Gasdynamics: Fundamentals and Applications
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Lecture 49
Small Perturbation Theory - I

So, until now for quite a long duration of this course we have been discussing about 1D flows quasi 1D flows where we make the assumption that the flow variables remain uniform across a certain cross section. We do not actually solve for entire flow field and the dominant approach used is control volume methods where we looked at what is happening across interfaces. And do not go into details of what is happening within the flow field.

But now in real problems in actual applications we may need to know these flow field details and. So, the approach has to change and. So, we look into the cases of solving the entire flow field. A particular approach of course the complete set of Navier stokes equation for compressible flows have to be solved numerically. There is no analytical method to do it. But there are many approximate methods which will give us useful results.

One among them is if the flow if changes in the flow produced by say very slender bodies or airfoils, thin airfoils they are all small with respect to the free stream flow. So, they introduce small changes to the flow. So, that kind of an approach is known as small perturbations to the main flow. So, a small perturbation theory but before we go into small perturbations. We really have to look at the flow equations themselves at looking at the entire flow field.

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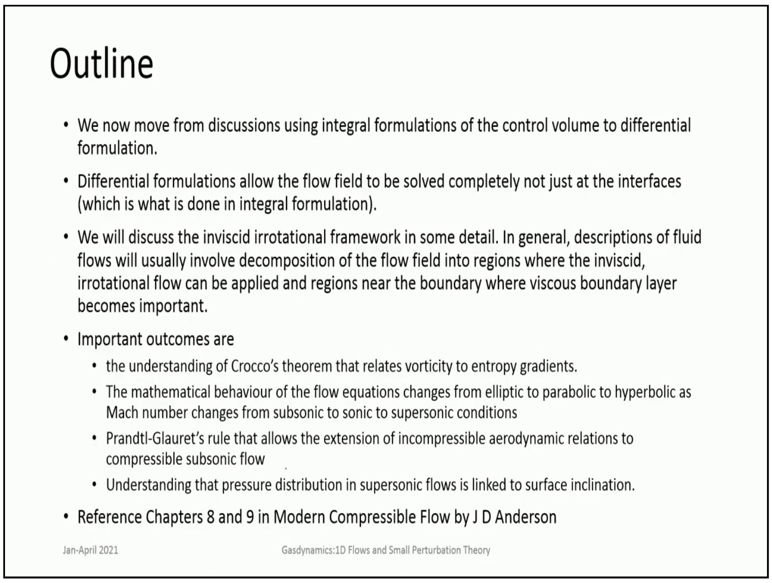

And therefore we have to get ourselves away from control volume approach, integral formulations to differential equations and look at details. Here we will be particularly looking into some details of inviscid irrotational flows and some good solutions, some important insights like Crocco's theorem and how do the flow equations behave as the flow changes from subsonic to supersonic.

Some important results like the Prandtl-Glauret's rule which allows extension of incompressible aerodynamic relations to compressible subsonic flows. And some idea of say pressure distributions over bodies in when they are very slender or the perturbation is very small. And in supersonic flows they can be linked only to the surface inclination. And largely this outline follows modern compressible flow by Anderson.

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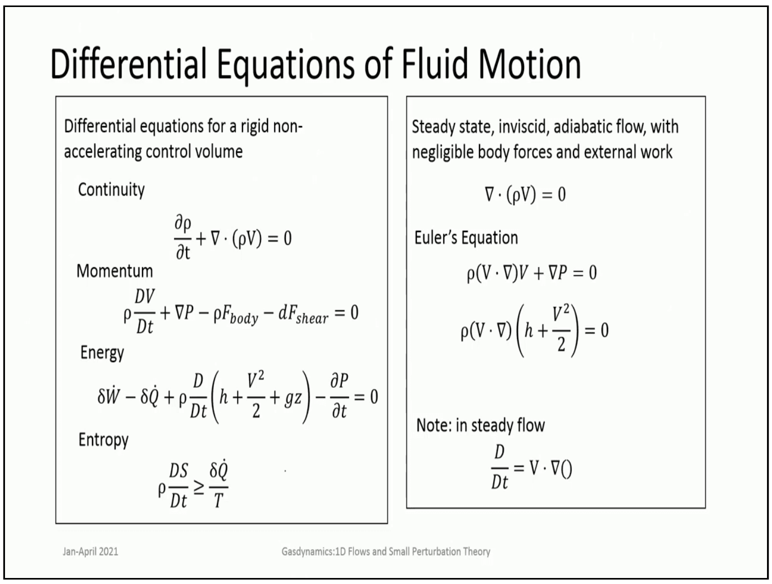

So, we will start looking at the differential equations of fluid motion and we are looking at inviscid flows. Therefore the shear forces are not there or they are absent . So, continuity equation here density is a variable.

