Gasdynamics: Fundamentals And Applications
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Lecture 53
Method of Characteristics 2D Supersonic flow - II

So, we are looking at a particular approach to solving the exact equations, velocity potential equations for Supersonic flow. It is possible in the case of Supersonic flow because, the nature of equations is hyperbolic, there is a particular method known as Method of Characteristics, which can be used to solve hyperbolic equations. In the last class we saw how Method of Characteristics is applied to the 2D velocity potential equation in Supersonic flow.

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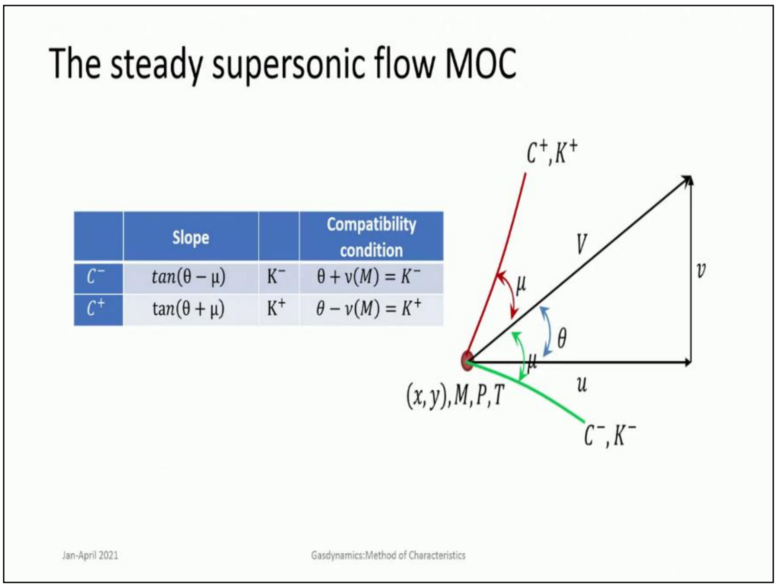

The total of what was done in the last class was that, if you consider any point, any general point in the flow field at any x and y having a certain Mach number, Pressure, Temperature, it is an isentropic flow, you find that it has a certain direction having an angle θ. There are 2 characteristics associated with this point, 1 is C+ characteristic which happens at $\tan(\theta+\mu)$, that is the slope of that characteristic.

Along the C+ characteristic there is a Compatibility condition, which is $\theta+\nu$, which is the Prandtl-Meyer angle $'\nu'$ of M that is a constant along this characteristic curve. Similarly, if you take the C- characteristics which happens at '$\theta+\mu$', where $'\mu'$ is the Mach angle, then along the C- characteristics there is another constant K-, which is $\theta+\nu$, which is a Prandtl-Meyer angle. So, now you have 2 sets of equations, 1 is equations for the slope at that point which is C- and C+, then the Compatibility conditions for those corresponding Characteristic curves.

This gives us a method to solve the Supersonic flow field. This is like or this is this kind of approaches fall in the domain of computational fluid dynamics which is CFD where we are solving for the entire flow field. We get in the entire flow field all values of Mach number pressure temperature and so on. If you happen to have done some CFD the normal approach nowadays is that you already have a mesh available for a certain geometry.

Then the flow equations are solved over that mesh but in this context in the Method of Characteristics you solve both the mesh as well as the flow solution marching from an initial condition. So, Forward marching from an initial condition, so, you solve the mesh in this case that kind of a mesh is known as a Characteristic Net, the mesh as well as the flow conditions at each point. You start from an initial line or initial condition.

