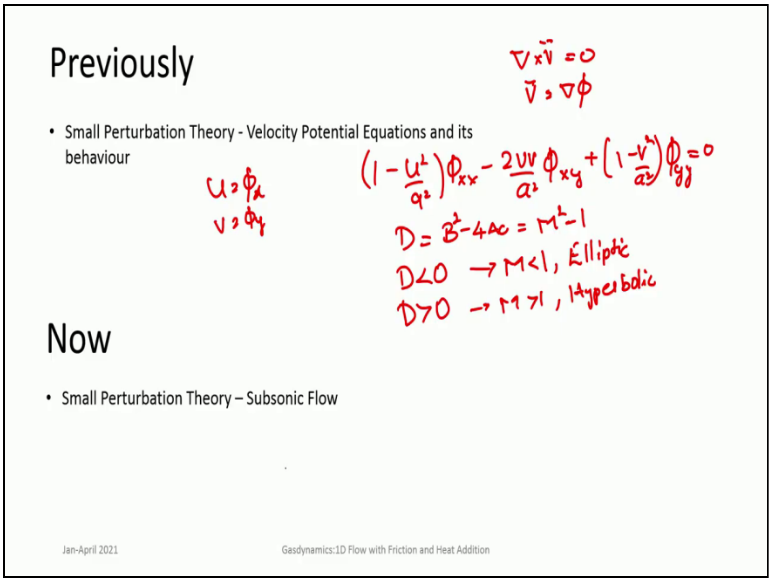
Gasdynamics: Fundamentals and Applications
Prof. Srisha Rao M V
Aerospace Engineering
Indian Institute of Science - Bangalore

Lecture 50
Small Perturbation Theory -II

So, now we are looking at solving the entire flow field in that context we are looking at inviscid irrotational flows and we are look specifically isentropic flows. So, then in the previous class we had looked at when we consider irrotational flows we know that the velocity can be expressed as the gradient of a scalar potential because it is irrotational or $\nabla \times \vec{V} = 0$ **(Refer Slide Time: 00:57)**



So, V can be expressed as gradient of a potential. And we have looked at the velocity potential equation in its full form in three dimensions. It is a nonlinear equation. But when specifically taken in along 2 dimensions then we looked at it was it specifically,

$$\left(1 - \frac{u^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{v^2}{a^2}\right)\Phi_{yy} - \frac{2uv}{a^2}\Phi_{xy} = 0$$

Of course $u = \frac{\partial \Phi}{\partial x}$, and $v = \frac{\partial \Phi}{\partial y}$. And we looked at the behaviour of this equations we found that the determinant is $D = B^2 - 4AC$ was turned out to be $M^2 - 1$.

Therefore when you consider subsonic flows determinant is less than 0. So, when M is less than 1 therefore it behaves in an elliptic manner and determinant is greater than 0 when M is greater than 1 for supersonic flows it behaves in hyperbolic manner. So, this distinction should be really appreciated when you look at solving subsonic flow problems and solving has supersonic flow problems and we will come to it again and again.

But now our question is, is there any approach by which see this in its full form it is a non linear equation and it has to be solved numerically but can we get some solutions for certain specific cases a normal approach.

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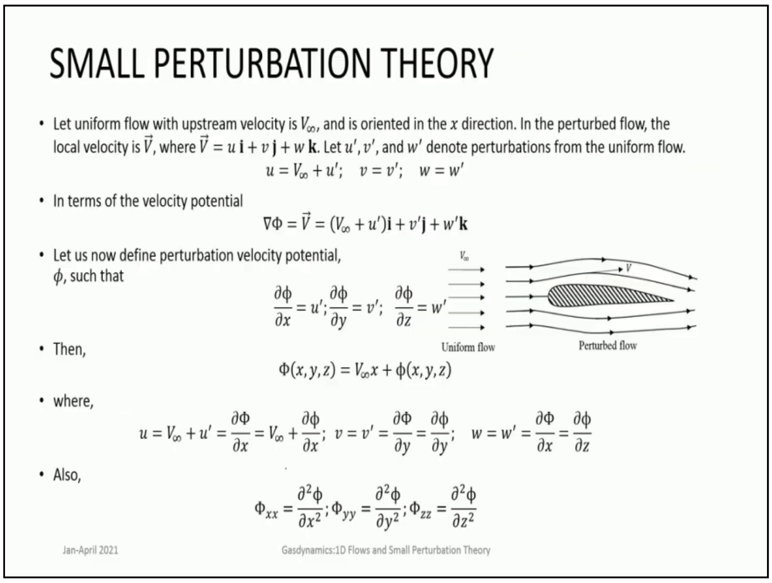

In such cases of looking at non linear equations is to see if it can be linearized and this particularly useful in cases where you have thin bodies in a mean flow for example an airfoil in a mean flow in a uniform flow . So, this is a very important problem for aerodynamics and it can be an airfoil which is placed in a uniform flow. Now this airfoil actually changes the flow around it by small values which is u' , v' , w' .

