Gasdynamics: Fundamentals and Applications
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Lecture 51
Small Perturbation Theory - III

We are looking at the Small Perturbation theory, and now looking at flow fields rather than integral quantities and in the previous classes we had looked at velocity potential equation and Small Perturbation theory applied to Subsonic flows So, we are dealing with inviscid irrotational flows and therefore these flows are isentropic also. We had looked at the main steps that are followed to convert the velocity potential equation, which is a non-linear equation in full to simpler linearized equations using small perturbations, that is, it is taken that a body in a flow.

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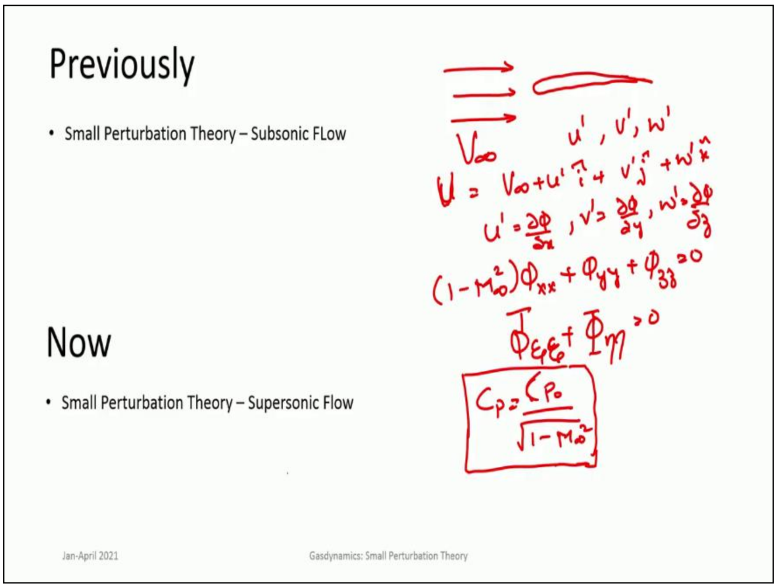

For example, it can be a thin air foil or a flat plate it produces a uniform flow is here. So, ‘V_∞’, it produces very small changes to the flow that is we have ‘’, v’, w’, they are very small changes. So, the velocity, V, is written as

$$V = V_\infty + u'\hat{\imath} + v'\hat{\jmath} + w'\hat{k}$$

and a perturbation potential was defined which is

$$u' = \frac{\partial\emptyset}{\partial x}, v' = \frac{\partial\emptyset}{\partial y}, w' = \frac{\partial\emptyset}{\partial z}$$

Using this we converted a non-linear equation into a linearized form which had the form

$$(1 - M^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

We looked at the subsonic case and there we looked at how we made a transformation of variables. So, that in a 2-dimensional flow this can be converted into '$\xi\eta$ coordinates' a simple Laplace equation equal to 0.

$$\phi_{\xi\xi} + \phi_{\eta\eta} = 0$$

Actually, these solutions are very similar to the incompressible Laplace equation and from there we got the idea that C_P can be represented as

$$C_p = \frac{C_{p0}}{\sqrt{1 - M_\infty^2}}$$

So, incompressible relations of C_p over any body such that it produces only small changes. If you consider such cases, then the incompressible relations can be extended to the domain of compressible flows, Subsonic compressible flows.

The relation is given here, and this is known as the Prandtl-Glauert rule, which is extensively used, applied and there have been corrections later but the basic idea has not changed. So, that is for the case of Subsonic flows, where if you know the coefficients non-dimensional coefficients for incompressible flows it can be extended to compressible domain using the Prandtl-Glauert rule. Now in this class we will look at Supersonic flows.

