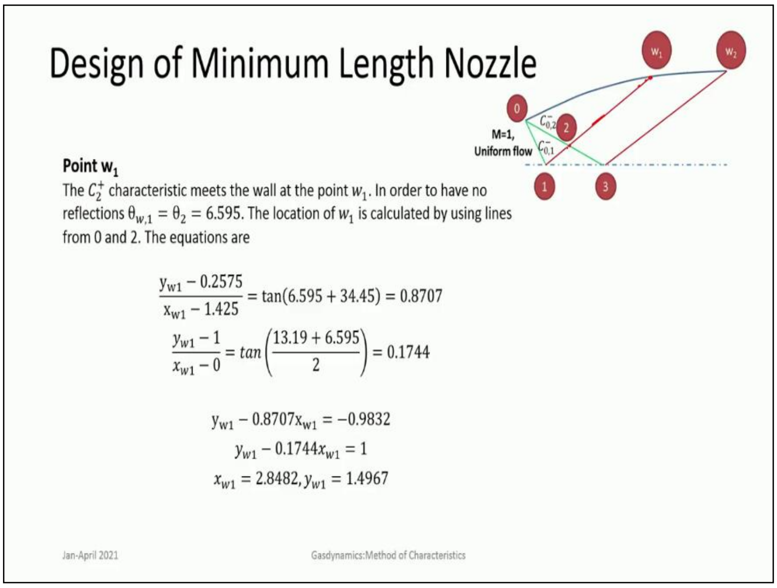

If you follow this characteristics C+ characteristic after point 2, it goes to the wall point, now to ensure that there is no wall cancellation the angle at the wall should be equal to the angle of the flow. The angle of flow at this point is 6.595°. So, the wall should match that. So, you get theta wall equal to theta 2 equal to 6.595 °.

So, now we know the wall angle at 0 this wall angle is $\theta_{w,max}$ this is known to us. Now we know the wall angle at this point that is 6.595°. So, this equation is also estimated to be a straight line. So, in this case these 2 points are far apart. So, we will get a small error here that is understood but when this is really done it is done in computers and several waves are used very large number say hundreds of waves are used.

Then the points are very close to each other and therefore you can get a smooth profile. But for class work's sake or classroom's sake we will just simply take a fewer number of points and therefore you just write the equation of the straight line between 0 and w_1 and 2 and w_1, we take average angles again and therefore, we get these 2 equations. The previous values are substituted here and in the locations for points of 2 and 0 and we can get the values of wall as x_{w1} and y_{w1}.

So, you are getting these numbers now. So, this number is away much away. So, this is about 1.4967 and the location is 2.8482.

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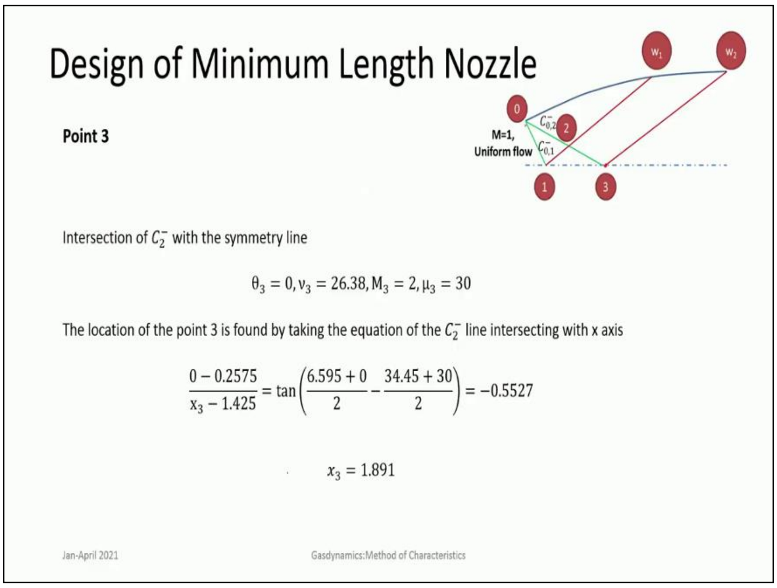

So, now we can continue this approach go to point 3 that is a symmetry point now this point is coming from the second wave after it has passed through point 2. So, we must take the C- characteristic from point 2 and let it come to point 3. Here at point 3, will find that the value of Mach number at 3 will be equal to 2 and angle is 0 this is exactly what is required and consequence the wall angle at w_2 should also be 0.