So, they are called as perturbations to the uniform flow V_∞ . So, this V_∞ is along x direction which is along u. So, u = $V_\infty + u'$, while v = v' and w = w' . So, now we look at, can we take this approach and get some useful results for the case of subsonic flows and then how is the velocity potential written. So, we define a perturbation velocity potential such that

$u' = \frac{\partial\phi}{\partial x}$, $v' = \frac{\partial\phi}{\partial y}$ $w' = \frac{\partial\phi}{\partial z}$. Where ϕ is perturbation velocity potential

Then the velocity potential in the x direction actually becomes,

$\Phi = V_\infty x + \varphi$

So, you can see that,

u = $V_\infty + u' = \frac{\partial\Phi}{\partial x} = V_\infty + \frac{\partial\phi}{\partial x}$

v = $v' = \frac{\partial\Phi}{\partial y} = \frac{\partial\phi}{\partial y}$; w = $w' = \frac{\partial\Phi}{\partial z} = \frac{\partial\phi}{\partial z}$.

So, you can just differentiate it and it will be you can easily see that because

$u' = \frac{\partial\phi}{\partial x}$, $v' = \frac{\partial\phi}{\partial y}$ $w' = \frac{\partial\phi}{\partial z}$.

So, now you can define,

$\varphi_{xx} = \frac{\partial^2\varphi}{\partial x^2}$; $\varphi_{yy} = \frac{\partial^2\varphi}{\partial y^2}$; $\varphi_{zz} = \frac{\partial^2\varphi}{\partial z^2}$

So, now we are considering small perturbations.

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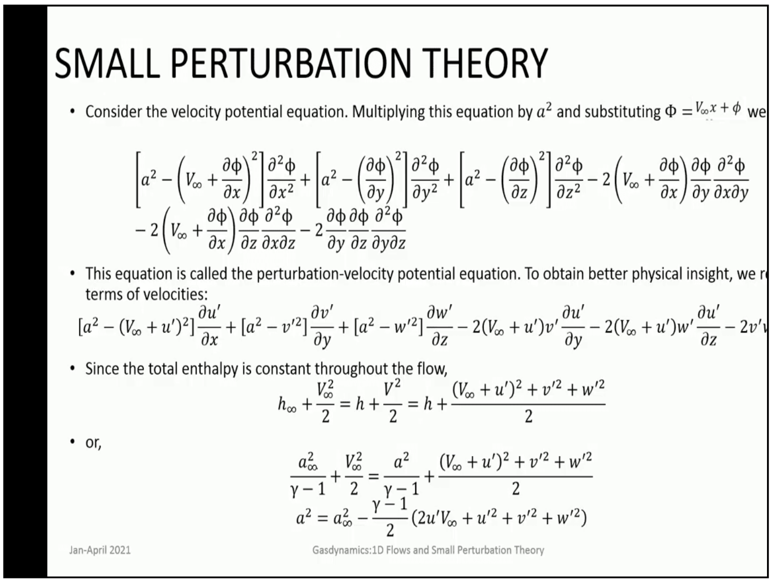

So, let us go back to the velocity potential equation itself and then substitute these perturbation potentials into the velocity potential equation. So, then if you substitute them you have,

$$\left(a^2-\left(V_\infty+\frac{\partial\phi}{\partial x}\right)^2\right)\frac{\partial^2\varphi}{\partial x^2}+\left(a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right)\frac{\partial^2\varphi}{\partial y^2}+\left(a^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right)\frac{\partial^2\varphi}{\partial z^2}-$$
$$2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}-2\left(V_\infty+\frac{\partial\phi}{\partial x}\right)\frac{\partial\phi}{\partial z}\frac{\partial^2\varphi}{\partial x\partial z}-2\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}\frac{\partial^2\varphi}{\partial y\partial z}=0$$

So, now this entire equation is the perturbation velocity potential equation ,you can write it in terms of velocities , u' , v' , w' ,to get better insight. Now we also get a^2 square here we have to relate a^2 square. So, for that we use the approach that total enthalpy is constant.

$$h_\infty+\frac{V_\infty^2}{2}=h+\frac{V^2}{2}=h+\frac{(V_\infty+u')^2+v'^2+w'^2}{2}$$

This is the way it is approached.

They can be written in terms of acoustic speeds. This is something we did early on in gas dynamics in the early chapters. So, it is written in terms of that,

$$\frac{a_\infty^2}{\gamma-1}+\frac{V_\infty^2}{2}=\frac{a^2}{\gamma-1}+\frac{(V_\infty+u')^2+v'^2+w'^2}{2}$$

So, this equation for a^2 is substituted in the main equation which is the full perturbation velocity potential equation.