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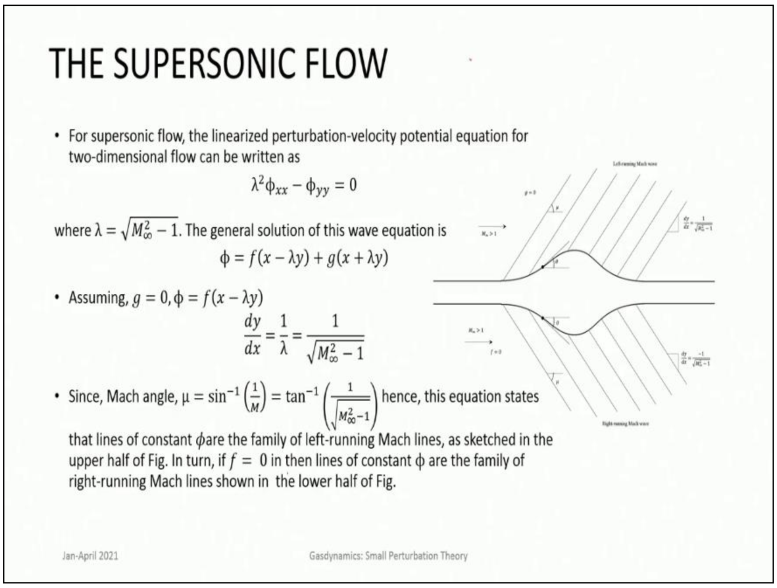

In Supersonic flows the value ‘$1 - M_\infty^2$ in that particular perturbation relation it goes negative because $M_\infty > 1$. So, consequently we make a corresponding change is made. So, it is written as ‘- $(M_\infty^2 - 1)$’. So, if you make it then it will become lambda square phi xx minus phi yy equal to 0 where lambda is square root of minutes M infinity square minus 1.

$$\lambda^2 \emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} = 0,$$

$$\lambda = \sqrt{M_\infty^2 - 1}$$

This is applicable for flows which are supersonic, that is $M_\infty > 1$. So, now this if you look at this equation it is a wave equation. So, a wave equation the general solution is given by

$$\emptyset = f(x - \lambda y) + g(x + \lambda y)$$

where ‘$x - \lambda y$’ and ‘$x + \lambda y$’ represent specific lines in the domain along which there is a propagation of information they are also called as Characteristic lines.

If you look at this, the angle which is

$$\frac{dy}{dx} = \frac{1}{\lambda}$$

you can take a general solution, say g =0, $\emptyset = x - \lambda y$

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

and that those are represented here, on the top they are left running waves.

We can then look at the definition of a Mach angle, the Mach angle is

$$\mu = \sin^{-1}(\frac{1}{M}) = \tan^{-1}\left(\frac{1}{\sqrt{M_\infty^2 - 1}}\right).$$

So, now this shows that these angles or these lines are nothing but Mach waves. So, in supersonic flow if you have a linearized supersonic flow this is now linearized it is a wave equation.

Then there are lines along which information propagation happens or there is a solution propagates in certain lines and these lines are nothing but Mach waves. Mach waves we have introduced very early on in the class. Now we see the actual implications of those waves. So, in any general arbitrary change or an initial profile is propagated along the Mach waves. There are 2 of them because you have '$x - \lambda y$' and '$x + \lambda y$'.

So, in general there are both left running Mach waves as well as right running Mach waves. So, the idea here is Supersonic flow the perturbation velocity potential equation becomes like a wave equation and the wave equation is Hyperbolic. So, this is inconsistent with the general characteristics of the flows that we saw that steady supersonic flow is hyperbolic in nature even you find the same thing in the velocity perturbation equations also.

