Gasdynamics: Fundamentals. And Applications

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Lecture 48
Generalized 1D Flows

So let us look at a generalized flow in quasi 1D which consists of a combination of various parameters like variations in area, there may be friction there can be heat transfer. There is also an additional factor which we had not accounted till now is if there is a injection of mass, if mass is getting added also. This which is typical in cases of say propulsion systems where there is some fuel getting injected into the flow then there is a mass addition also happening.

So, we will look at these cases until now the analysis we were doing used quasi 1D assumption which is that flow properties remain uniform across cross sections. And we looked at different drivers like either only area was changing or there was only friction and no other effects were there it was a constant area duct. Similarly, only heat addition or heat removal. But in real life problems and applications there is always combination of these effects. So, how do we analyze such flows.?

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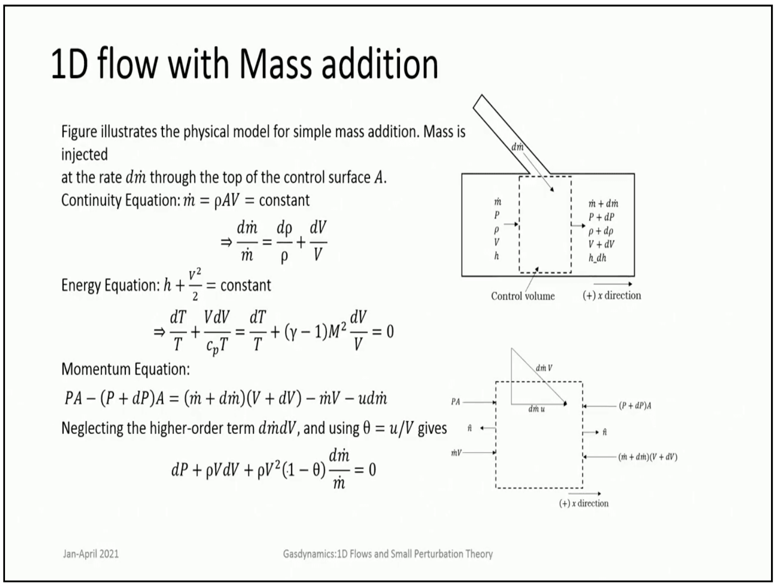

Before we go there we will just look at quickly look at what happens when there is mass addition without going much into details. Because now this process of doing a quasi 1D analysis with a control volume must have become quite familiar and this can be now you can do this continuously. So, now if you look at this what is this simple control volume that is taken. You have a constant area duct and at some place the mass is getting added certain d$\dot{m}$ is getting added.

So, across this control volume there is an increase in mass. So, $\dot{m}$ is not a constant anymore. There is a variation of $\dot{m}$ and you can get the differential equation from this quantity.
$\dot{m} = \rho AV$ is the mass flow rate through a particular cross section that is $\dot{m}$.
So, $\frac{d\dot{m}}{\dot{m}}$ can happen due to changes in density, change in area and change in velocity.

$$\frac{d\dot{m}}{\dot{m}} = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V}$$

So, there can be domain if you consider a constant area duct then $\frac{dA}{A} = 0$. So,

$$\frac{d\dot{m}}{\dot{m}} = \frac{d\rho}{\rho} + \frac{dV}{V}$$

So, A changes in mass flow rate will get affected in terms of change in velocity and density. And if you consider the energy equation here we are considering that this particular mass that is getting added is having the same stagnation enthalpy as the main flow.

So, both of them have same stagnation enthalpy therefore there is no change in stagnation enthalpy when additional mass is getting added, that is the assumption being done here. In general this can be different we will soon see that one also. So, $h + \frac{V^2}{2} = constant$

So, you can write now in terms of derivatives. Now if you look at the momentum equation you should be careful here because in general this mass that is getting added can have a velocity which can be in any general direction need not be along the same axis.

So, it need not be along the same axis. So, we are looking at only 1D flow. So, we are looking only along x axis therefore we should take the component of velocity in the x axis which is u here to look at momentum equation, x momentum equation, solve that.
$PA - (P + dP)A = (\dot{m} + d\dot{m})(V + dV) - \dot{m}V - u\, d\dot{m}$

Now this $\frac{u}{V}$ the component of the force in the direction in x direction for the injected flow, $\frac{u}{V}$ that is given the term $\theta = \frac{u}{V}$, it is a parameter in this problem. So, now you can divide the entire equation by $\dot{m}$ and this equation can be got from here.

