Gasdynamics: Fundamentals. And Applications
Prof. Srisha Rao M V
Aerospace Engineering
Indian Institute of Science – Bangalore

Lecture 52
Method of Characteristics 2D Supersonic Flow - I

So, we are looking at solving flow fields using Differential equations. In previous classes, we were looking at a particular way in doing them using the velocity potential equation, we introduced a Small Perturbation. This was applicable for small changes, slant very thin bodies in subsonic and supersonic flows. We looked at Prandtl-Gluaert rule. Also, in supersonic flows we found that Cp depends only on the inclination of the surface.

So, that was a particular approach now if you really want to solve the flow fields it is involved. It is not very easy to do it analytically, but in Supersonic flows because of their hyperbolic nature. They have the characteristic that information propagates in certain directions which are known as the Characteristic lines, we can use the Method of Characteristics, which is used typically in any hyperbolic equation.

It is a method to solve hyperbolic partial differential equations in short it is also called as MOC, so, Method of Characteristics.

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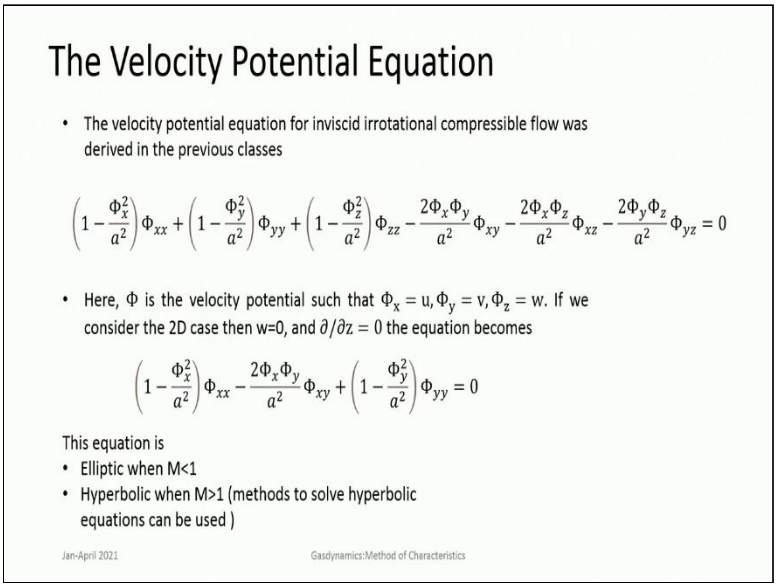

So, we look at how to solve these equations in the context of supersonic flows only. This is the general velocity perturbation equation which is exact and three dimensional. Now we if you take it in the context of 2-dimensional cases, we have already discussed this equation which is exact equation for 2-dimensional case, inviscid irrotational compressible flow. We also had discussed that when M > 1 the flow is hyperbolic right.

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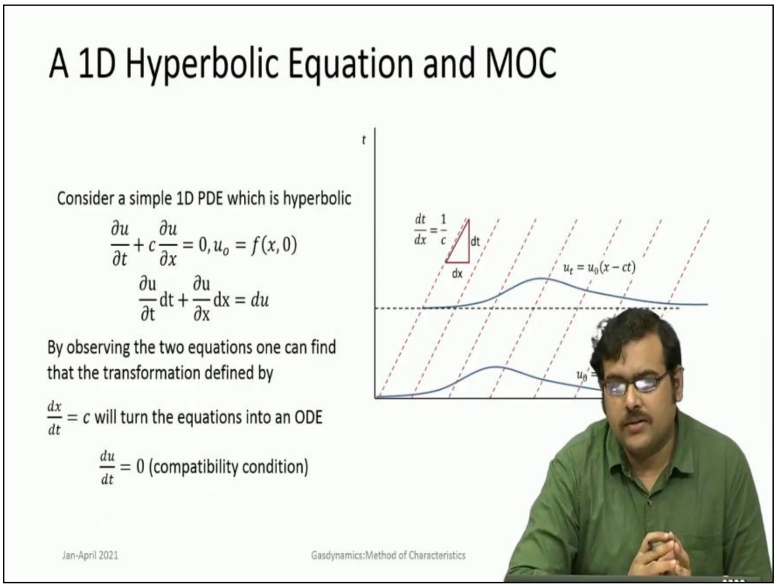

So, now let us look at these hyperbolic equations, partial differential equations, and Method of Characteristics, we have been coming across this at various times in the due course of this lectures of these lectures we encountered it very early on when we were looking at unsteady flows. You found there are lines across which the solution propagates which is ‘u + a’ and ‘u – a’ for isentropic flow that was the unsteady case.