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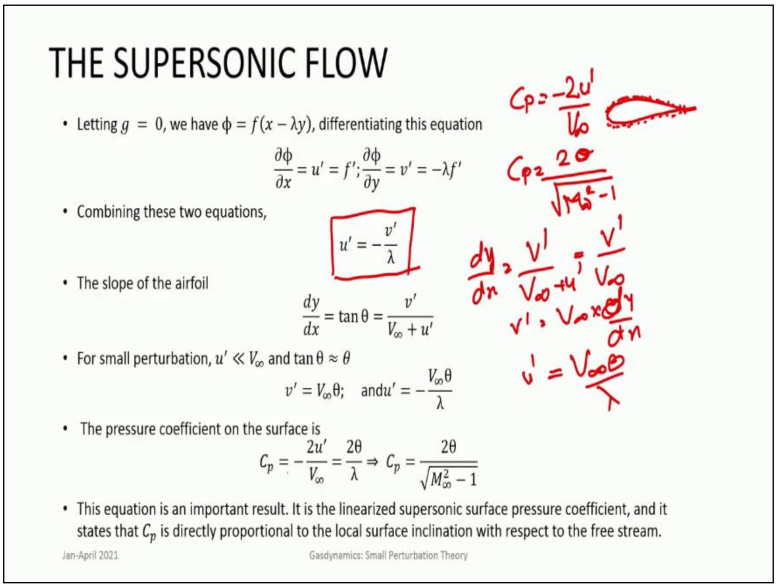

So, like the Subsonic flow, we are interested in trying to find out what is the pressure over the surfaces and therefore we can calculate what is the force that is occurring on such surfaces example like Airfoils, and so on. So, now that we have a general solution '$f(x-\lambda y)$' and '$g(x+\lambda y)$'. Let us see what we can achieve from them by differentiation you can take for letting g = 0.

You can just take $f(x-\lambda y)$ and you get

$$\frac{\partial \emptyset}{\partial x} = u' = f';\ \frac{\partial \emptyset}{\partial y} = v' = -\lambda f'$$

So, if you compare these 2 equations, we can see that

$$u' = \frac{v'}{\lambda}$$

So, this is what we get, and we know the boundary condition at the surface of the Airfoil, that is it should be tangential.

So, the velocity is tangential to this

$$\frac{dy}{dx} = \frac{v'}{V_\infty + u'} = \frac{v'}{V_\infty}$$

u prime is very small. So, we are looking at small angles, small perturbation. So, therefore we get

$$v' = V_\infty \frac{dy}{dx}$$

'dy/dx' is nothing but tan (θ) theta, but tan (θ) $\approx$ θ for small angles.

$$u' = -\frac{V_\infty \theta}{\lambda}$$

now we had seen in the previous class that C_P for small perturbation can be written as

$$C_P = -\frac{2u'}{V_\infty}$$

So, now the u' value that we had just arrived at which is '$-v'/\lambda$'. If we substitute into this equation, we get

$$C_P = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

So, here we get a very important result

So, what we find here is that the pressure on the surface is just related to the angle of the surface at that point or it is like a local surface in inclination that is required. If we know the equation of the surface, some equation of the Airfoil, then its pressure at any point in a supersonic flow in the domain of linearized supersonic flow that is under small perturbation theory which implies thin airfoils small angle of attacks.

So, under that condition the pressure depends only on the surface inclination or dy by dx at every point. How much does that surface inclination occur with respect to the free stream because here we took the free stream to be parallel to x axis. So, this is a very important result.

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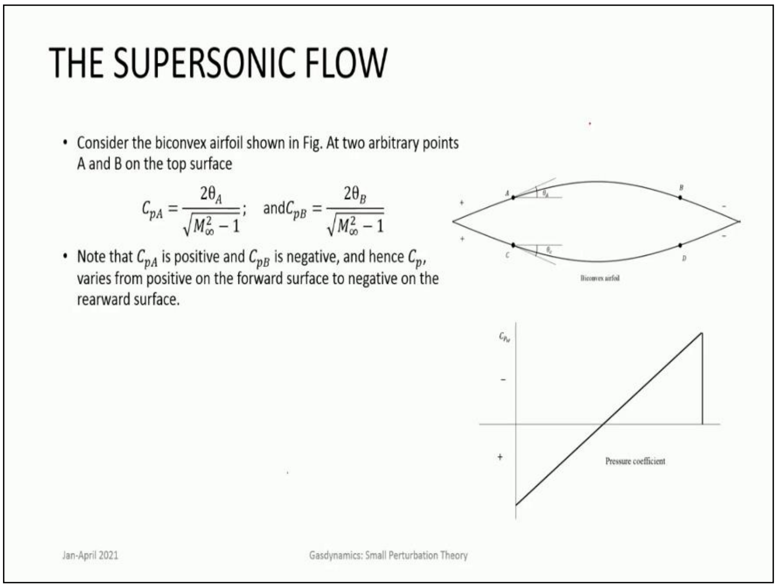