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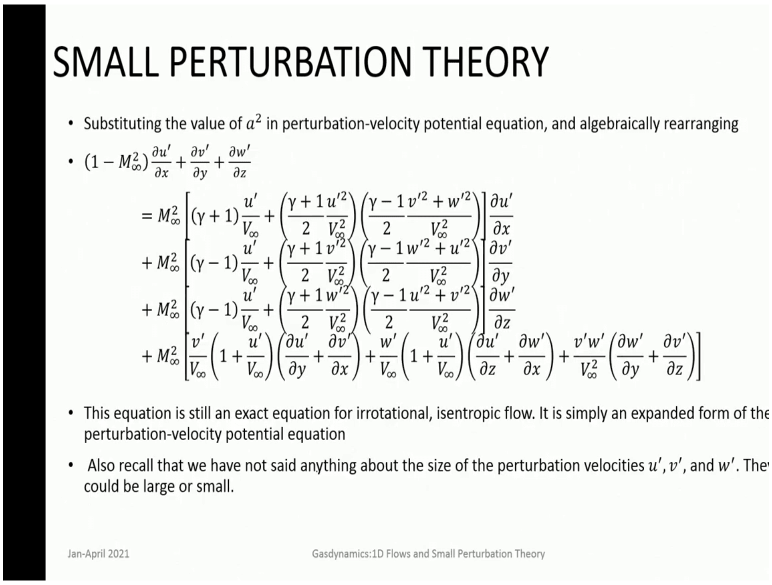

So if you do that and collect various terms and rearrange them and collect various terms then you get this particular form which is

$$\left(1-M_\infty^{2}\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} = M_\infty^{2}\left[(\gamma+1)\ \frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\ \frac{u'^2}{V_\infty^{2}}\right)\left(\frac{\gamma-1}{2}\ \frac{v'^2+w'^2}{V_\infty^{2}}\right)\right]\frac{\partial u'}{\partial x}$$

$$+M_\infty^{2}\left[(\gamma-1)\ \frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\ \frac{v'^2}{V_\infty^{2}}\right)\left(\frac{\gamma-1}{2}\ \frac{u'^2+w'^2}{V_\infty^{2}}\right)\right]\frac{\partial v'}{\partial y} \quad +$$

$$+M_\infty^{2}\left[(\gamma-1)\ \frac{u'}{V_\infty}+\left(\frac{\gamma+1}{2}\ \frac{w'^2}{V_\infty^{2}}\right)\left(\frac{\gamma-1}{2}\ \frac{u'^2+v'^2}{V_\infty^{2}}\right)\right]\frac{\partial w'}{\partial z}$$

$$+M_\infty^{2}\left[\frac{v'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)+\frac{w'}{V_\infty}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)+\frac{v'w'}{V_\infty^{2}}\left(1+\frac{u'}{V_\infty}\right)\left(\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\right)\right]$$

Now this is a full equation, full exact equation for irrotational isentropic flow in terms of perturbation velocity potential. But Now what we have to do is simplify this equation by considering an analysis where we see how which of them are really important because you have u', v' , w'.

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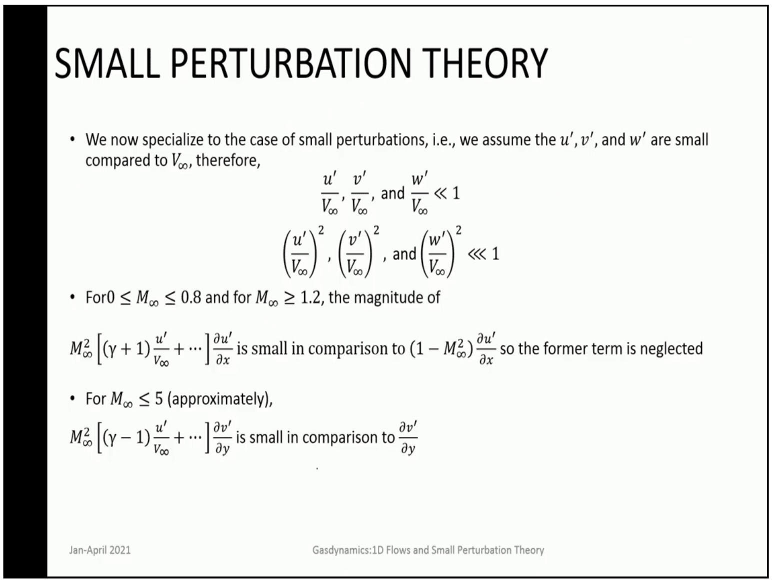

And what we say is they are small perturbations that mean they are small if you consider squares or multiples of $\frac{u'}{V_\infty}$, $\frac{v'}{V_\infty}$, $\frac{w'}{V_\infty}$, $\frac{u'^2}{V_\infty^2}$, $\frac{v'^2}{V_\infty^2}$, $\frac{w'^2}{V_\infty^2}$. So, all these parameters are even smaller. So, if you consider such order of magnitude analysis for subsonic flow and supersonic flow not in between not in the range of transonic flows then you find that the various terms that are there on the right hand side are very small in comparison to the corresponding terms on the left hand side.