So, that should be the case. So, that should be the end of the wave cancellation section and you write the equation of C- characteristic again and here y is 0. So, you can solve for this starting from the point 2 and you get x_3 is 1.891. So, you got value of these different numbers now.

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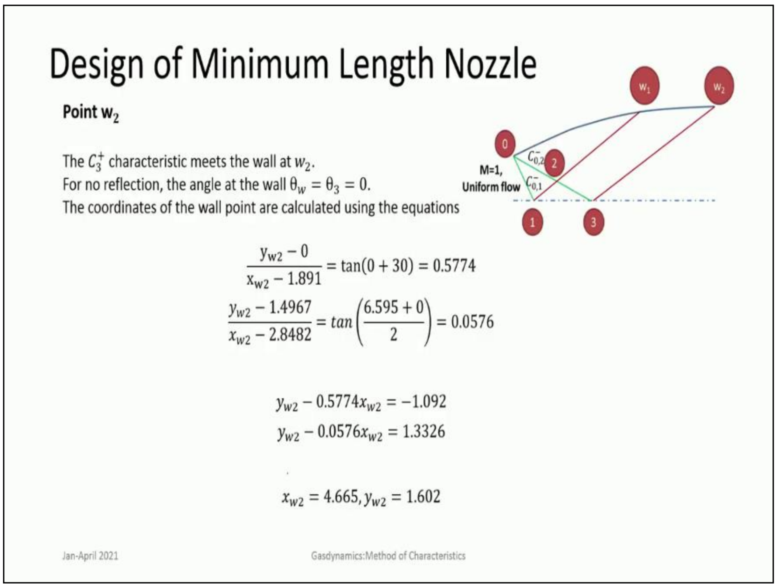

Now you can further go to wall point 2 at wall point 2 the value of θ is 0. So, it is the end. So, it has become parallel to the x-axis. So, you get a uniform flow here Mach number is 2.0. You can get the x and y values by considering the previous ball point and a straight line from there and a straight line from θ_3 and the intersection of those 2 and here you get y wall is 1.602.

So, if you if we had looked at the area ratio for Mach number equal to 2, we would find that y should be from isentropic area ratio relations it should be 1.6875 of course we have considered only very few waves here. If we take a good number of waves, we will get a more accurate solution. But even considering, 2 waves we have got a solution which is very close to the expected values.

So, now by this set, now we have completed all the different things and we have got the points w_1 and w_2 if you take many numbers of waves, we can get the smooth contour. So, for people who are going to use this they need to develop a small code to compute for large number of waves for their problem. So, I hope with this example you get an idea of how to use MOC for solving supersonic flow fields it can be used for analysing supersonic flow fields or it can be used for designing supersonic Nozzles.

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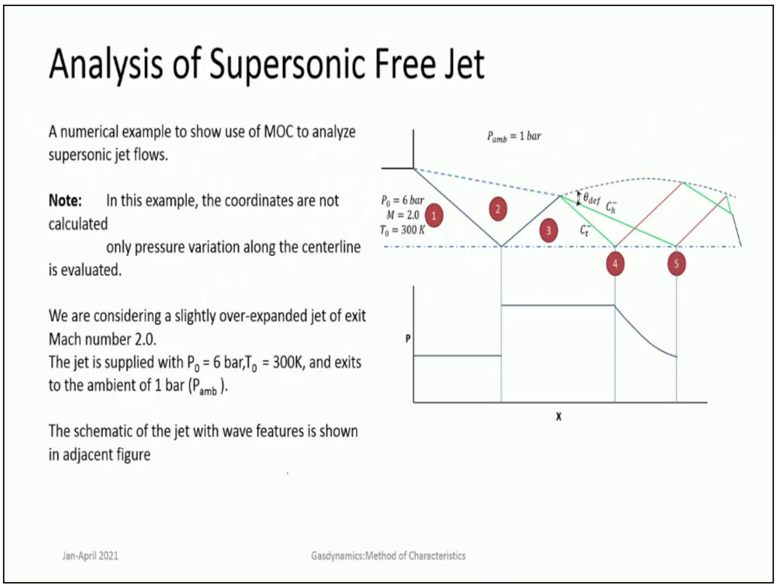

This is a particular case where we are looking at analysing a flow field understanding how flow field looks like by knowing the initial values. This is the case of analysing a supersonic free jet that is the jet agent from that it can be a contoured Nozzle it is coming out into ambient. The initial conditions of the jet are given P_o = 6 bar, T_o = 300 K and Mach number is 2.0.