So, if we want to find out what should be the forces on such in typical bodies, for example here it is a Biconvex Airfoil is shown, which is symmetric. So, in the initial portion it has a wedge angle which is small θ. So, over there initially the angle θ, it is moving the flow towards turning the flow towards itself. So, it sort of compresses the flow here and this θ is positive therefore you get

$$C_{PA} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}}$$

here this value becomes positive. But on the towards the rear part there we find since its symmetric you are getting similar theta values but in the negative direction now. So, this is θ_B which is negative.

So, C_P is negative. So, in general you can consider many different such shapes. Earlier we had looked at shock expansion theory and we had looked at Diamond Airfoil, of this kind you can consider the same in the context of linearized flows also. This is also straightforward the relationships will be very similar to what we are discussing over here just right over here.

But one more thing that one has to understand here is that here obviously because on the y-axis everything is symmetric. So, it produces no lift, but at the same time if you look at the x component of the force then you will find that in this case there is a finite drag being produced. If you consider symmetric airfoils in Subsonic flow, inviscid irrotational flows, then in such cases in those cases there is complete symmetricity.

So, both lift and drag are 0, it is the D Alembert's paradox because there the drag is being produced mainly by skin friction which is not being considered in inviscid cases. But here, it is an inviscid flow, even if it is an inviscid flow if you integrate on along the x direction you will find that there is a drag. A drag is created, that drag because if it is a supersonic flow and there are waves, and the behaviour is completely different from Subsonic flow.

This is often referred to as wave drag. So, it is a pressure-based drag wave drag and it is not a skin friction drag.

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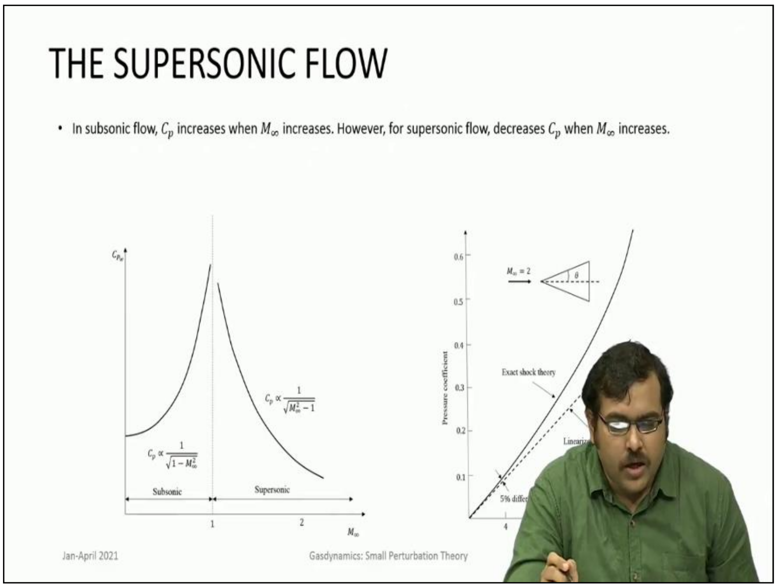

So, if you also look at how the C P for subsonic flow. So, as Mach number increases in the subsonic case you find that higher and higher pressures or higher and higher C_P is achieved because $'1 - M_\infty^2'$ goes smaller and smaller.