So, the left hand side becomes important. So, you have these various terms, $\frac{\partial u'}{\partial x}$, $\frac{\partial v'}{\partial y}$, $\frac{\partial w'}{\partial z}$. So, these terms if you take a look at them the order of these are small in comparison to the left hand side which is $\frac{\partial u'}{\partial x}$, $\frac{\partial v'}{\partial y}$, $\frac{\partial w'}{\partial z}$ because they are getting multiplied by various small quantities.

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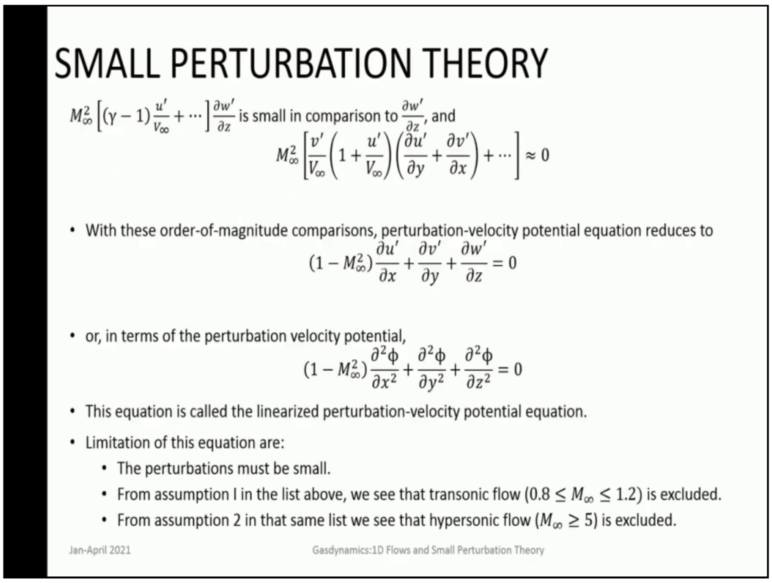

So, if you consider such an approach then we find that this perturbation velocity potential reduces down to,

$$\left(1-M_\infty{}^2\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

which is much, much simpler and not only is it simpler it is also linear in terms of if you now put back the velocity potential, M_∞ is a constant for a given problem. So, now this is a linear equation. So, this is a linearized perturbation velocity potential equation.

So, this is applicable for very small perturbations. So, perturbations are small that corresponds to that airfoils are thin and similar such arguments and it is for subsonic and supersonic flows that is transonic flow is excluded. In one of the conditions you also say that for mach numbers which are less than 5 which is that is it is in supersonic flow but not so, high speed that you can consider it as hypersonic flow then changes in V is also small.

So, that means hypersonic flow is also excluded. So, if you consider only subsonic flow and supersonic flow then the linearized velocity potential equation must hold good. So, it should hold good.

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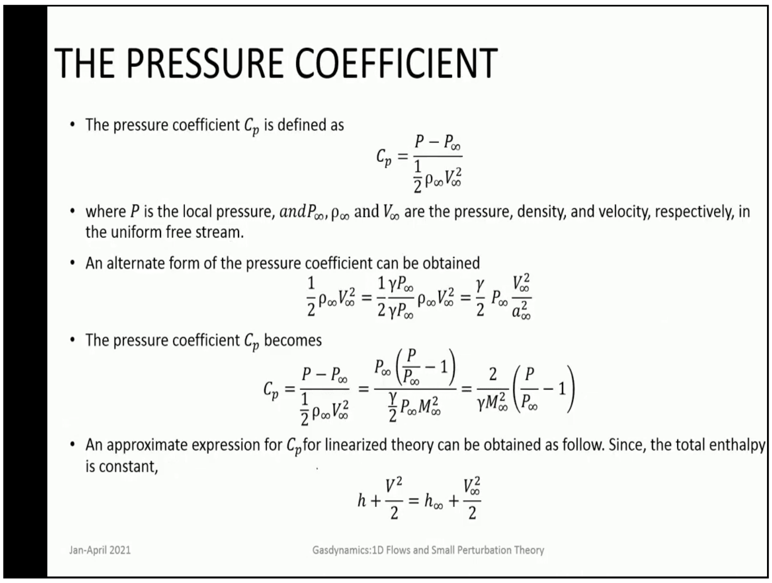

So, now we can see how this can be analyzed for subsonic flow. Before going there what we really are interested if you consider an airfoil is how does pressure change over the airfoil. So, that from such an information we will be able to gather information on the lift or aerodynamic coefficients. So, we really need to know what is the coefficient of pressure ?