So, if you calculate this at this exit location what would be the pressure corresponding to Mach number 2.0, it can be shown that this pressure is less than 1 bar that means this Nozzle is working in the over expanded condition, consequently it will develop Oblique shock waves. Now this Oblique shock wave will come to the centre line. At centre line of course the flow must maintain θ equal to 0 condition.

So, to ensure that that happens because if you consider a streamline here this streamline is deflected towards the flow but at this region the flow must be parallel again to x-axis therefore, another shock develops. Because of 2 shocks, you get region 3 which has much higher pressure. So, if you look at the centreline pressure variation initially you have a pressure which is lower than ambient pressure.

Then suddenly the pressure will increase higher than ambient pressure in the region 2 it will be exactly ambient pressure. This is a case where we are looking at reflection of waves from a free pressure boundary and we know for to achieve that always the pressure at the boundary should be 1 and we find that pressure at 3 is higher therefore it develops expansion fans.

And then further you can see how pressure varies here flow is non uniform and pressure varies in a smooth manner.

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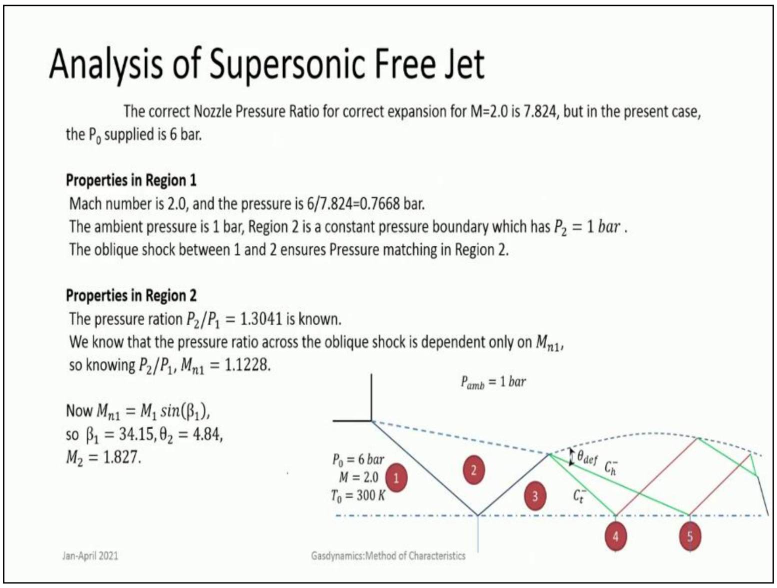

So, this analysis can be carried out using Method of Characteristics and shock waves together. In regions those shock waves are discontinuity so across the shock wave there is an entropy jump but once you get into uniform regions, they are isentropic. So, those regions can be still analysed using MOC. At the shock you will have to apply the shock jump conditions. So, if you look at that. So, at in region 1 if you find out what is the pressure by looking at Nozzle P_0/ P for Mach 2, you find pressure is 0.7668 bar.

Now region 2 has a pressure equal to P ambient equal to 1 bar. So, we know the value of P_2/P_1 across the shock this value is known. Since we know the pressure ratio across the shock, we know that that for an oblique shock it is dependent only on the normal Mach number upstream of the shock. So, we can get M_{n1} which is 1.1228, if you know the Mach number M_1 is known and M_{n1} is known, we know M_{n1} is equal to M_1 sin β, therefore you can get β, now M and β is known we can get θ also.

So, we get how much the flow is deflected it is deflected by 4.824^{o} pressure in this region is 1 bar. But to make sure that the flow remains parallel in at the line of symmetry this shock wave develops between region 2 and 3. So, Mach number in region 2 is 1.827 this is just by Oblique shock equations you can solve for it.