So, there is an increase here in the subsonic domain but in the supersonic domain as M_∞ decreases, if it starts increasing this value increases therefore C_P will decrease. A comparison of these approaches with more exact theories shows that linearized theories are good for small angles of attack. For example, this is a taken from the typical textbook it is from John D Anderson both these images.

And here they have looked at the difference between linearized theories and more exact solutions. And at angle of attack of around 4^o there is about 5 below that it is very much same as the exact theory. So, such equations we can use for small angles of attack. So, or small angles and small angles of attack which is thin airfoils small angles of attack. So, but it is a useful theory to quickly evaluate aerodynamic coefficients in both the subsonic and the supersonic domain before we go into other kinds of approaches which have better accuracy.

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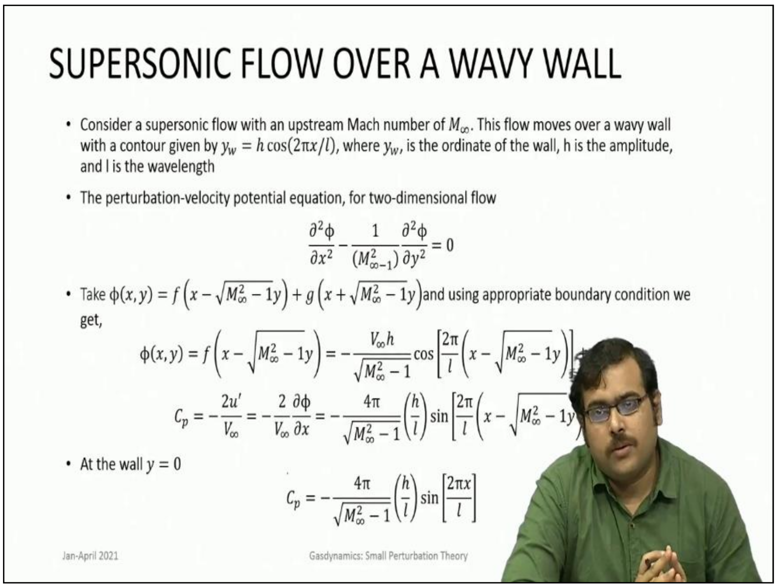

So, towards the end of this topic we look at a typical solution over a wavy surface. If you have a supersonic and a subsonic flow over a wavy surface something of this kind then how does how do, the solutions. So, here you have 'V_∞' over here. The purpose is to distinguish between what happens in subsonic flows and what happens in supersonic flows understand the characteristics of these different flows these different domains.

Also look at the solution how are in general solutions of the velocity perturbation equation here of course we consider h is very small. So, h is small. So, because it is a small perturbation, we are doing supersonic flow. So, we will just do the supersonic flow initially and then later look at subsonic flows and see how they compare and here. So, we already have seen that in supersonic flows the equation is a wave equation it is hyperbolic in nature.

I n general, the solution is

$$\emptyset(x,y) = f\left(x - \sqrt{M_\infty^2 - 1}y\right) + g(x + \sqrt{M_\infty^2 - 1}y)$$

So, you have 2 characteristics left running and right running and the appropriate boundary conditions are there which is related to the gradients or dy/dx at the wall. So, you can put the appropriate boundary conditions and solve for $\emptyset$. So, solution of the wave equation is done.

Here for an initial contour which was, $y_w = h\cos(\frac{2\pi x}{l})$. $\emptyset$, we get as it runs as cos of this along the lines '$x - \sqrt{M_\infty^2 - 1}y$'. So, you can look at C_P and here you find that C_P is going as

$\sin\left(\frac{2\pi x}{l}\right)$, C_P at the wall. $\frac{\partial \phi}{\partial x} = u'$. So, you can get that over here and at the wall it will be going as sin. So, the wall has a shape which is cos, but cp goes as a sine function.