Now we are discussing steady do 2-dimensional case even here in the case of the steady 2-dimensional case if it is a Supersonic flow, then the flow is hyperbolic. Here also it goes along particular lines now, what is the method to get these lines we are looking at how do we get these characteristic lines. So, always it is useful to go look at some simple equation try to understand how it works. Then further extend it to more complex situations that we are facing.

So, take a simple 1D PDE which is hyperbolic,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$u_0 = f(x,0)$$

If you take the total derivative ‘du’ , here the u is a function of x and t only. So, it can be written as

$$du = \frac{\partial u}{\partial x} dt + \frac{\partial u}{\partial x} dx$$

By simple observation we can find that if you take the lines dx/ dt = c, then you can convert the equation into an ODE which is du/ dt = 0.

So, this equation dx/ dt = c, which is also an Ordinary differential equation is known as the Characteristic equation. The consequence of which with the PDE turns into an ODE, du/dt = 0 this equation is known as the Compatibility condition. So, now this Partial differential equation with the initial condition gets transformed into 2 Ordinary differential equations.

It is easy to solve Ordinary differential equations. In this case, it is quite simple you just find that ‘u’ is constant along the lines ‘x – ct’. So, the solution just propagates along those lines.

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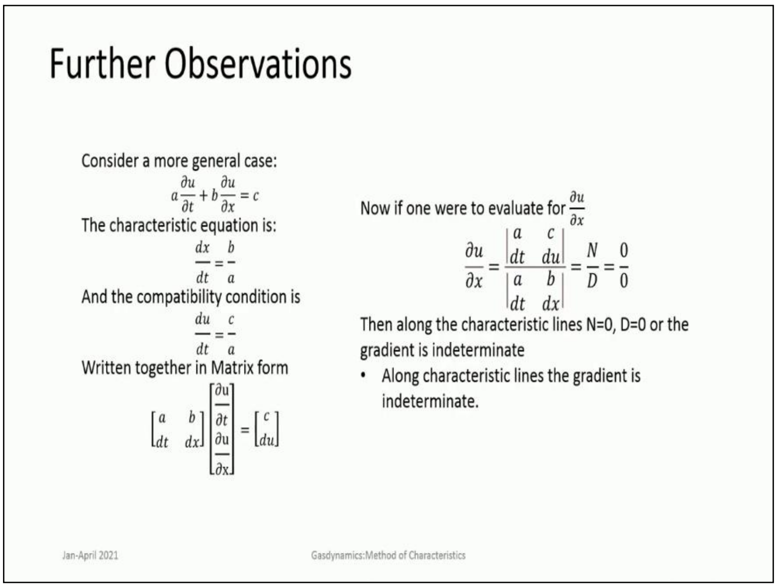

Now even if we go to a more general case where you have a, b, and c, which are more general. So, this is a linear equation of course. So, a, b, and c are consistent with the definition of the equation being linear. Then you find that the Characteristic equation, here is dx/ dt = b/ a. The Compatibility condition is du/ dt = c/ a, you just have to divide by ‘a’ to get to this form. Now these 2 conditions written together in a matrix form gives you these 2 equations, so, gives you this kind of matrix.

$$\begin{bmatrix} a & b \\ dt & dx \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} c \\ du \end{bmatrix}$$

From this if we were to evaluate what is $\frac{\partial u}{\partial x}$ or the gradient partial derivative $\frac{\partial u}{\partial x}$along the Characteristic lines along, we are interested in trying to find this out.

So, you can solve this is by using Cramer's rule and we know that we are looking at Characteristic lines. So, we get this condition the bottom is the determinant. So, a*dx – b*dt = 0 along a characteristic line. Similarly, if you take the numerator which is a ‘du - c dt’ that is nothing but the Compatibility condition, that is ‘N’ numerator is also 0. So, you find that along Characteristic lines, this gradient is indeterminate.

So, this gives us a method to find the numerator and the denominator of a gradient. Then equating them to 0/ 0 form this gives us a method to get Characteristic lines in general.