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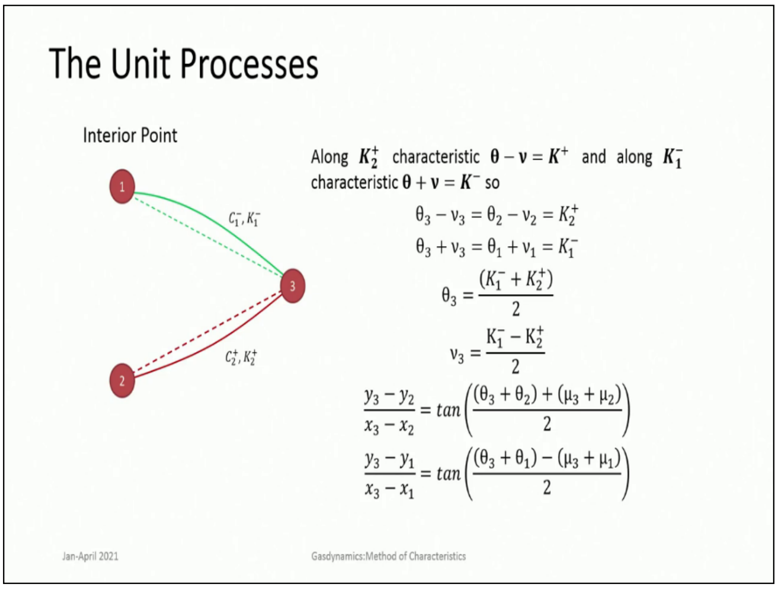

So, to do this, there are certain set of sorts of procedures that has to be done for certain points within the flow or at the boundaries. So, what are these processes, we will see they are called the Unit process. If you consider an interior point that means it is in well inside the flow all its neighbours around it are flow points, they are all inside the domain of the flow. Then how do we solve the equations? how do we get Mach number and the location of this point?

So, let us take 2 points which we know point one. We know the values say M_1, θ_1. And M_2, θ_2 these are known to us already. Then our interest is to find what is M_3, θ_3? If you know M_1, know the Prandtl-Meyer angle ν_1, and you also know ν_2. So, we are interested to find M_3, θ_3. How do we do that?

We take the corresponding characteristics. So, point 3 is formed by the intersection of 2 characteristics, it is an intersection of the C+ characteristic from point 2 and C- characteristic from point 1. So, along C- characteristic K- is constant. So, we can write

$$\theta_3 - \nu_3 = \theta_2 - \nu_2 = K_2^+$$

$$\theta_3 + \nu_3 = \theta_1 + \nu_1 = K_1^-$$

So, use the corresponding equations, from here it is easy to calculate θ_3, it's just

$$\theta_3 = \frac{(K_1^- + K_2^+)}{2}$$

$$\nu_3 = \frac{(K_1^- - K_2^+)}{2}$$

So, we can solve for θ_3 and ν_3, if you get the Prandtl-Meyer angle ν_3, you can invert this, you can look at the charts and invert them or use a calculator you can get M_3. So, you have solved for the Mach number at that point. You also have solved for the direction θ_3.

So, you know both Mach number and direction at point 3, but where is this point 3 located. So, this point is the location of each intersection of 2 curves each of them starting from 1 and 2, respectively. Now in general the characteristics they are not straight lines but if you take good number of discretized points which are close to each other then we can assume that they are almost a straight line in between the points 1, 3 and 2, 3 respectively.

To sort of give a better accuracy the angle that is taken is taken as an average. So, we know that the equation for C_2^+ is the dy/ dx is $\tan(\theta + \mu)$. So, we take the averages of angles at 3 and 2 it is a sort of you can use a predictor corrector method where you have an initial estimate of what should be you can use an initial estimate.

Then come back and revaluate also or just take the averages. So, dy/ dx is now a straight line it is approximated as a straight-line

$$\frac{y_3 - y_2}{x_3 - x_2} = \tan\left(\frac{(\theta_3 + \theta_2) + (\mu_3 + \mu_2)}{2}\right)$$

$$\frac{y_3 - y_1}{x_3 - x_1} = \tan\left(\frac{(\theta_3 + \theta_1) - (\mu_3 + \mu_1)}{2}\right)$$

So, you will know μ_3 already. So, this is the case for an interior point, it is with well within the flow field any such point, we can solve using this set of equations. Using the Compatibility conditions, we can get θ_3 and what is Prandtl-Meyer angle at 3, from Prandtl-Meyer angle at 3, we can invert and get Mach number at 3.