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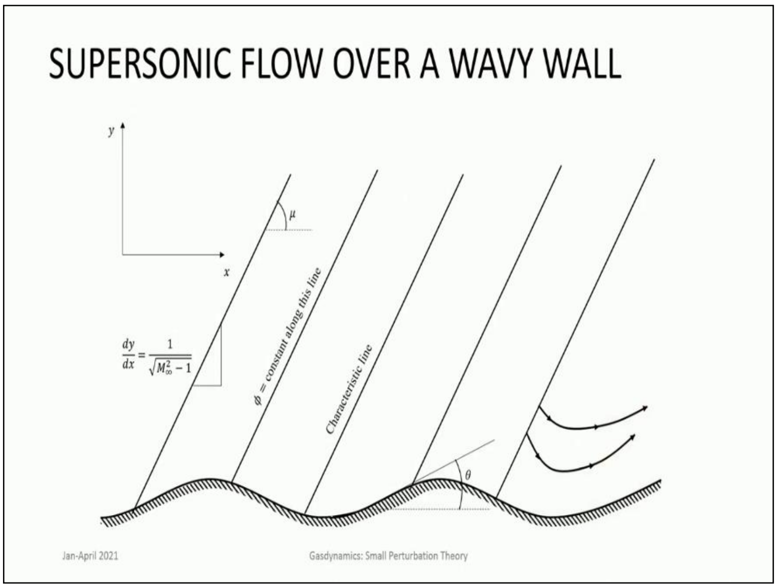

So, now if we try to look at the physical picture of this problem what we find is that there are Mach waves that occur along the whole flow domain starting right from the wall going all right up to infinity. And the solution is just moving over this or propagating over these lines along these characteristic lines. So, if you consider the crest. So, if you consider the point of the crest and this is y-direction.

Then the solution is going along a line of this kind. So, the solution is symmetric along phi equal or characteristics lines which are $x - \sqrt{M_\infty^2 - 1}$, they are not symmetric about the y axis at any point So, they are getting displaced or stretched by a certain amount along this Characteristic lines.

So, because of these effects, so, you can see that it is written over here the way the streamlines behave. The consequence is that the pressure, if pressure is integrated over the wall, then you always get a finite drag for this particular solution. Also, this solution carries over all the way up till infinity that is the effect of this wall in the supersonic flow is felt along these characteristic lines all through the domain but only along certain directions and this produces a certain drag.

This is something that we also saw in the previous discussion that in supersonic flows the nature is such that you will get a drag even if all things look the geometry looks symmetric that is because of the wave nature of the flow.

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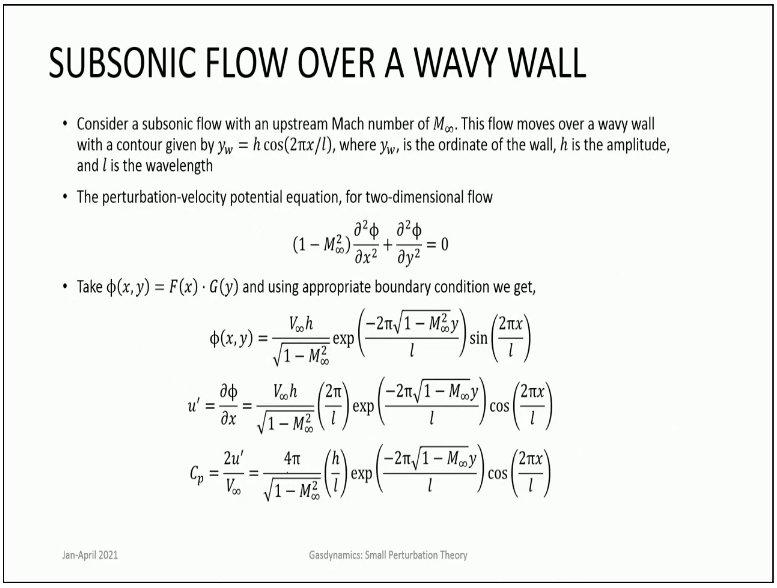

Now if you consider a subsonic flow of the same problem where the same contour is given, but now this equation is elliptic in nature. The solution, if you look at the solution, you find that it contains an exponential term you have some constants which are $\frac{V_\infty h}{\sqrt{M_\infty^2-1}}$ but here you have a term 'e' power $-2\pi\sqrt{1-M_\infty^2}\,y\big/_{l}$.