$\frac{\partial \rho}{\partial t}$ $+\nabla.(\rho\ \vec{V})=0$

For a steady flow, $.(\rho\ \vec{V})=0$

Momentum equation we hav,

rho DV by Dt, material derivative of velocity, is the change in flux momentum flux change.

$\rho\frac{D\vec{V}}{Dt}\ +\nabla P - \rho F_{body}\ -\ dF_{shear}\ = 0$

Rate of change of momentum is equal to all the forces that appear on the on the fluid which is due to pressure, it can be due to body forces ,it can be due to shear forces. In this particular case we consider only pressure forces and the others are negligible. Therefore we come to the Euler's equation which is,

$\rho\frac{D\vec{V}}{Dt}\ +\nabla P=0$

for steady flow,

$\rho\ (\ \vec{V}.\nabla)\ \vec{V}\ \ +\nabla P=0$

Then you consider the energy equation where we are considering work done, heat added and change in total enthalpy changes in total enthalpy this is the differential equation where $\frac{D}{Dt}$ corresponds to a material derivative. If we consider adiabatic flow, no work done, no external work then the corresponding and steady flow then essentially the total enthalpy should remain a constant.

$\rho\ (\ \vec{V}.\nabla)\ (h+\ \frac{V^2}{2})=0$

So, you have continuity, momentum gives you Euler's equation in the differential form and the condition of adiabatic flow with other effects being neglected no work done gives you that the total enthalpy remains constant. So, now we look at how these equations can be manipulated.

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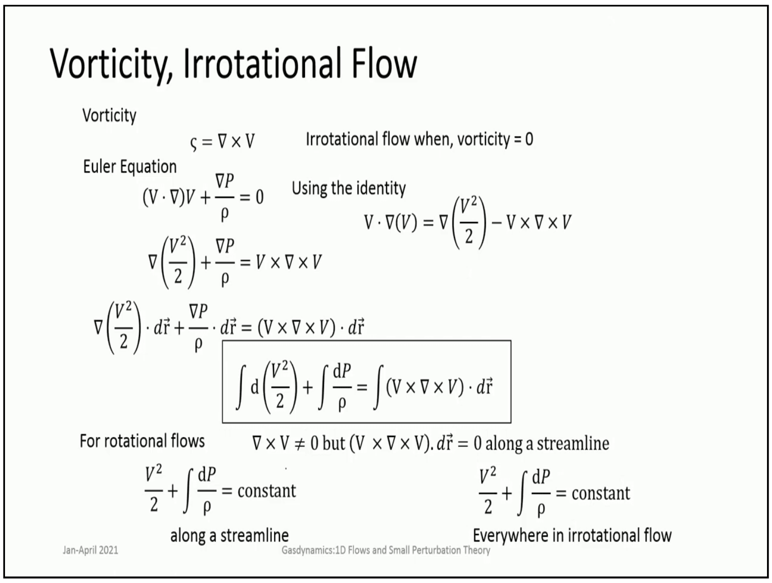

To look at that we said we are looking at inviscid a rotational framework what do you mean by irrotational framework? We are looking at vorticity which corresponds to rotation in the fluid element. When there is no rotation then we say that the flow is irrotational or vorticity is 0. Vorticity is related to the velocity field by curl. So, this is the relation for vorticity.