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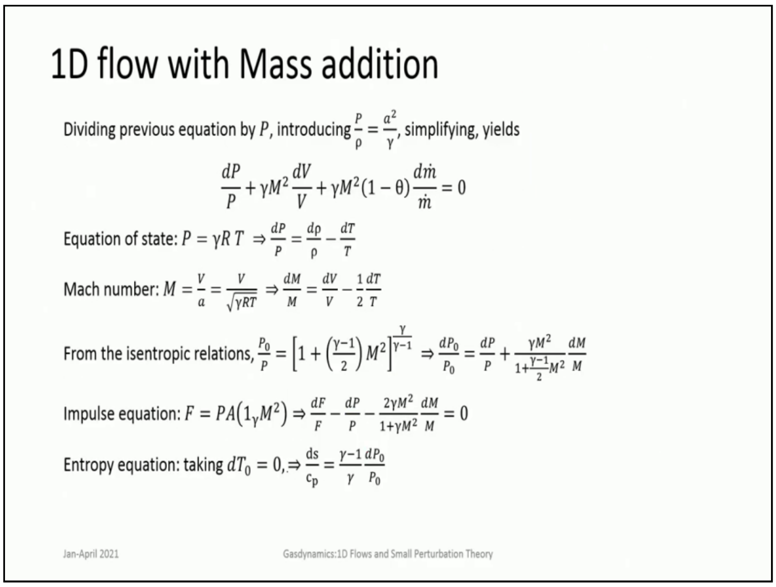

$dP + \rho V\, dV + \rho V^2 (1-\theta) \frac{d\dot{m}}{\dot{m}} = 0$

Now as we did in all other cases our intention is to represent the main governing equation here. It is a momentum equation you express momentum equations only in terms of Mach number. This can be done by series of algebraic manipulations of the different equations which is how pressure is related to the equation of state, P =ρ RT and

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Then you have Mach number, M = $\frac{V}{a}$ How is that related. Then you have $\frac{P_0}{P}$ this relation.

From here you can get the relationship between $\frac{P_0}{P}$. And similarly the impulse function and also you consider the dT_0

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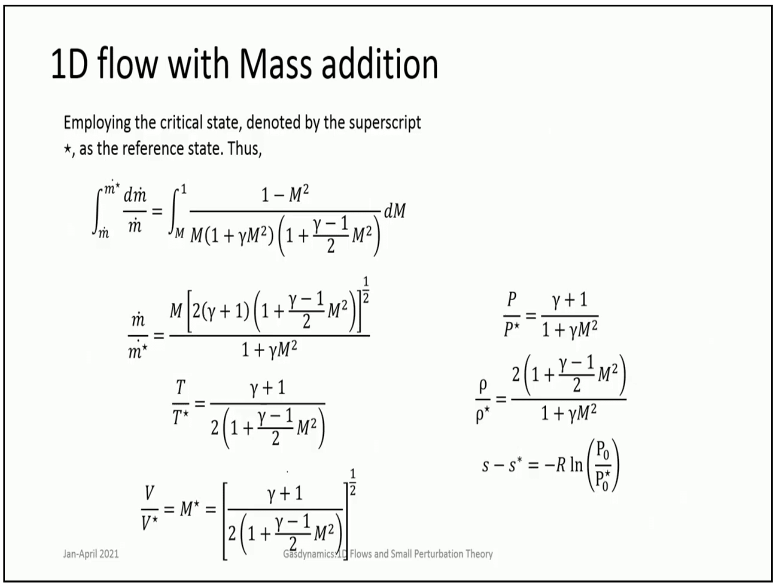

So, you can plug all of them and then look at integrating this with $\frac{d\dot{m}}{\dot{m}}$ as the driving parameter and similar to all other cases you can also have the star point which is the reference point and you can get closed form solutions for mass addition also.