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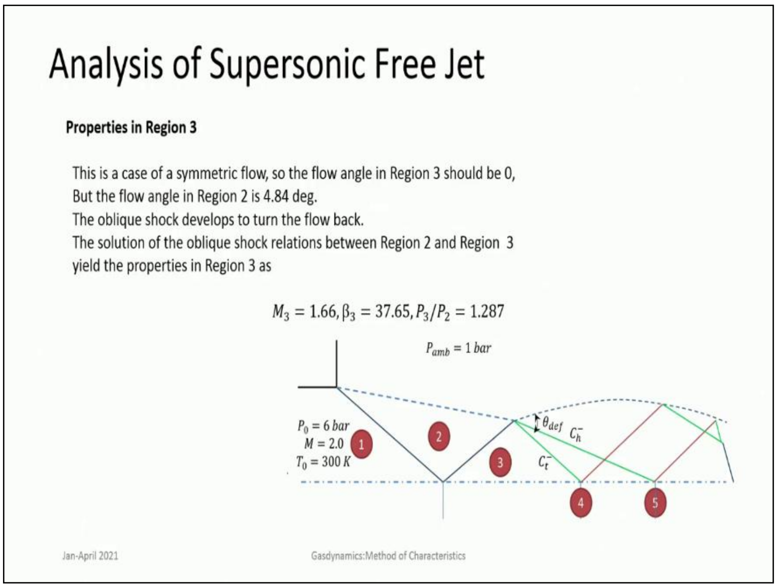

Now it is a symmetric flow. So, between 2 and 3 another shock forms again you can calculate it you know Mach number in M_2 then you know what the flow deflection should be as the same 4.84 degrees. Therefore, you can know M and θ is known you can find out all the other parameters what is P_3/P_2 and so on. So, here pressure becomes 1.287 bar. So, now pressure has increased in point 3 but again you require in this region a constant pressure of 1 bar.

So, across these 2 points regions between 3 and the region that is represented over here we will have an expansion fan and that expansion fan can be solved using Prandtl-Meyer theory.

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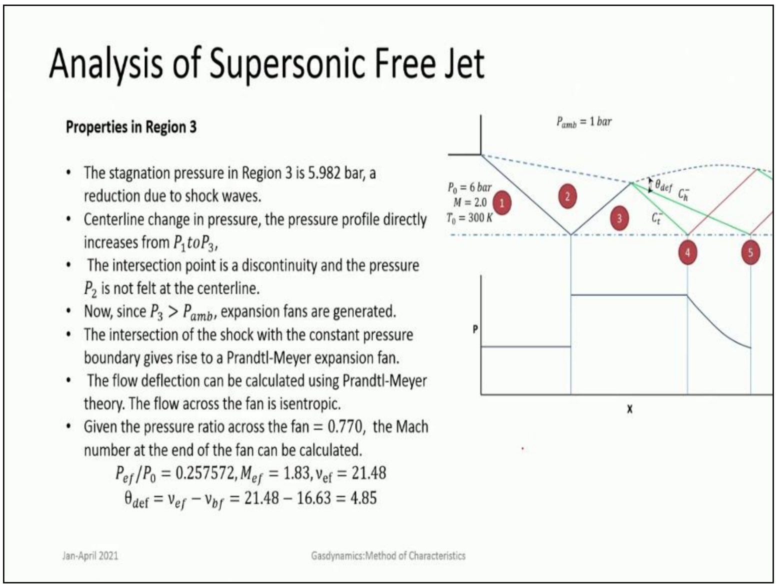

Here Prandtl-Meyer expansion is an isentropic expansion, and we need to find what is the stagnation pressure in region 3, P_{03} this can be found it is 5.982 bar because of 2 shock waves. You see that the stagnation pressure has reduced from 6 bar this due to entropy generation in shock waves. Now this pressure is higher than ambient. So, you will get an expansion you know the pressure over here which is 1 bar.

Therefore, you know pressure across the Prandtl-Meyer expansion fan and therefore since the flow is isentropic, we can calculate what should be the Mach number of the final wave after which it again gets into uniform flow and that Mach number is found out. So, that Mach number is 1.83 and corresponding flow deflection can be found it is 4.85 degrees. So, now this Prandtl-Meyer expansion now if you see this region, it is very much similar like the analysis of the supersonic Nozzle that we just conducted right now.