If you know Mach number, you can get the Mach angle at 3. What we need to know is the location at of 3. Location of 3 is found out by assuming the points that are connected they are connected by straight lines. This will give us a good solution with the least error if these points are very close to each other. Then you write the equations of the straight line with average angles and then you can get the solution for x_3 and y_3.

So, through this process we get $x_3,\ y_3,\ M_3, \theta_3, \mu_3, and\ \nu_3$. So, now again at 3 we can define K+ and K-. So, this process can be taken forward.

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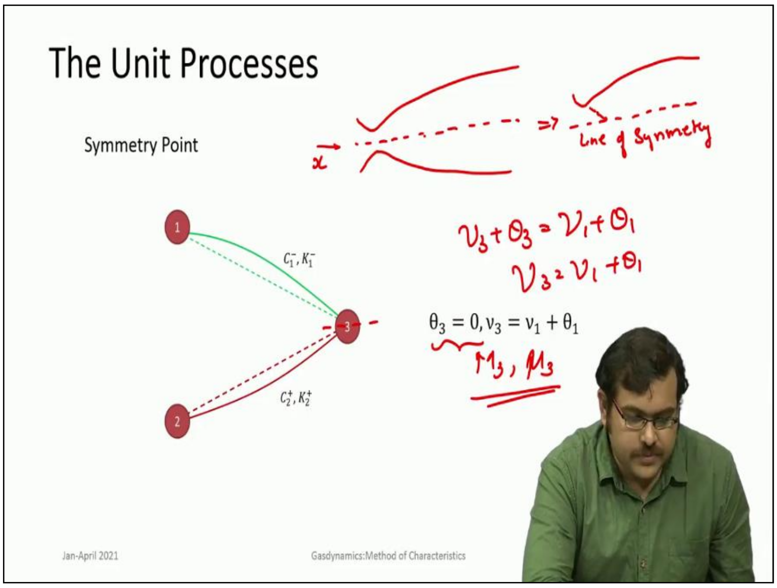

Now what happens if we have a point of symmetry, that point 3 falls in the symmetry line, it can be for example, you can take a nozzle of this kind which is exactly symmetric about the centre line. Then generally we do not solve for the entire plane because it is symmetric about centre line. We solve only 1 half of this. So, there is a symmetry line or line of symmetry.

So, when you consider such a line of symmetry, then you will have lines or characteristic lines coming from certain points and at 3 we find that it is a line of symmetry. So, if you look at a symmetric point, for example, in this case it is symmetric at along x-axis then the angle is 0. So, θ_3 is equal to 0, you can apply this then if you take any, for example, you are taking the C- characteristic which is coming over here then you know that along C- characteristic

$$\theta_3 + \nu_3 = \theta_1 + \nu_1$$

$$\theta_3 = 0$$

$$\nu_3 = \theta_1 + \nu_1$$

So, now you have got the solution for ν_3, you also know θ_3. From here, you can get M_3. You get μ_3 also. So, you can appropriately apply a symmetry boundary condition. You must look at the problem in this case we found that the symmetry was along the x-axis and correspondingly we took θ_3 is equal to zero at the x-axis.