So, if you look at this term there is an exponential which has a negative term that means this kind of a function will try to reduce the amplitude as you increase y. So that effect is there. So, you get sin ($2\pi x\big/_{l}$) in this case you see that the C_p they also follow the cos function which is the same as the profile that means they are symmetric over the profile though there will be a phase difference.

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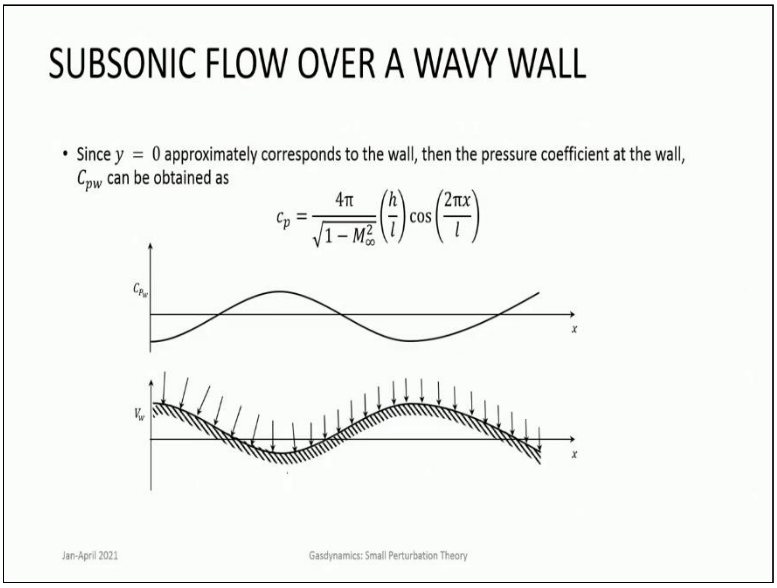

So, there can be a phase difference, but it will be symmetric across the profile. So, consequently if you integrate along the x direction, they all the forces will cancel each other and hence you get drag is 0. So, this is something that again we had discussed earlier in subsonic cases invisible flows potential flows you do not get a drag due to the pressure effects mainly it is the skin friction tag. So, that is the consequence here.

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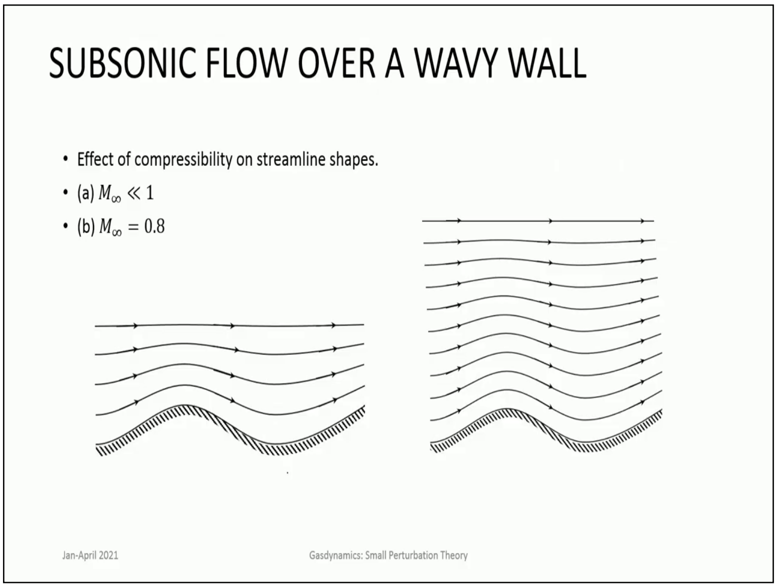

But the effect is more predominant in the streamlines you see that there is a term e power something here. So, ‘e’ power and a function of y some say ‘q(y)’. So, this function reduces the amplitude as you go away from the wall. So, that at a certain distance the effect of the wall is completely forgotten. So, if you look at the streamlines, they look like this near the wall they follow the wall very much.