$\zeta = \nabla \times \vec{V}$

Essentially for irrotational flows curl is 0 ,curl of velocity is 0, $\nabla \times \vec{V} = 0$ where velocity is a vector. And if we consider a three dimensional field it is $\vec{V} = u\, i + v\, j + w\, k$. So, this is the general form. Now let us take a look at the inviscid flow which is represented by the Euler equation , $(\vec{V}.\nabla)\vec{V} + \frac{\nabla P}{\rho} = 0$

Now $(\vec{V}.\nabla)\vec{V}$,this term can be represented as,

$(\vec{V}.\nabla)\vec{V} = \nabla(\frac{V^2}{2}) - V \times \nabla \times \vec{V}$

So, curl is over here this is an identity, it is a vector identity and $V \times \nabla \times \vec{V}$

So, we can substitute this into Euler equation and we get here,

$\nabla(\frac{V^2}{2}) + \frac{\nabla P}{\rho} = V \times \nabla \times \vec{V}$

$\nabla\left(\frac{V^2}{2}\right).\ \vec{dr} + \frac{\nabla P}{\rho}.\vec{dr} = (V \times \nabla \times \vec{V}).\ \vec{dr}$

Now if you try to integrate it along a certain path dr if you try to integrate this particular equation along a certain path and then this will become a total derivative,

$\nabla\left(\frac{V^2}{2}\right).\ \vec{dr} + \frac{\nabla P}{\rho}.\vec{dr} = (V \times \nabla \times \vec{V}).\ \vec{dr}$

$\int d\left(\frac{V^2}{2}\right).\ \vec{dr} + \int \frac{dP}{\rho}.\vec{dr} = \int(V \times \nabla \times \vec{V}).\vec{dr}$

It is a directional derivative now and this integral for the component $(V\ x\ \nabla\ x\ \vec{V})$ in the direction dr. So, if we consider an irrotational flow then $d\nabla\ x\ \vec{V}\ =\ 0$ everywhere then this term the right hand term completely drops out therefore we get,

$$\int d\left(\frac{V^2}{2}\right).\ \vec{dr} + \int \frac{dP}{\rho}.\vec{dr}\ = 0$$

If you consider the case of an incompressible fluid this must be quite familiar to all of you it is nothing but,

$$\frac{V_2^{\ 2}}{2} - \frac{V_1^{\ 2}}{2} + \frac{P_2 - P_1}{\rho} = 0$$

Or

$$\frac{V^2}{2} + \frac{P}{\rho} = \text{constant}$$

which is nothing but the Bernoulli's equation, that is for a incompressible constant density flow.

So, this is it turns out to be the Bernoulli's equation. But here is general Euler equation we cannot do this integration $\int \frac{dP}{\rho}.\vec{dr}$. So, $\frac{V^2}{2}$ can be integrated. So,

$$\frac{V^2}{2} + \int \frac{dP}{\rho}.\vec{dr}\ = constant$$

In an irrotational flow it is constant everywhere but if there is rotational if it is rotational and vorticity exists then it is constant along directions where $\left(V\ x\ \nabla\ x\ \vec{V}\right).\vec{dr} =\ 0$ which is true for a stream line.

So, it is for rotational flows, this parameter or this particular combination is constant along a stream line. So, that distinction must have been familiar to you in the context of Bernoulli's equation but it is true in the case of Euler's equation only thing now it is a compressible flow, density is a variable. So, it cannot be you cannot easily integrate this as you did for the Bernoulli's equation.

So, the consequence of having vorticity and irrotational flow should be understood in this context.

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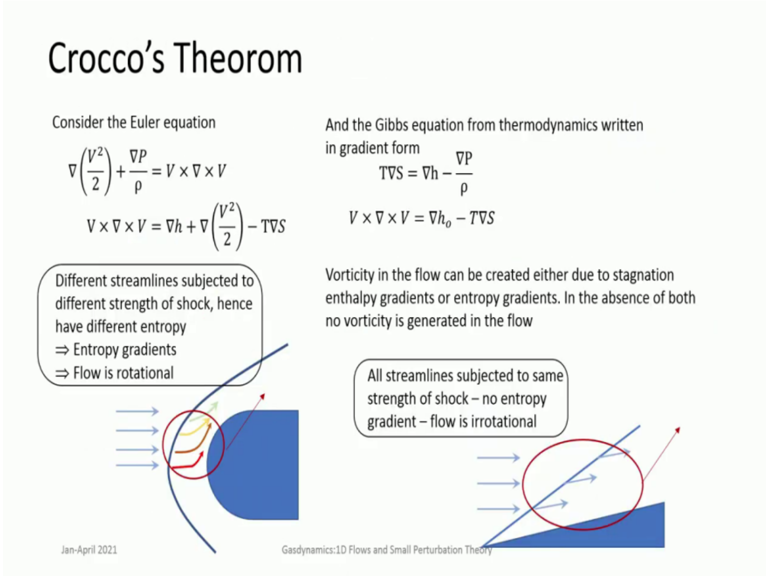