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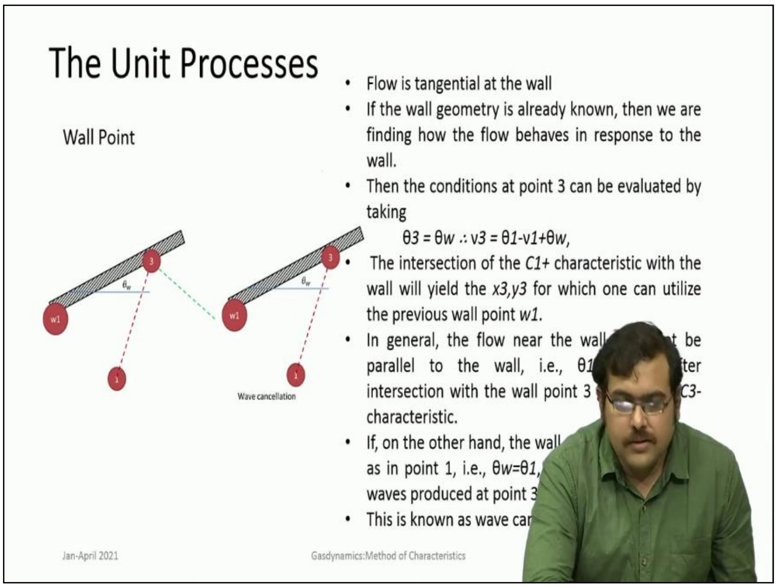

Now the other case is if these characteristics curves go and meet the wall that is known as the wall point. Then we apply the wall boundary condition it is an inviscid flow. In an inviscid flow the ball boundary condition is flow is always tangential at the wall. Now there are 2 cases usually. If you are analysing a certain case, then you are looking at a problem where you already know the geometry of the wall, but you are trying to find out what is the flow field.

Then what we know is this point 1 is some point within the interior in the flow. We know a previous wall point this is a certain section of the wall of the entire wall geometry. This is a previous wall geometry w_1. 3 is the line which lies on the wall that is a now this can be a C+

characteristic coming from 1 and impinging on the wall at 3. What we need to know is what happens at 3.

So, since the always the flow must be parallel to the wall. If we know the contour of the wall, then we know the angle of the wall at 3. So, we know θ_{wall}, wall basically so, $\theta_3 - \nu_3$ so, this C+ characteristic. So, in a C+ characteristic it is,

$$\theta_3 - \nu_3 = \theta_1 - \nu_1$$

$$\theta_3 = \theta_{wall}$$

Therefore,

$$\nu_3 = \theta_{wall} - \theta_1 + \nu_1$$

So, this has this equation needs to be corrected appropriately. But here in general what happens is at point 3 another set of characteristics can be produced which is now in the C- direction C- characteristic. So, the characteristics that is a C+ characteristic can get go to 3 and then get reflected from that point.

So, but if we are looking to see that; we do not get any C- reflection of these characteristics. This is useful when we are looking to design a contour such that the wall changes in such a way that it will always produce a uniform flow. So, if you must do that then it is possible to do that if the $\theta_{wall} = \theta_1$. Flow in the neighbourhood of the wall, if it is parallel to the wall already parallel to the wall then it needs to take no more turn.

So, consequently there will be no more reflection this is like the wave cancellation that we discussed in the Oblique shock case. Mach waves are nothing but the weak limit of oblique shocks. So, if the angle of wall is such that it matches the flow direction then there will be no reflection. So, exactly if at point 3, $\theta_{wall} = \theta_1$ at point 3 then there is no wave cancellation.

This method is approach is used to produce very smooth contoured surfaces for supersonic nozzles. Otherwise in general if we look at the wall point, we should see whether there is a reflection or not.