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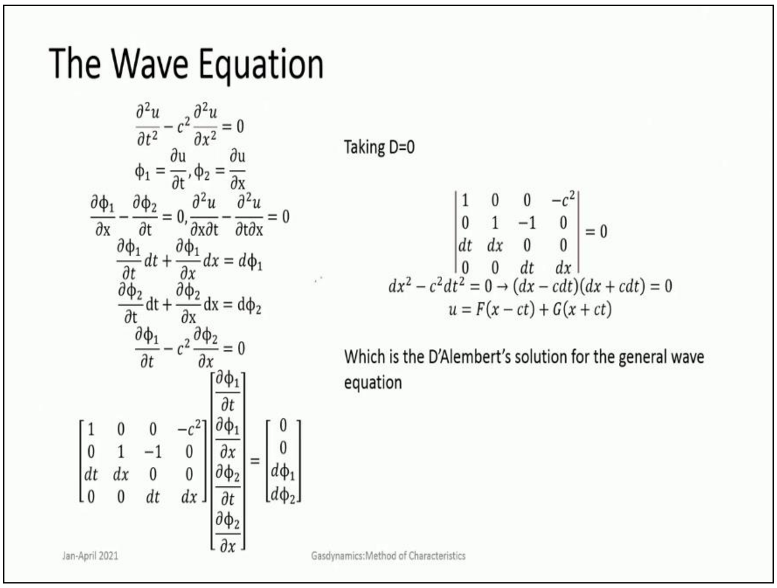

So, we go ahead and apply this here, it is applied for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

You can follow these steps it is written as a system of equations. Using the system of equations, you find out set that determinant is equal to 0 or that also is the denominator. If you set determinant equal to 0 you get 2 characteristics which is x - ct and x + ct.

So, this method if you follow the method, you also get correct characteristics for the wave equation. So, in a general wave equation also you will get it.

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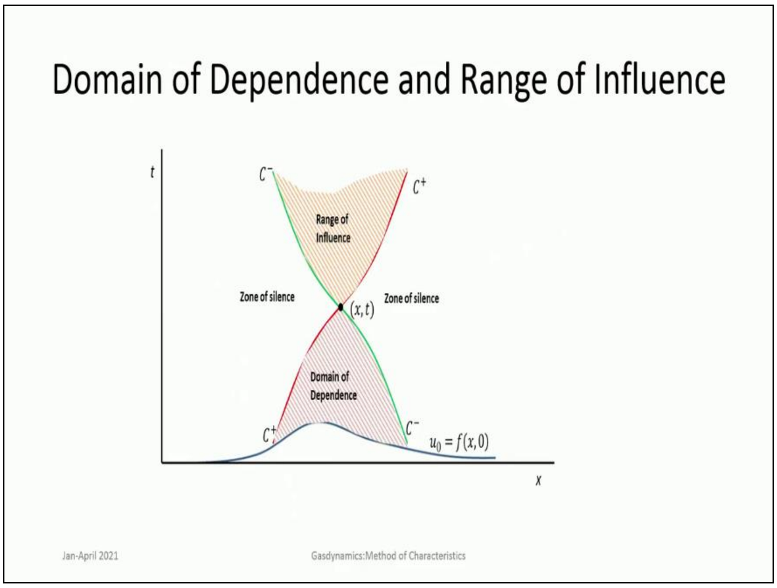

So, we can apply the same principles for the velocity potential equation in Supersonic domain. Before we go there, we also must understand that if you take any point in such a general hyperbolic flow or hyperbolic situation, hyperbolic equation then the particular point solution at that particular point is influenced by all the regions which are bound by the 2 characteristics.

We saw the wave equation has 2 characteristics, you can call them c+ and c-, left running and right running waves. So, there are 2 characteristics. So, the region that is bounded by the 2 characteristics is the region which influences a particular point in the domain in the entire solution domain. So, that region is known as the domain of dependence that is this point is dependent on the initial values or initial values that are within bound by the 2 characteristics which pass through that point.

Similarly, if you go ahead and push these characteristics out into the other regions of the flows. So, these characteristics originate from x, t and then look further in both x and t directions then this point influences all other points after the point or ahead of the point only along certain directions which are bound by again the Characteristic curve.