But as you go further and further away, they change and then again, they come back to being like the streamline itself. The distance from the wall; at which this change takes place it varies as Mach number changes and as Mach number is increased the distance also increases. So, at very low velocities it will be quite close to the wall but at higher velocities the influence is carried quite a long distance away from the wall. But eventually it dies down but this is not the case with Supersonic flows.

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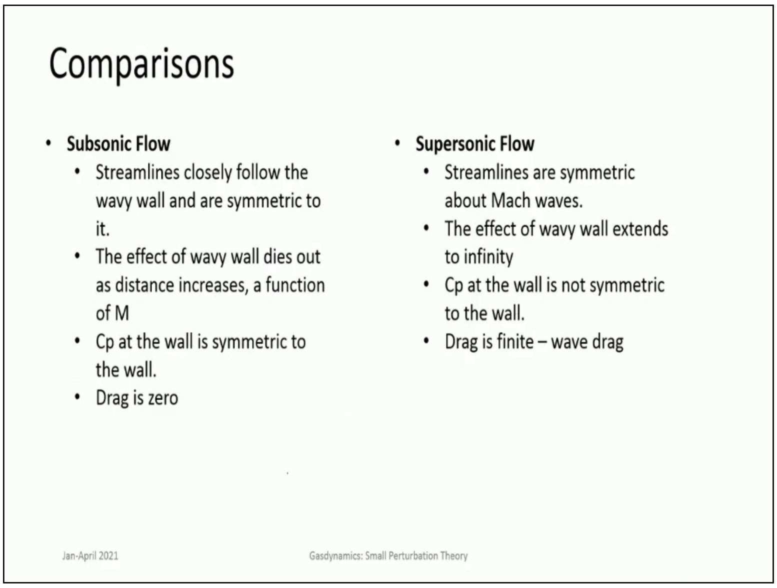

So, if you now look at the comparisons between the 2, the streamlines closely follow the wavy wall are and are symmetric to the wavy wall in subsonic flow. But in supersonic flow the streamlines are symmetric about the Mach waves not the wall. So, and the effect of the wavy wall it dies out as distance increases and that is a function of the Mach number in the subsonic case but in supersonic flow there is no such exponential multiplying the solution therefore it extends to infinity.

Here we also know that drag is 0, C_p symmetric to wall in Subsonic flows but in Supersonic flows drag is finite this is the wave drag that occurs. These differences are arising due to the very nature of these flows. In subsonic flows it is an elliptic equation, and the information propagation happens in all directions. But in supersonic flow it is a hyperbolic equation and information propagation happens only along certain directions which are Mach waves.

So, with this we end on the small perturbation theory it was useful to understand about Subsonic and Supersonic flow fields. Their characteristics how they behave what are the differences in the context of inviscid irrotational flows. Now Supersonic flows, if you look at Supersonic flows, they are hyperbolic in nature and there are certain methods by which they can these hyperbolic equations can be solved exactly.

They are like the wave equation, and we saw that there is the D' Alambert solution which follows certain characteristics. So, the exact velocity potential equation for Supersonic flows can be solved in using such methods they are known as Method of Characteristics. We will look at these Methods of Characteristics for the supersonic flow because they are used for certain applications particularly to look at the design of Supersonic nozzles.

The divergent you know that a Supersonic nozzle has a convergent and divergent portion, that is if you are trying to accelerate from Subsonic to Supersonic flows. So, a Supersonic nozzle is the divergent portion can be designed such that we get uniform flows at the end at the exit and that has to be carefully done by solving the flow field. So, that is done using method of characteristics and we look at method of characteristics in the coming classes, thank you.