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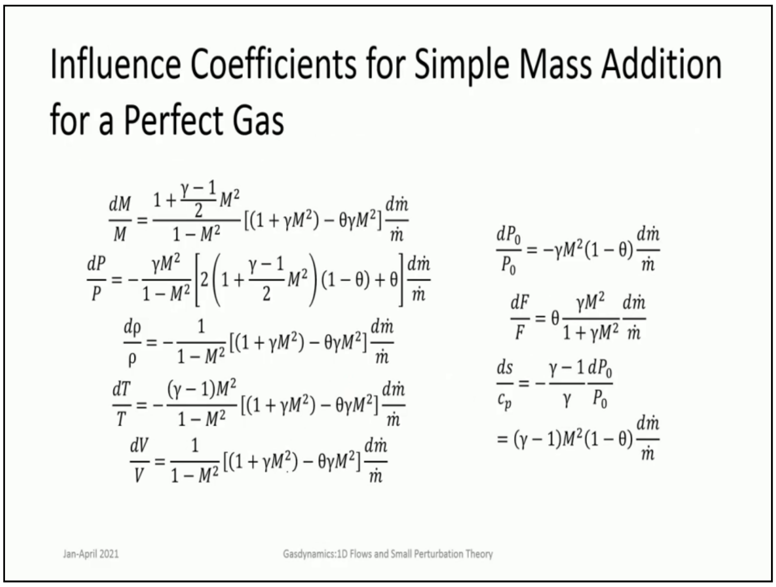

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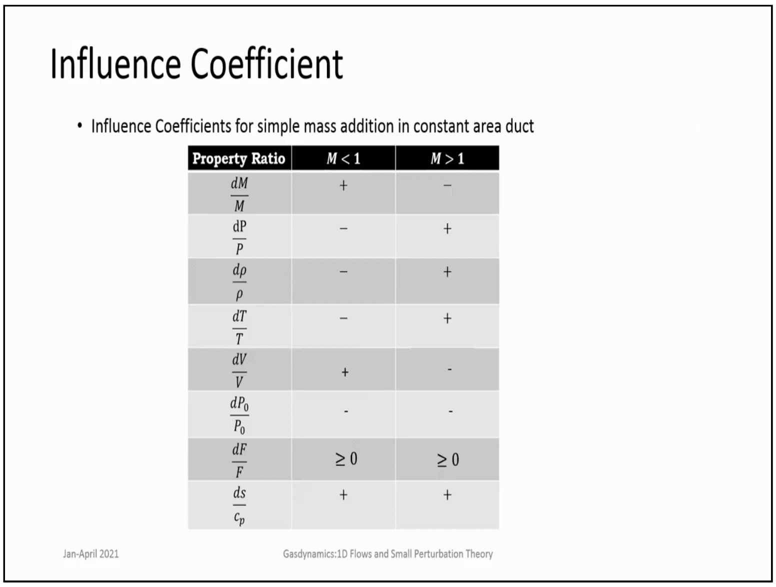

Property Ratio	$M < 1$	$M > 1$
$\frac{dM}{M}$	+	−
$\frac{dP}{P}$	−	+
$\frac{d\rho}{\rho}$	−	+
$\frac{dT}{T}$	−	+
$\frac{dV}{V}$	+	-
$\frac{dP_0}{P_0}$	-	-
$\frac{dF}{F}$	≥ 0	≥ 0
$\frac{ds}{c_p}$	+	+

So, let us directly go and look at how mass flow mass addition affects various parameters when the flow is subsonic. Similar to all other cases when flow is subsonic Mach number increases ,velocity increases, pressure, temperature, density decreases and entropy always increases. So, mass addition entropy increases. There is but in supersonic flow it goes the other way around which is Mach number decreases, pressure, temperature, density they increase and velocity decreases again entropy will increase.

So, that is a quick introduction that we did not consider mass addition but mass addition is also can be a possibility in specific applications. So, just before we go to the generalized case where we consider all possible variations of the driving forces or driving drivers of these equations this was a quick introduction to mass addition. The general analysis tools are similar to what we had done for the previous cases.

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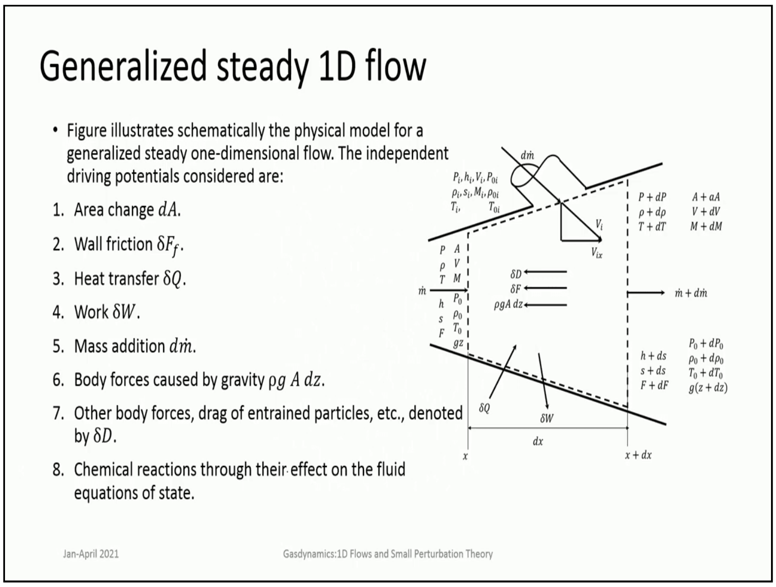