So, this is the range of influence for this point. If you take any other domain region which is away which is not bound by the characteristic curves they are not influenced, or those points are not influencing that point. So, they are known as Zone of silence. This is true for supersonic cases where we have seen that a body in Supersonic flow. So, something of this nature it affects the flow only along Mach waves.

So, if unless this microwave passes over a point for example, a point over here is not influenced at all by a body over here moving at supersonic speeds if M infinity is greater than one then this point is not influenced by such a body while the other point which is lying after or which has been influenced by the characteristics or microwaves there you get the influence of them. So, this is what is known as Domain of Dependence and Range of Influence.

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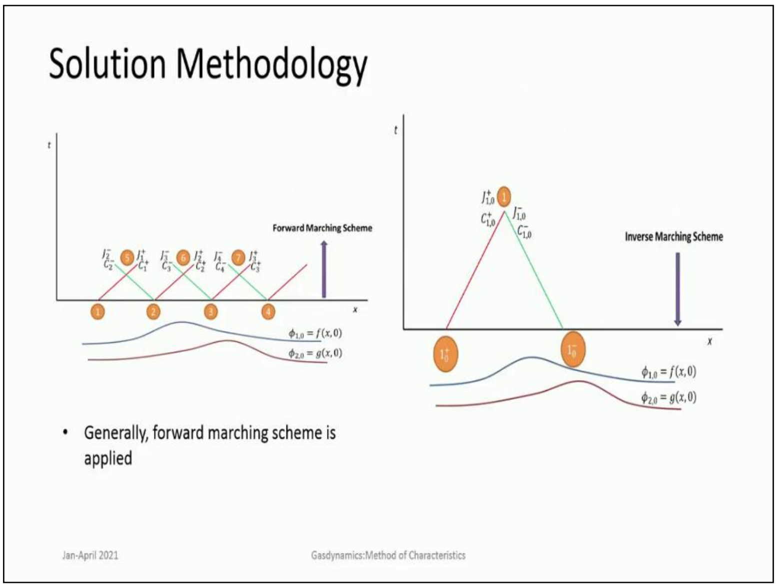

So, how to go ahead now, we know that there are these characteristics. We can use to solve them; how do we solve them across each along each characteristic there is a Compatibility condition. So, you have 2 things one is a Characteristic equation the other one is compatibility condition. You can use 2 of them together to solve both the flow field as well as what is known as a characteristic net character.

So, you draw characteristics from one point and then you can progress. So, you can progress in forward direction starting from an initial value. So, this kind of an approach is known as Forward marching. While you can also have a case where you can take an arbitrary point and try to draw the characteristics back towards the initial position this kind of solution is Inverse matching. It is difficult to do inverse matching if the characteristics keep changing in both x and as both x and t which is the general case.

If it is constant, then yes, it is very easy to do it, but it is not always the case. So, generally Forward marching scheme is applied where from the initial values you progress up.

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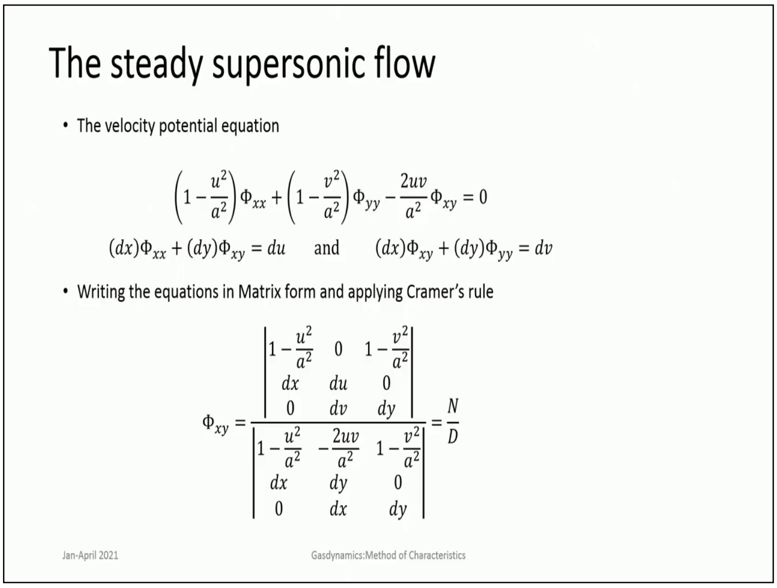

So, now let us come to take the specific case of the Steady Supersonic flow, this is the velocity per potential equation written here right. You can also take what is du and dv? These are the equations for du and dv they are functions of only x and y. So, you know that 'u' is $\frac{\partial \emptyset}{\partial x}$ and v is $\frac{\partial \emptyset}{\partial y}$.