So, another important sort of theorem which comes about from this inviscid considerations and what we had discussed previously, the Euler's equation written in terms of vorticity. Here we have the term $\frac{\nabla P}{\rho}$

Now we consider the equation for enthalpy the Gibbs equation,

$T\nabla S = \nabla h - \frac{\nabla P}{\rho}$

So, this $\frac{\nabla P}{\rho}$ can be written in terms of $T\nabla S$, ∇h that entropy gradient and gradient of enthalpy.

So, if you replace that. So, then you can get a relation between vorticity, enthalpy and entropy gradients along with of course kinetic energy gradient.

$$\nabla\left(\frac{V^2}{2}\right) + \nabla h - T\nabla S = (V \times \nabla \times \vec{V})$$

But these two together taken, $(h + \frac{V^2}{2}) = h_0$.

So, it you get.

$$(V \times \nabla \times \vec{V}) = \nabla h_0 - T\nabla S$$

If you had considered an adiabatic flow which is what we have been considering here

$h_0 =$ constant.

So, gradient of h_0 is not there, $\nabla h_0 = 0$.

So, therefore we get the information that if there are any gradients of entropy, if it is an isentropic flow and entropy is constant everywhere then the consequence is that there is no vorticity. So, if you consider that is, adiabatic, isentropic flows. So, isentropic flows

essentially are irrotational flows. But if you consider there are there is rotational effects in the flow or rotationality is there in the flow.
Then directly from this equation it is seen that there should be entropy gradients. Where is this sort of applied ? If you consider any generic, say, body in a supersonic flow. So, this is a certain body placed in supersonic flow then we know there should be a shock and if the body is blunt you will have a bow kind of a shock where the shock has a curvature. So, you see that the shock has curvature along this direction. That means at every point along the shock the shock strength varies.

So, now we know that across a shock there is an entropy jump. So, there is an entropy jump. So, if you take any particular stream line as it goes across the shock ,there is an entropy jump it has different entropy. If you take different stream lines they have different entropies. So, downstream of the shock in this region which is marked by the brown circle in front of this bow kind of in front of this blunt kind of a shape having a bow shock you have entropy gradients that means the flow is no longer irrotational it has rotational, it is rotational.

So, flow is rotational there are entropy gradients in this kind of a case. But if you consider a case like a planar shock that is suppose you consider a wedge kind of a shock then here you see that you have an oblique shock and oblique shock turns all the stream lines by the same angle. So, lines were parallel earlier they are parallel after the shock they have just been deflected by a certain angle there is a entropy jump across the shock but after the shock there is no entropy gradient.

So, before the shock there it is a constant entropy, after the shock it is another constant entropy. Only that entropy is changing. That means these are cases where flow is irrotational. So, this is very important as to how we can apply different approaches in solving such flow field problems and Croco’s theorem connects vorticity with entropy gradients.
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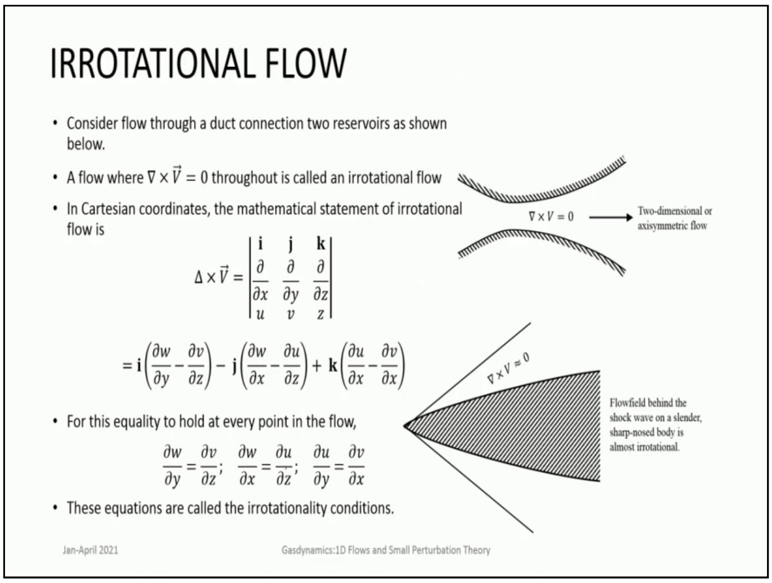