So, now let us look at the generalized steady 1D flow where you can have all kinds of effects that is happening. One is that it is a varying area duct. So, area change is happening then we are considering that there is wall friction so there is a frictional force along the walls. So, friction is considered there is heat that is getting added or removed. So, heat transfer is there. External work can be done or done by the system or on the system.

There can be mass addition with different sort of temperature, stagnation temperatures and pressures for the mass being added. You can have body forces and also in some cases you can have some entrained particles which induce some drag. So, another drag force is also present and there can be effects like there can be chemical reactions which can change properties of the fluid that can also be considered.

For the case that we are discussing here we will still consider calorically perfect gas. If chemical reactions are happening that assumption is not completely right but for the sake of a analysis in this class we will just consider the calorically perfect gas.

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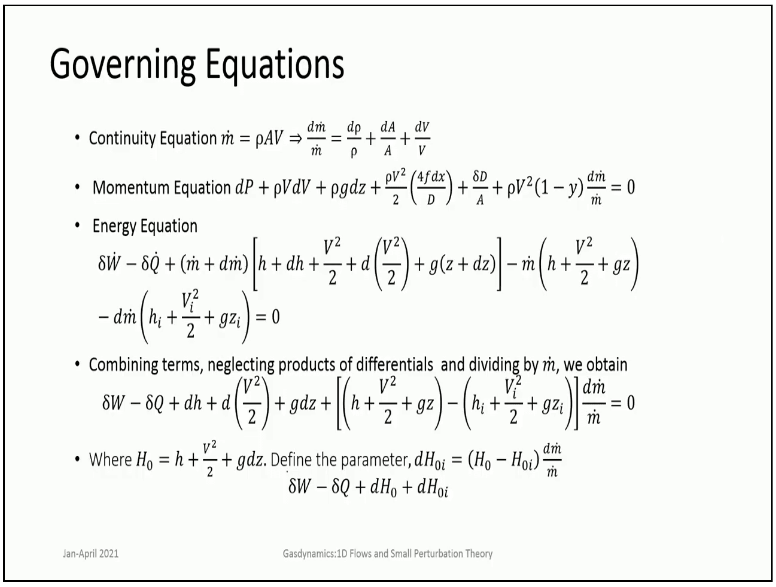

So, what do we do now, we write the conservation equations. Now considering all effects, so, now you have,

$\dot{m} = \rho AV$ and therefore area is also changing.

$$\frac{d\dot{m}}{\dot{m}} = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V}$$

Now momentum equation you can consider, you have all the various parameters,

$$dP + \rho VdV + \rho g\, dz + \frac{\rho V^2}{2}\frac{4fdx}{D} + \frac{\delta D}{A} + \rho V^2(1-y)\frac{d\dot{m}}{\dot{m}} = 0$$

How we reach that each individual part of these terms, we have already discussed in those corresponding sections how we got this was dealt with in Fanno flow. Similarly this is δD by a reference. So, force by area. So, that it is consistent in dimensions where D is a drag force. And this particular quantity $\rho V^2(1-y)$ where y is the same as $\theta = \frac{u}{V}$.

We have just now discussed in this particular section. So, this is due to mass addition. Similarly if you like to take a look at the energy equation,

$$\delta W - \delta Q + (\dot{m} + d\dot{m})\left[h + dh + \frac{V^2}{2} + d\left(\frac{V^2}{2}\right) + g(z+dz)\right] - \dot{m}\left[h + \frac{V^2}{2} + + gz\right] - d\dot{m}\left[h_i + \frac{V_i^2}{2} + + gz_i\right] = 0$$

Again you come to work done, heat added or removed and then here is the change in total enthalpy is a difference between what is outgoing and what is incoming. So, outgoing is given here $\left[h + dh + \frac{V^2}{2} + d\left(\frac{V^2}{2}\right) + g(z+dz)\right]$.