So, you have these three equations we can put them into matrix form and find the N/D for say $\emptyset_{xy}$. So, for $\emptyset_{xy}$ a gradient you find N/D. The idea is to put D = 0, then you get Characteristic equations. If you put N = 0, you get the Compatibility condition.

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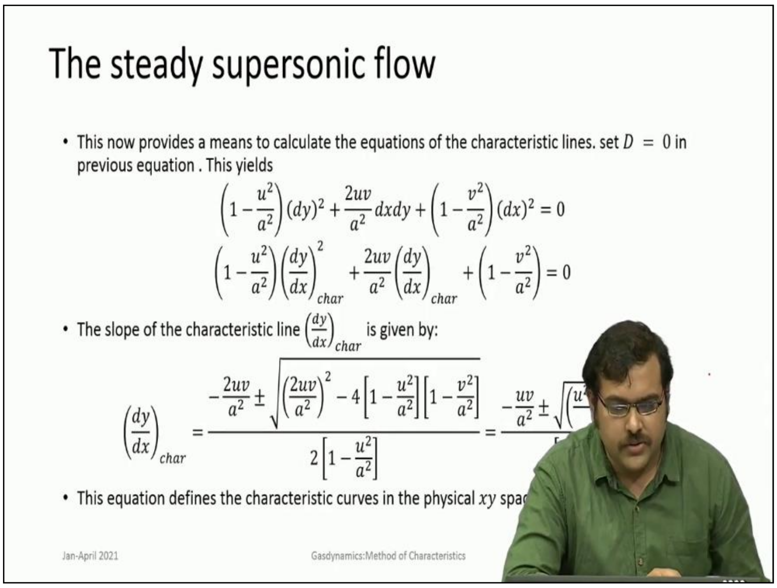

So, you take the denominator put it equal to 0, you get this particular form divide the entire equation by dx. So, this becomes a quadratic equation in ‘dy/dx’ which is the slope of the Characteristic equation or the Characteristic lines that can be solved. This is just the solution of a quadratic equation and you get it in terms of u and v. So, now this ‘dy/dx’ is solved in terms of u, v and a^2. Is there a way to convert them into more useful form which can yield us something quicker?

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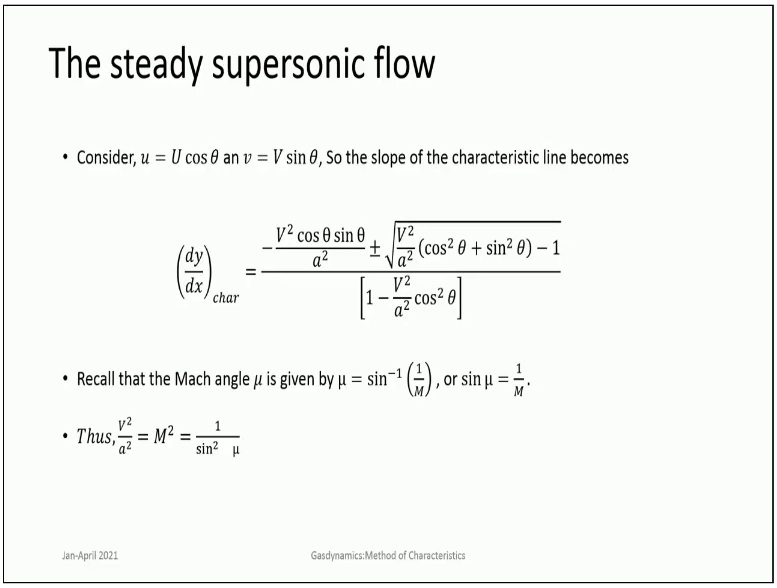

So yes, we can do it by converting it into polar coordinates, where ‘u’ is general vector, ‘v’ is the vector, or v has magnitude of v vector is v. And u = v cos θ and v = v sin θ. So, we apply this converting it into polar coordinates. Then what we get is that we can get in terms of cos θ, sin θ, and V^2/a^2.

V^2/a^2 is nothing but M^2. Also, we know that $1/M^2$ is nothing but $\sin^2 \mu$, where μ is the Mach angle.