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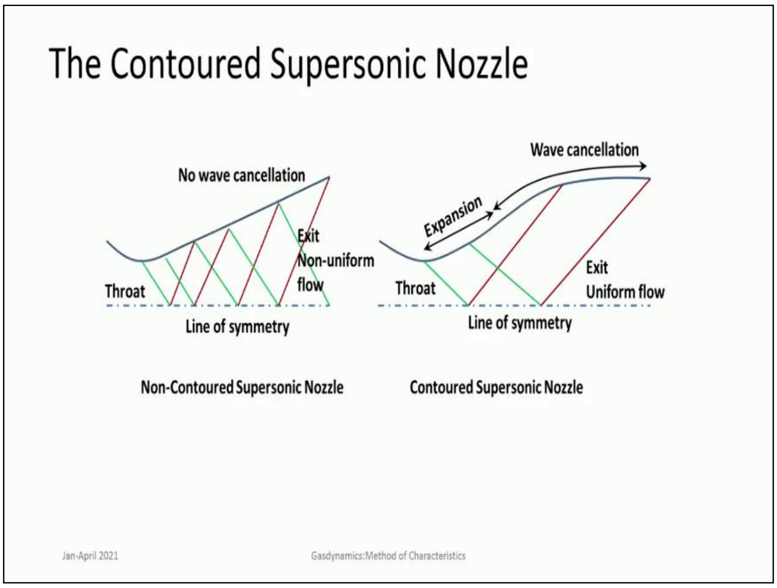

So, now let us look at how this procedure is applied. This is applied extensively for the case of design of Supersonic nozzles, for particularly contoured Supersonic nozzles. If you take as any arbitrary shaped Supersonic nozzle, this is a Convergent-Divergent nozzle. The subsonic portion is generally made smooth then you have the throat. And we know from our previous discussion on nozzles that Mach number of 1 is achieved at the throat.

After that you have a divergent section to accelerate to Supersonic velocities. Now if you take any arbitrary divergent passage, then we have now had just had the discussions. So, expansions produce a set of Mach waves, and these Mach waves get go and hit the wall. If it is any arbitrary contour, then they will reflect off the wall.

Then they keep reflecting of the walls everywhere this is considering a line of symmetry. So, in that case so, at every point if you see here at every point, you have some different Mach number and so on. But what we require in certain cases specifically for say Wind tunnel applications you need a uniform flow coming out of your supersonic nozzle. So, that we are sure that we are giving a specific correct flow or wind to the model which is placed inside the test section.

Then non-uniform flows are not possible, or you should not apply a non-unformed flow. If you take just a short of conical or wedge shape nozzle it will always give a non-uniform flow because now, we can understand what is happening to the flow field within the nozzle. So, then we look at how we can design such nozzles. So, that we always get uniform flow at the exit of the nozzle.

Now this based on the principle just we have discussed now on the characteristics and wave cancellations at the throat there is an expansion you can have a smooth expansion. Then you have for these nozzles you have an expansion section, where the flow is smoothly expanded. Here, Expansion waves, these are nothing but Mach waves they are produced. They come now and then go and impinge upon the walls.

Then in such contoured supersonic nozzles there is a section after the expansion section known as the wave cancellation section, where these Mach waves when they impinge on the wall the wall is designed in such a manner it is contoured in such a manner that the angle of the flow and the angle of the wall match. Therefore, the waves do not reflect again away from this point.

Once that kind of a contour is achieved after each such subsequent interaction with the wall the flow always continues to be uniform with respect to the x direction. So, at the exit you get a uniform flow of constant Mach number.

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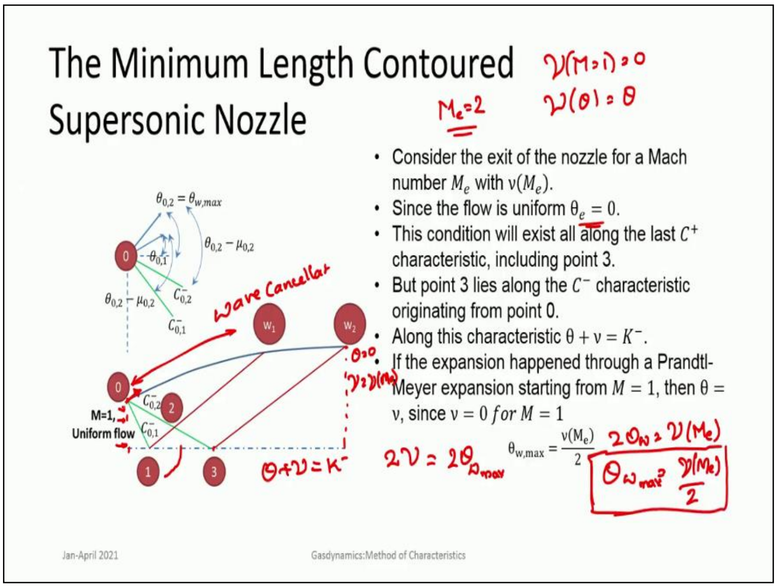