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty V_\infty{}^2}$$

Now if you take P_∞ out of this it will become ,

$$C_P = \frac{P_\infty\left(\frac{P}{P_\infty} - 1\right)}{\frac{1}{2}\rho_\infty V_\infty{}^2}$$

Use the fact if you multiply and divide by γ then you have,

$$C_P = \frac{2\left(\frac{P}{P_\infty} - 1\right)}{\gamma M_\infty{}^2}$$

Now can we express $\frac{P}{P_\infty}$ in terms of the velocity potentials or perturbation velocity potential then we will get Cp in terms of the linearized theory.

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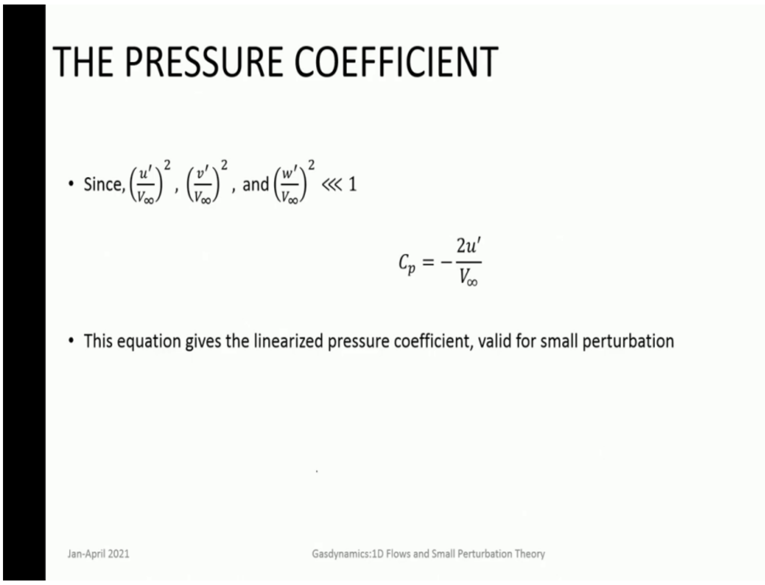

So, for this we can use again the total enthalpy is constant,

$$h_\infty + \frac{V_\infty^{\,2}}{2} = h + \frac{V^2}{2}$$

$$T + \frac{V^2}{2Cp} = T_\infty + \frac{V_\infty^{\,2}}{2Cp} \rightarrow T - T_\infty = \frac{V_\infty^{\,2} - V^2}{\frac{2\gamma R}{\gamma - 1}}$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\,\frac{V_\infty^{\,2} - V^2}{\gamma R T_\infty} = \frac{\gamma-1}{2}\,\frac{V_\infty^{\,2} - V^2}{a_\infty^{\,2}}$$

$$V^2 = (V_\infty + u')^2 + v'^2 + w'^2$$

So, if you substitute that and you look at,

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\,\frac{V_\infty^{\,2} - V^2}{\gamma R T_\infty} = \frac{\gamma-1}{2}\,\frac{2u'V_\infty + u'^2 + v'^2 + w'^2}{a_\infty^{\,2}}$$

So, now we know that it is a isentropic flow

$$\frac{P}{P_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P}{P_\infty} = \left(1 - \frac{\gamma-1}{2}\,\frac{2u'V_\infty + u'^2 + v'^2 + w'^2}{a_\infty^{\,2}}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P}{P_\infty} = \left(1 - \frac{\gamma-1}{2} M_\infty^{\,2} \left[\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{a_\infty^{\,2}}\right]\right)^{\frac{\gamma}{\gamma-1}}$$

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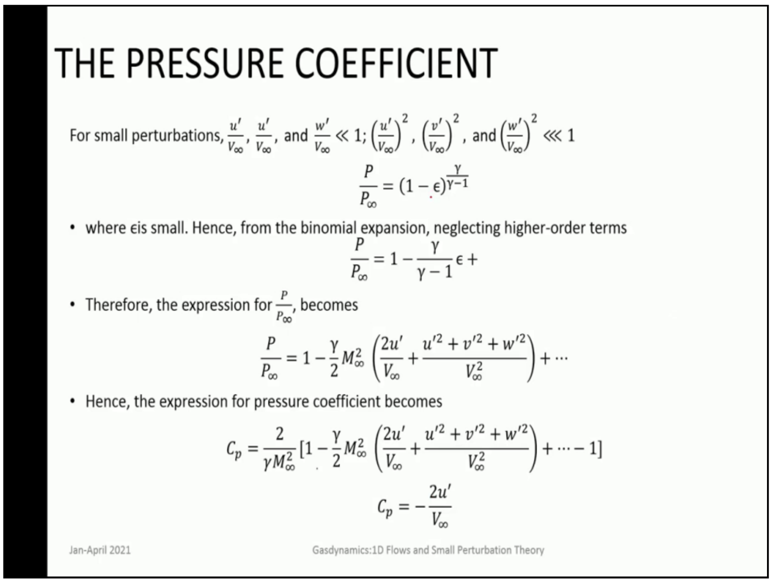

So, now this entire term the complete term can be written as an epsilon.