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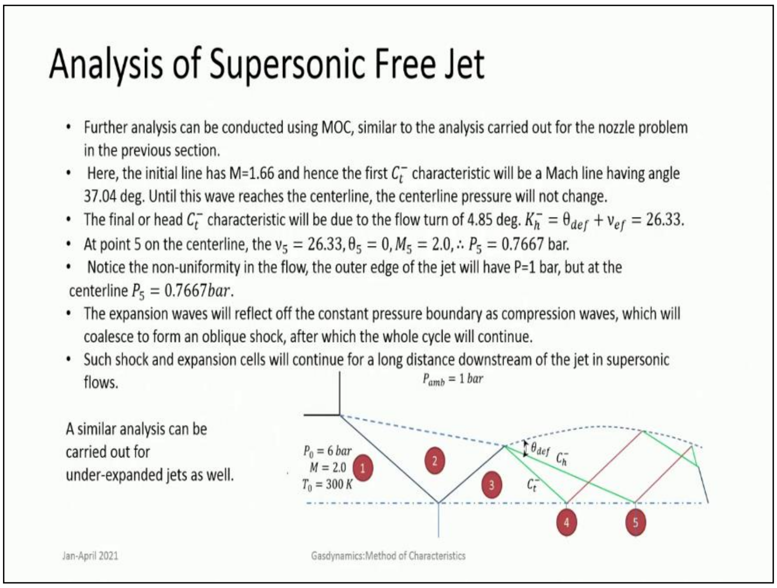

So, we will not go into all the details here but just understand how we can get the pressures at these points 4 and 5 the initial line. So, the head of the expansion fan is here, and head is here, and tail is here. So, these are Mach waves at those values. So, we can consider 2 different set of waves again like the way we did for supersonic Nozzle we divide the total turn into 2 turns. So, use these are C- characteristics.

So, you have $\theta + \nu$ is constant and at the point 4, it is a line of symmetry point 4 and 5 therefore you get the value of Mach number at 4 and 5 because θ is 0. If you know the value of ν_5 if you know M_5, we can calculate P_5. So, by this way we can continuously find out this and this in turn continues as a whole cycle. In fact, if you take a picture of these supersonic jets, you find these shock cells and expansion cells and shock waves and expansion waves they continue for a very long duration.

So, this just the initial part of it we have analysed in greater detail than what we were doing using 1 dimensional theory. So, similar analysis can be carried out for under expanded jets as well. So, that can be done is a very similar process can be carried out now. Once the expansion phase reflects of the constant pressure boundary, we know they become Compression waves and ultimately can coalesce to form a shock wave which is what happens over here.

So, we saw now that Method of Characteristics is a very useful technique not only to look at the design of contoured supersonic Nozzles but to solve supersonic flows in general. If you can find regions where flow is isentropic, and it is also non uniform in general still that can be solved using Method of Characteristic provided you know some initial conditions and you can march from that initial condition into the flow field and solve the entire flow field.

Here unlike what currently we do in many cases of CFD where we have a given mesh and we try to solve the equations over the mesh here both the equations and the solutions are done simultaneously. So, you solve the characteristic net as well as you solve the flow properties at every section as you move in forward direction. With this we end of discussions on the flow field kind of analysis in subsonic supersonic flows.

We saw 2 kinds of approaches 1 was the small perturbation theory and in supersonic flow we saw Method of Characteristics a useful approach again. This covers majority of the contents that we had planned for this course the intention also here is that to introduce you to several special topics in that we find in compressible flows very many interesting facts or very interesting flow features are present.

But it is not possible to cover all of them in a short course they are full courses in themselves, but a brief idea can be presented of specific topics like what do you mean by hypersonic flow or how does flow look like in real cases where there are viscous effects along the walls which forms boundary layers. There are also shock waves in supersonic flows how do they interact with each other.

We have talked about reflection of shock waves but often we sometimes find for example even in the jet problem that 2 shock waves interact with each other those are shock-shock interactions. These are very much present in all practical applications of compressible flows. So, we will just introduce these topics towards the end of this course in the subsequent classes.

So, that people interested in them can take them and learn more from other references with that we close this class, thank you.