So, generally the condition of irrotationality, irotational flow is that $\nabla \times \vec{V} = 0$. So, you get relationships between the different components of the velocity that is, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ etc. which might be familiar for a 2 dimensional flow. We always have this particular form $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. But in a general three dimensional flow it can be evaluated for all other components also. So, we are looking at irrotational flows.

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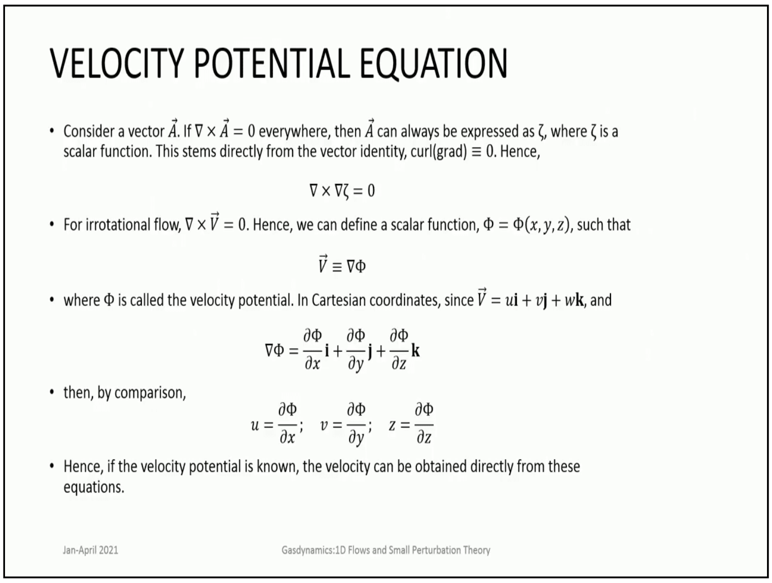

So, if you consider irrotational flows, then a vector, if you have a vector and it is solenoidal or does not have a curl then that particular vector a can be represented as the gradient of a scalar. So, this is also an identity. This is a vector identity. Curl of gradient is equal to 0 means there you can always define a scalar function or scalar potential. So, if you consider rotational flows then we can define a velocity potential Φ such that velocity is a gradient of Φ.

$\nabla \times \nabla \vec{V} = 0$

So, $u = \frac{\partial \Phi}{\partial x}$, $v = \frac{\partial \Phi}{\partial y}$, $w = \frac{\partial \Phi}{\partial z}$. You would have become familiar with this kind of approach in your fluid mechanics classes on potential flow theory and so on. This is the same approach.

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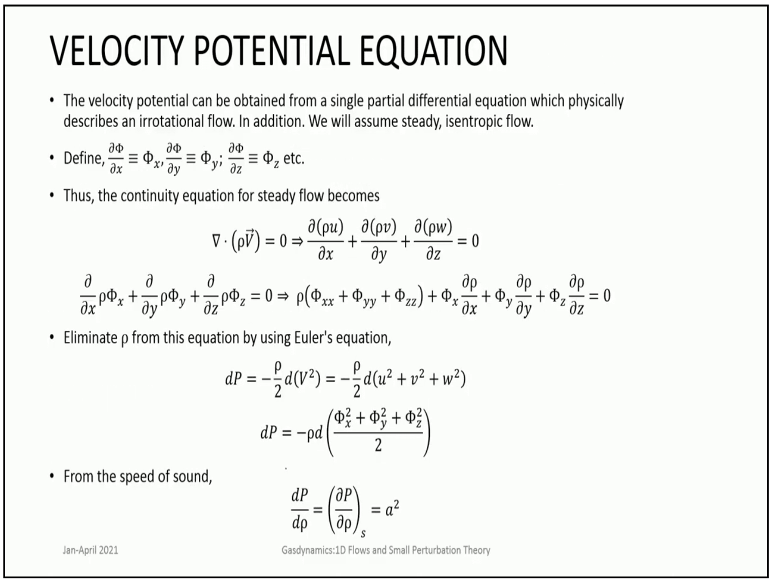