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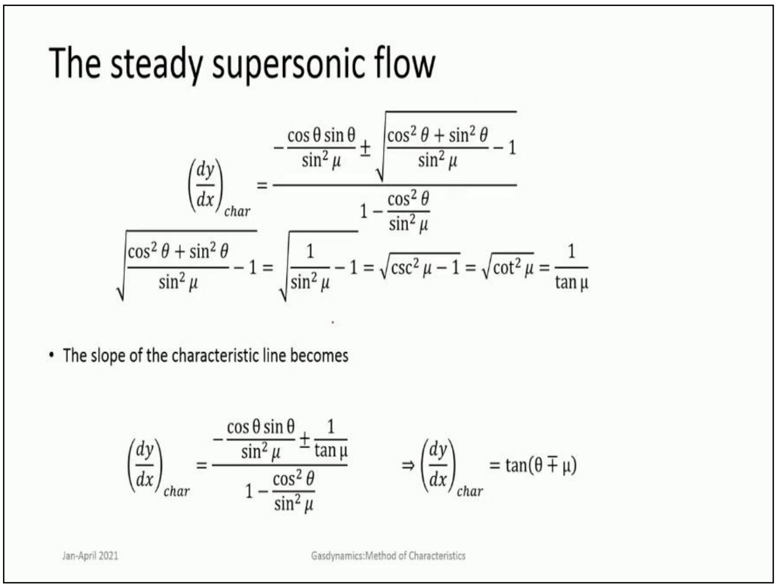

So, we can use those definitions of back angle together with these equations and if you combine all of them together and do some algebraic manipulation you get that the characteristic equations are defined by tan $(\theta \pm \mu)$ that is the c- characteristics, one set of characteristics is tan $(\theta - \mu)$. The other set which is known as c+ characteristics is tan $(\theta + \mu)$.

So, if you have a general velocity vector v having a certain angle theta there are 2 waves each of them at an angle μ with respect to that point. So, the solution propagates along Mach wave. So, this is something we already know. And therefore, the characteristics are located at $(\theta + \mu)$ and $(\theta - \mu)$.

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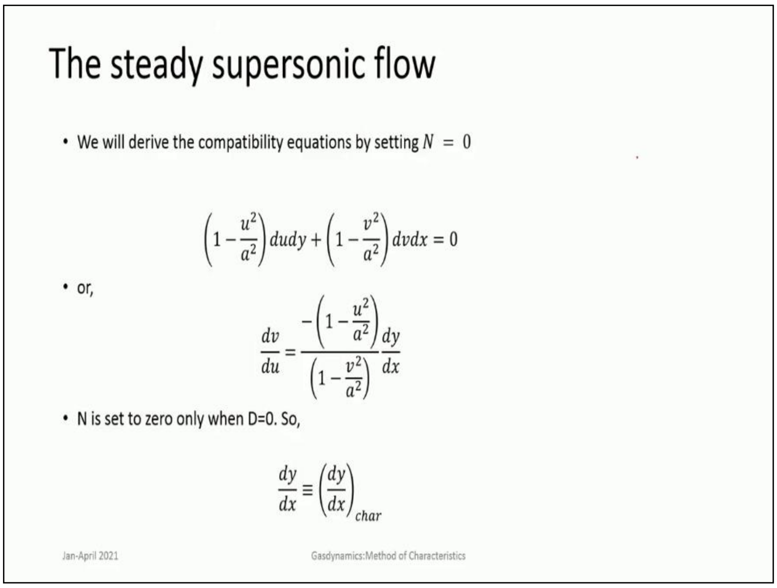

Now what happens to compatibility condition we can set numerator equal to 0. Along the direction dy/dx is the characteristic curve.