$$\varepsilon = \frac{\gamma-1}{2} M_\infty^2 \left[\frac{2u'}{V_\infty} + \frac{u'^2+v'^2+w'^2}{a_\infty^2}\right]$$

Now what we are saying is that this is small perturbation. So, $\frac{v'}{V_\infty}$, $\frac{u'}{V_\infty}$ all of them are very small similarly their squares are also very small. Therefore,

$$\frac{P}{P_\infty} = (1 - \varepsilon)^{\frac{\gamma}{\gamma-1}}$$

So, this can be expanded and we are considering ε square and higher terms they are very small they are negligible. You can neglect those terms and you can get it only in terms of ε.

$$\frac{P}{P_\infty} = 1 - \frac{\gamma}{\gamma-1}\varepsilon$$

Now we know $\frac{P}{P_\infty}$ in terms of the perturbation potentials perturbation velocities and M_∞^2. Now this can be substituted in Cp ,

$$C_P = \frac{2\left(1 - \frac{\gamma}{2} M_\infty^2 \left[\frac{2u'}{V_\infty} + \frac{u'^2+v'^2+w'^2}{a_\infty^2}\right] + \ldots - 1\right)}{\gamma M_\infty^2}$$

$$C_P = -\frac{2u'}{V_\infty}$$

So, for a linearized coefficient of pressure can be just expressed in terms of the velocity perturbation potential. So, this is an important result that comes out of this analysis this enables a certain way to solve the equations.

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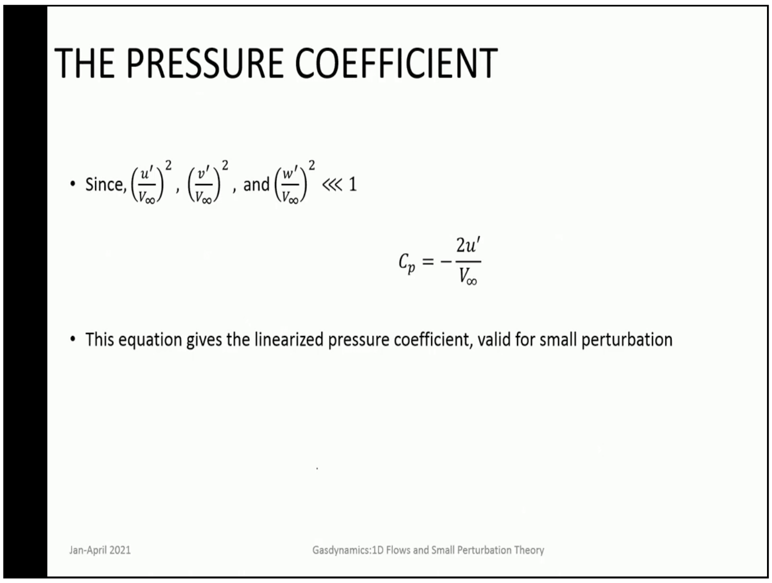

So, this is valid for small perturbations any small perturbations and it is valid both for supersonic and subsonic flow.

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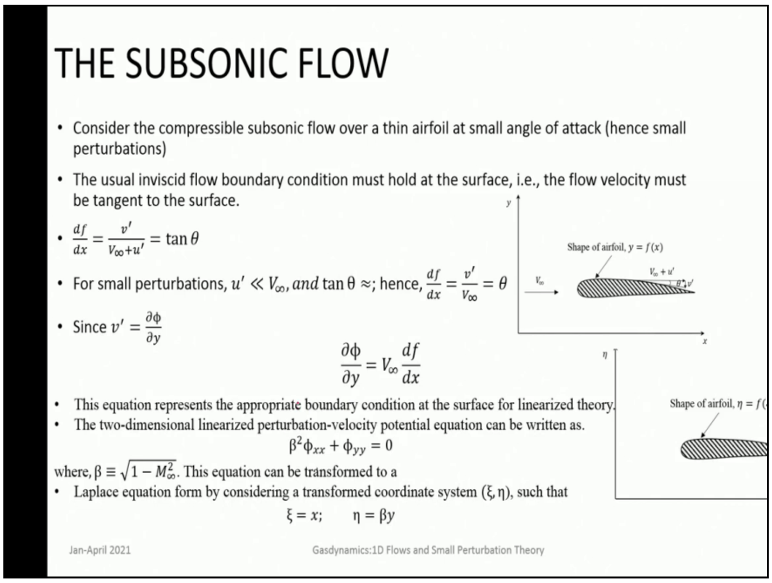

Now let us look at subsonic flow in particular or a thin airfoil at small angle of attack. So, then your velocity potentials or are small perturbations are small. It is an inviscid flow. So, the appropriate boundary condition is that the flow is tangential to the shape. So, if you know the shape of the airfoil y = f(x) then the tangent is $\frac{df}{dx}$ and the flow at the surface should be tangent to it i.e. $\frac{df}{dx} = \frac{v'}{V_\infty + u'} = \tan\theta$

So, now it is very, very small.