But now it is being applied to a compressible flow and in a compressible flow density is changing. So, if you look at the continuity equation which is essentially,

$$\nabla.(\rho \vec{V}) = 0 \rightarrow \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

And consider the velocity potential Φ and u is then $u = \frac{\partial \Phi}{\partial x}$. we can substitute those terms here in the equation and we get

$$\frac{\partial \rho \Phi_x}{\partial x} + \frac{\partial \rho \Phi_y}{\partial y} + \frac{\partial \rho \Phi_z}{\partial z} = 0 \rightarrow \rho(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) + \Phi_x \frac{\partial \rho}{\partial x} + \Phi_y \frac{\partial \rho}{\partial y} + \Phi_z \frac{\partial \rho}{\partial z} = 0$$

Now we can use the Euler's equation to try and remove ρ from this particular expression .

$dP = \frac{\rho}{2} d(V^2) = \frac{\rho}{2} d(u^2 + v^2 + w^2)$

And that can be written in terms of the derivatives of potential $u = \frac{\partial \Phi}{\partial x}$, $v = \frac{\partial \Phi}{\partial y}$, $w = \frac{\partial \Phi}{\partial z}$

And we use the definition of speed of sound because it is an isentropic flow, steady isentropic flow. So, you can use,

$a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$ at constant entropy.

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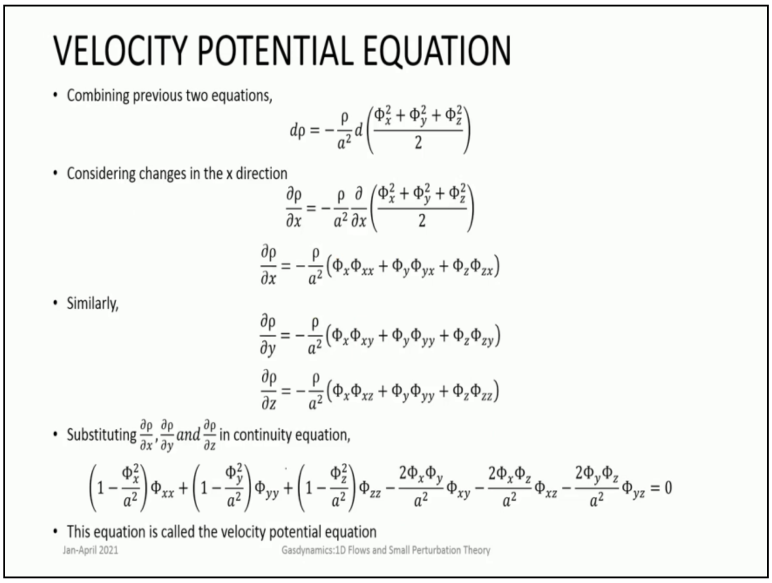

Combine them together and we get,

$d\rho = \frac{-\rho}{a^2} d\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right)$

So, now considering changes in every direction which is x direction ,y direction, z direction separately ,we have to differentiate this separately and put them into the continuity equation and arrive at the final equation for the velocity potential in case of compressible flows.

$\frac{\partial\rho}{\partial x} = \frac{-\rho}{a^2}\frac{\partial}{\partial x}\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right)$

Here you can see this is the final equation which is,

$$\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right)\Phi_{zz} - \frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy} - \frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz} - \frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz} = 0$$

So, now this equation written only in terms of the velocity potential, is the velocity potential equation is a general equation. Of course it has the term a^2. We have to look at a way to calculate a^2.

But otherwise it is only in terms of the velocity potential. So, now how to get a^2 into this?