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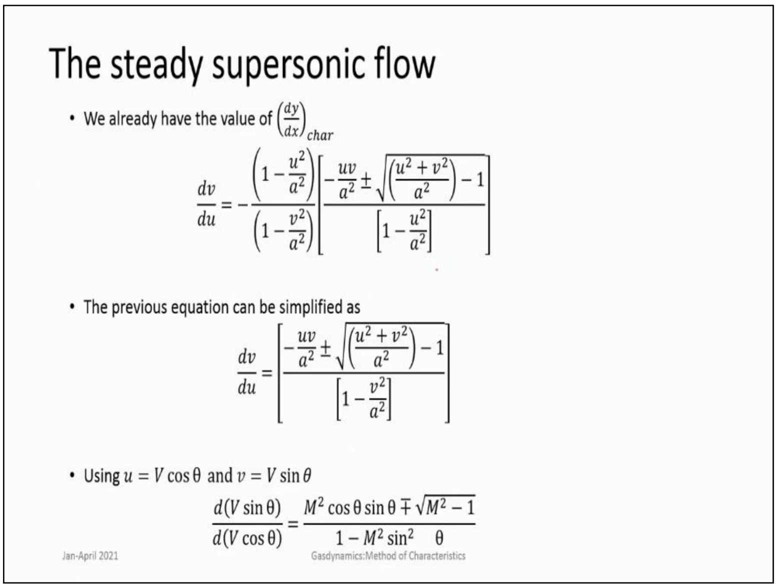

We already know what is dy/ dx for the characteristic curve, we can solve for this equation what is dv/du but u = b cos (θ) and v = v sin (θ).

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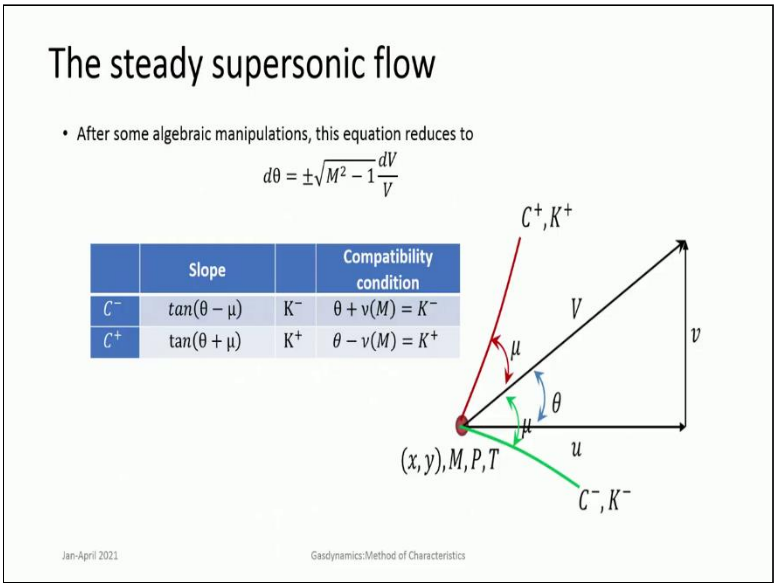

Then what we get is that d theta is equal to plus or minus square root of M^2-1 dv/v this form should be familiar to all of you by now $\sqrt{M^2 - 1}$, dv/ v is nothing but ν, where ν is the Prandtl-Meyer function. So, you get d theta is equal to plus or minus d nu. So, if you integrate it, you just get the condition. So, if you take the slope c minus.

So, that is tan theta minus nu. So, if you look at this this is a general vector. So, in the flow field it is at any point x, y having a Mach number pressure and temperature. So, this is the velocity vector. With respect to this velocity vector there are 2 characteristics one goes both go at angles ν. So, let me just revisit the yeah. So, that ν must be with respect to the velocity vector.

So, you get θ + ν here, with respect to the horizontal θ + ν. With respect to horizontal this angle is θ - ν. So, this is c- characteristics which is coloured in green. The one that is coloured in red is c+ characteristic. Along c- characteristic the compatibility condition is given by k-, k- = θ + ν, $'\nu'$ is the Prandtl-Meyer mirror function at that Mach number.

Similarly, you can take c+ characteristics which is θ + μ. Along them θ + ν is constant. So, this is k+, which is θ + μ. This is k- = θ + μ this is constant along those directions. So, for the case of a supersonic 2D flow we can get the complete integration for of course for perfect gases in closed form solutions.

This gives us now simple equations. So, now we have converted a rather daunting looking velocity potential equation to now they have been converted to a set of algebraic equations of course they are in terms of angles and tan and so on. So, but that can be solved it is not difficult to solve this. So, the flow field in the supersonic flow 2D supersonic flow can be solved using this set which we will do in the coming classes for specific cases.

There we follow certain methods known as the unit processes. We look at Supersonic nozzles how they can be the contour of the wall how they can be designed based on Method of Characteristics in the next class, thank you.