So, we say tan θ is approximately equal to θ which is equal to $\frac{df}{dx}$. Also, we also use the fact that u' is much smaller than V_∞ therefore you can express $\frac{df}{dx} = \frac{v'}{V_\infty}$ and $v' = \frac{\partial\phi}{\partial y}$.

$$\frac{\partial\phi}{\partial y} = V_\infty \frac{df}{dx}$$

So, this is an appropriate boundary condition to be put along the walls for the linearized theory.

Now if you take the equation in 2 dimensions,

$\beta^2 \varphi_{xx} + \varphi_{yy} = 0$

$$\beta = \sqrt{1 - M_\infty^2}$$

So, the idea is can we transform this equation to something that we already know ,we know solutions already exist. So, in the transformed coordinates, so, this is you are applying a transformation to

$\xi = x \,;\, \eta = \beta y$

So, now what we have to do we have to convert this equation by doing differentiation along those.

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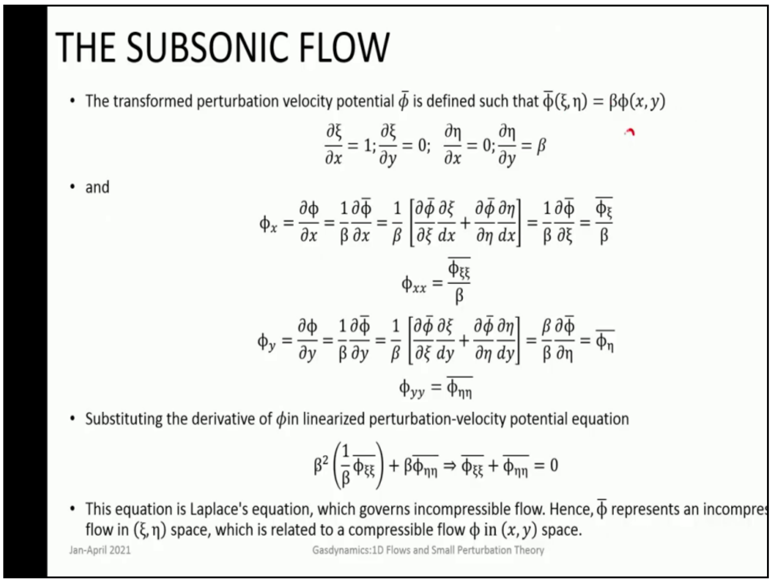

So, now $\bar{\varphi}$ is expressed in (ξ, η) coordinates. So, the same equation gets converted into (ξ, η) coordinates. So, when you do the conversions do the differentiation and do the various conversions which is sort of listed over here and the final expression that you get is,

$\bar{\varphi}_{\xi\xi} + \bar{\varphi}_{\eta\eta} = 0$ which is a Laplace equation. So, this equation is a potential equation its Laplace equation.

And it is valid or it governs the incompressible flow which is something we already know about and we know many solutions of these incompressible flows, Laplace equations we have done that for airfoil sources things and so on. So, now can we then utilize the results that we already have in incompressible flow for that we should know that what happens to the shape of the airfoil in the transformed coordinates.

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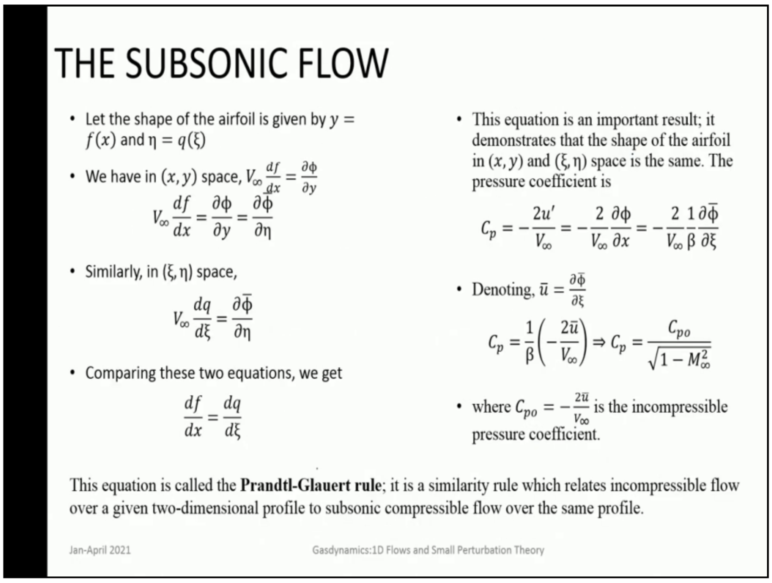

So, let us look at that. So, in xy space we know the boundary condition corresponding to that is $\frac{\partial\phi}{\partial y} = V_\infty \frac{df}{dx} = \frac{\partial\bar{\varphi}}{\partial\eta}$ =. Now if you consider it in (ξ, η) coordinate, the shape of the airfoil is $\eta = q(\xi)$. So, $V_\infty \frac{dq}{d\xi} = \frac{\partial\bar{\varphi}}{\partial\eta}$ .If you do the math by differentiating you will approach get this particular solution and you compare it with the previous solution they are exactly the same. Or what you get is d $\frac{df}{dx} = \frac{dq}{d\xi}$ or in other words what it shows is that the shape of the airfoil does not change as you move from x y coordinate to zeta eta space. So, it remains the same. So, whatever solutions we get in (ξ, η) coordinates which is for a particular shape of the airfoil in incompressible flow that can be used as a solutions in the compressible domain but there will be additional terms that will come and that term is due to β. So, when you put the term denoting β.