So, we shall look at this in a detailed manner for what is known as a minimum length contoured Supersonic nozzle. The minimum length is used because if you do look at such contoured nozzles, they are really very long they have very long extent. They are also used these principles are also used for design of Rocket nozzles also. In such nozzles we do not require long length because it affects the weight of the entire structure.

So, it is good if it can be made short. There the principles use this; if this expansion section can be made very sharp not made. So, long but rather it is made to turn at a very sharp corner then this can be achieved. That is done by placing a sharp corner. Turn flow is turned at the sharp corner by a Prandtl-Meyer expansion. It is not an extended expansion. So, we will see the design of such a Minimum Length Supersonic nozzle.

Then for example it is given here it is considered in this graph. So, we are only looking at the wave cancellation section. So, all the section over here is only wave cancellation. All the waves are produced at this point, which is a Prandtl-Meyer expansion. This is also the throat. So, here Mach number is 1. We are assuming that you have a uniform flow at Mach number equal to 1.

So, what should be the initial wall angle? So, that we get the correct exit velocity at the end of the wave cancellation section. So, that is the problem of how to design these nozzles. If you are considering that I want to design a Mach 2 nozzle, M_{exit} equal to Mach number 2, then what is the correct angle of expansion that needs to be provided and how the wall should be contoured. So, we can just look at that by looking at these characteristics over here.

So, if you look at the final characteristic that impinges at w 2. And trace the way back all the way towards the point 0 then we should be able to comment on that particular angle. Now at the exit we know that the flow is uniform, and theta is 0. So, if you consider these characteristics $\theta_{wall} = 0$. ν corresponds to Mach number of M_{exit}.

So, this lies along this C+ characteristic, but at point 3 a part of a C- characteristic originating from point zero. So, along this C- characteristic you get that theta minus nu is constant is a constant which is K-. So, the expansion is happening through a Prandtl-Meyer expansion. The angle of turn it gets is a certain value theta.

If the initial flow is Mach number equal to 1 then we know that ν of Mach number equal to 1 is 0, therefore ν that is due to the flow turn of theta should be equal to θ itself. So, it is equal to θ itself. Since it is turning that you apply the Prandtl-Meyer equations, and it is turning from Mach number equal to 1. So, what we get is theta + nu that is equal to 2ν. This is equal to $\theta + \nu$ is 2ν or $2\theta_{wall}$, because ν and θ are equal at maximum turn, the wall turn that is given over here.

So that is equal to now from the K+ characteristic it is equal to nu that is the nu of the exit angle. So, you get 2 θ wall is equal to ν of Mach number at exit. So, therefore θ_{wall} maximum the flow tone is ν max exit by 2. So, this is the relationship to find what should angle should the wall turn at point zero to get the exact Mach number at the exit. Now this flow turns which is turning by a certain wall angle theta wall maximum can be divided into several waves.

Then each of these waves impinges upon the wall at certain points. The wall is designed so, that it does not reflect any of these this reflection is not done therefore you get this wave con cancellation contour. So, if this is correctly done then we will get uniform Mach to flow at the exit of this nozzle. So, we will do a solution for 1 case and see how it can be achieved.

We will also see certain applications once we understand the Method of Characteristics in Supersonic flows then we should have a better understanding of Supersonic flow fields and we can look at that in the next class.