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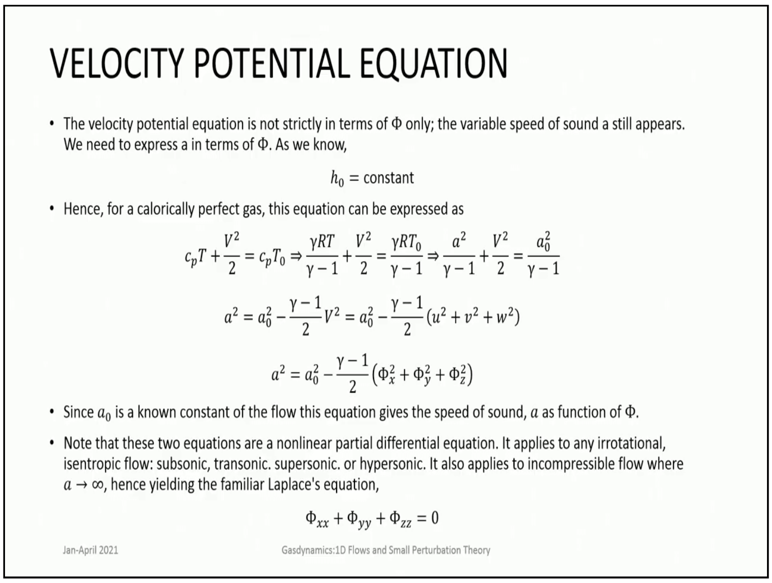

We can use the fact that it is an adiabatic flow h_0 = constant. So, if you take a calorically perfect gas then you can express the equation,

$$Cp\,T + \frac{V^2}{2} = Cp\,T_0$$

So, this can be expressed in terms of a_0 and this we had done very early in the class ,

$$\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a_0^2}{\gamma-1}$$

where now $V^2 = u^2 + v^2 + w^2 = \Phi_x{}^2 + \Phi_y{}^2 + \Phi_z{}^2$.

So, since you can know a_0, it is a constant within the flow. So, now you have an equation to relate a^2 with a_0^2. So, one should notice that this equation now is a non linear equation and it is general equation. It applies to any rotational isentropic flow. It can be subsonic transonic supersonic or hypersonic and if you consider that a goes to infinity. So, if you consider that which is corresponding to an incompressible flow.

The speed of sound goes to infinity in an incompressible flow then all these terms drop off slowly these the different terms which contain $\frac{1}{a^2}$ drop off and then you get only

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

which is the familiar Laplace equation, potential flow equation for a irrotational flow, so, incompressible flow. So, it is a general equation and it contains the incompressible part also. So, this is the velocity potential equation.

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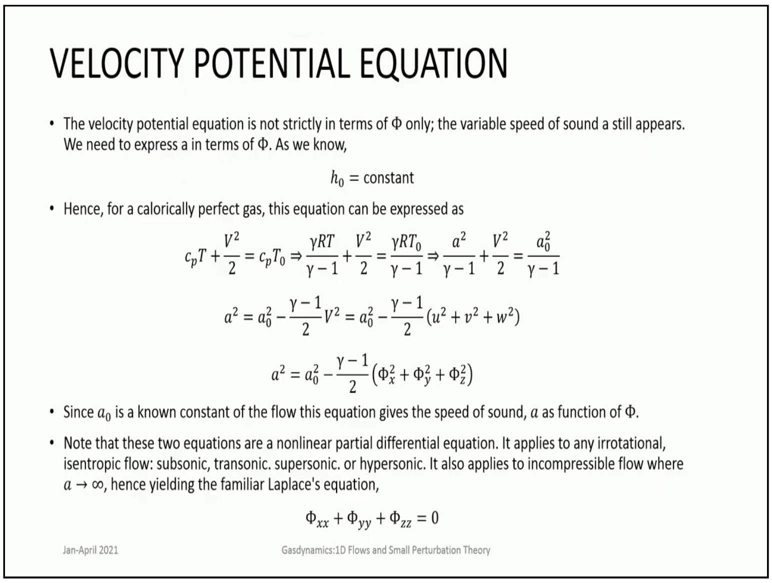

And if we; consider just the 2D velocity potential equation. Let us try to see how this velocity potential equation behaves . So, and look at how the partial differential equation is seen. So, writing,