So, you get Cp $= \frac{1}{\beta}\frac{2\bar{u}}{V_\infty}$ where $\bar{u}$ is the perturbation velocity in (ξ, η) coordinate which is an incompressible flow solution. And that can be represented as Cp_0 or Cp incompressible which is already known. So Cp incompressible if it is known then the Cp compressible is known, you can get it, Cp = $\frac{C_{P0}}{\sqrt{1-M_\infty^2}}$

So, this result is very important very famous also and used to extend aerodynamic relations that is known in incompressible flow to compressible flow.

This is the Prandtl-Glauert rule it is a similarity rule and extensively used to relate incompressible flow relations to subsonic compressible flow for the same shape and used extensively in aerodynamics. But there are obviously certain it is a linearized problem, actual flow is not linearized. So, people have looked at other ways to overcome this also.

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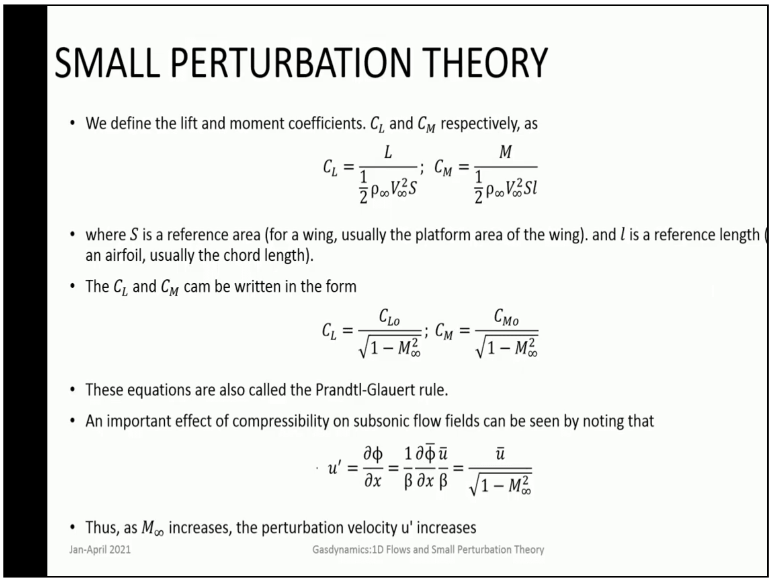

Now if you use that is C_L for a small that for a section or its for an airfoil similarly you can look at lift and moment coefficients which are the integration of Cp. And here also since all other parameters are constants you get you can write $C_L = \frac{C_{L0}}{\sqrt{1-M_\infty^2}}$.

Similarly C_M = $\frac{C_{M0}}{\sqrt{1-M_\infty^2}}$. So, what you see is that in compare to the incompressible lift coefficient as Mach number increases the lift coefficient also will increase because $1 - M_\infty^2$ is there.

So, it will increase. So, what you also see is that the effect of compressibility is to increase the perturbation velocity as M_∞^2 increases. But this here what is happening is you are considering M_∞^2 . So, that is a free stream flow but we know that as the flow passes over an airfoil it accelerates. So, it accelerates. So, it accelerates over the foil. So, you are expecting that the Mach number will increase. This effect is not considered, people try to consider it there are some improved compressibility corrections like the Laitone's equations or Karman-Tsien rule.

So, they are also applied and these show this graph shows a comparison of various experiments with these different kind of rules where it is seen that Karman-Tsien rule.
rule is somewhat more it applies closely or follows closely to the experimental values while Prandtl Glauert rule lie in the bottom part of it while Laitones lie on the upper part of it. But they are all good approximations when you want to make some quick calculations of aerodynamic coefficients.

Then Prandtl Glauert rule can be easily applied and it is quite useful. So, the highlight of this small perturbation analysis that we saw is the end result resulting in Prandtl Glauert rule and this particular approach where we see that you take small perturbations and linearize a non linear equation and then try to get some results out of it. So this is for subsonic flow and similarly we look at supersonic flow.

And in supersonic flow we find. Now it is going to be a hyperbolic equation we saw that earlier. So, that it behaves more like a wave equation. So, there are consequences similar consequences to the linearization also. And we will see what can be what is the result expected out of it in the next class. So, thank you.