$u^2 = {\Phi_x}^2$, $v^2 = {\Phi_y}^2$, $w^2 = {\Phi_z}^2$ and $\Phi_x \Phi_y$ = uv . So, you get,

$$\left(1 - \frac{u^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{v^2}{a^2}\right)\Phi_{yy} - \frac{2uv}{a^2}\Phi_{xy} = 0$$

To understand the behaviour of these equations second order partial differential equation we look at the determinant which is D = B^2 - 4 AC and you can express this and it comes out as $M^2 - 1$. Now we know that if the determinant is negative then the partial differential equation behaves in an elliptic manner. So, elliptic example of ellictic partial differential equation is the Laplace equation and a canonical form which you would have come across in your studies is the Laplace equation.

And characteristic of such kind of equations is that any change in any part will influence the entire domain that we are considering which is true when we consider steady subsonic flows. Information sort of propagates everywhere in subsonic flows. Changes in pressure can be felt, the effects of that can be felt everywhere. While if we take M = 1 then it is exactly 0 it behaves in a parabolic manner which is similar to the heat equation which you would have come across in your studies.

While if D is, if you take M > 0, M greater than 1 then determinant is greater than 0 or there are real roots to this determinant. So, that means this behaves more like it is a hyperbolic behaviour. So, supersonic flows steady supersonic flows behave in hyperbolic manner which is very important and a very well known equation which behaves similarly is the wave equation.

So, we find that in supersonic flows the behaviour is very much like that of a wave equation in that sense information propagation in supersonic flows happen only along specific direction is like a wave propagating. So, all regions of the supersonic flow are not affected by small changes happening or changes in pressure happening at particular points. Only when the wave or say a mach line passes along a particular point they connect the 2 points then they affect those points.

So, there is directionality in supersonic flows we will see this again and again in subsequent classes. Somewhat mixed behaviour is seen in transonic flows where both subsonic and supersonic parts can be present. But essentially what you have to understand is that the velocity potential equation in its full form which is what is given over here is a nonlinear equation.

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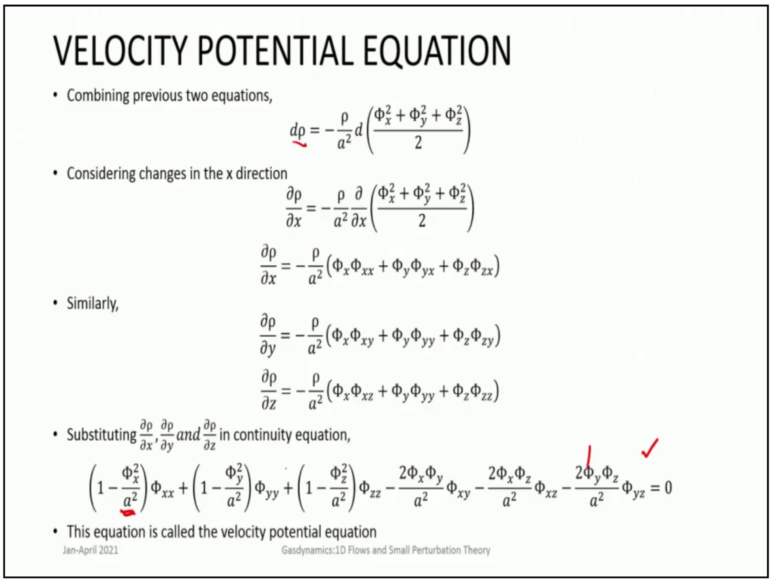

 It has to be there is no analytical approaches to solve it and you can only do it in a numerical manner. So, an approach that is taken when we can consider that the flow that we are trying or we are interested to solve introduces only small changes to a mean flow can be taken and that approach is known as the small perturbation theory. And we will look at how these equations change if we consider such a small perturbation.

And then first we look at subsonic flows, subsequently after that we look at how things happen in supersonic flows. This is a very important information that subsonic flows behave in an elliptic manner similar to Laplace equation and supersonic flows behave in a hyperbolic manner similar to wave equation. That you have to really understand. And it has consequences in how these equations are solved.

So, next class we will move towards application of small perturbation theory in subsonic flows. So, thank you.