While incoming is one is due to the core mass flow ,the other one is due to an injection $d\dot{m}$ and here it can have different in a total enthalpy which is for the injectant. So, that is possible. Now in this energy equation you can combine different differentials together and here is the change in h_0. It is dh_0, this entire term dh_0 while this is due to heat , due to the mass addition term which is also a change in h_0, for the mass addition dh_{0i} .

So, finally this is the term here which comes out which is nothing but the statement that work done and heat added or removed it will be changing the total enthalpy of the system.

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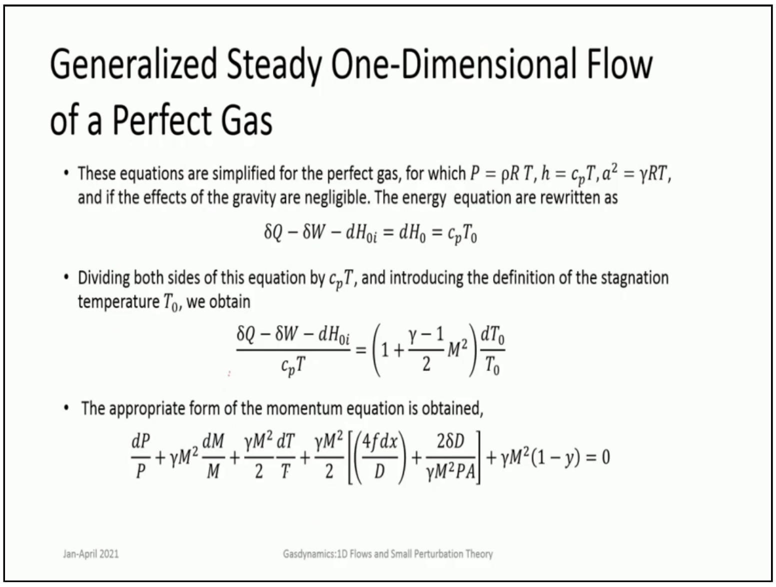

So, here the parameters are the driving parameter here is a combination in the energy equation is a combination of heat that is added ,work that is taken out shaft work and also the work the total difference between total enthalpies of the injectant fluid and that of the main fluid and if you divide that entire equation by CpT. So, this is a driving force or driver of the equations and that is related to change in stagnation temperature. Similarly we can write down for momentum equation.

You can write down in terms of $\frac{dP}{P}$, $\frac{d\dot{m}}{\dot{m}}$, $\frac{dT}{T}$ and so on.

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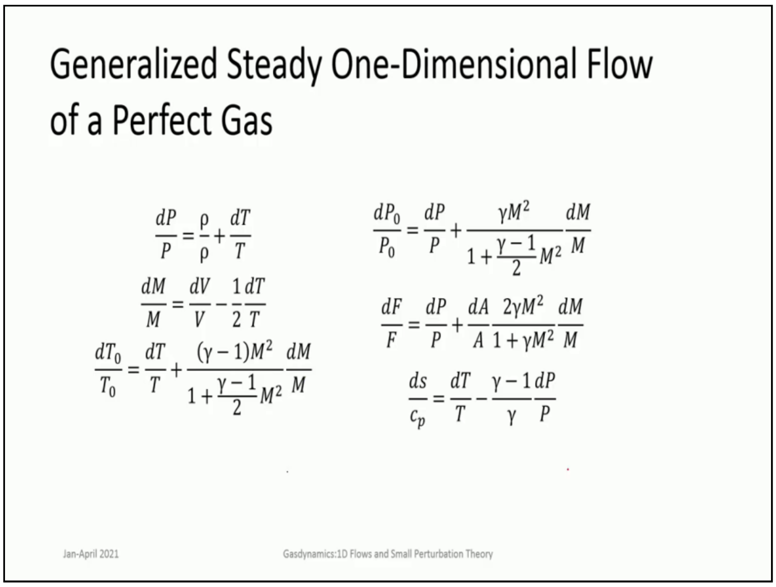

So, and the definitions of this comes from P = ρ RT. So,

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

And,

V =Ma, a = $\sqrt{\gamma RT}$ . . M = $\frac{V}{a}$ So, we get,

$$\frac{dM}{M} = \frac{dV}{V} - \frac{1}{2}\frac{dT}{T}$$

Then you can use ,T_0 = T(1+ $\frac{\gamma-1}{2} M^2$)

and from there you get this equation.

$$\frac{dT_0}{T_0} = \frac{dT}{T} + \frac{(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Similarly P_0,similarly the definition of impulse function F,

F = (1+ γM^2) PA

And the definition of entropy.

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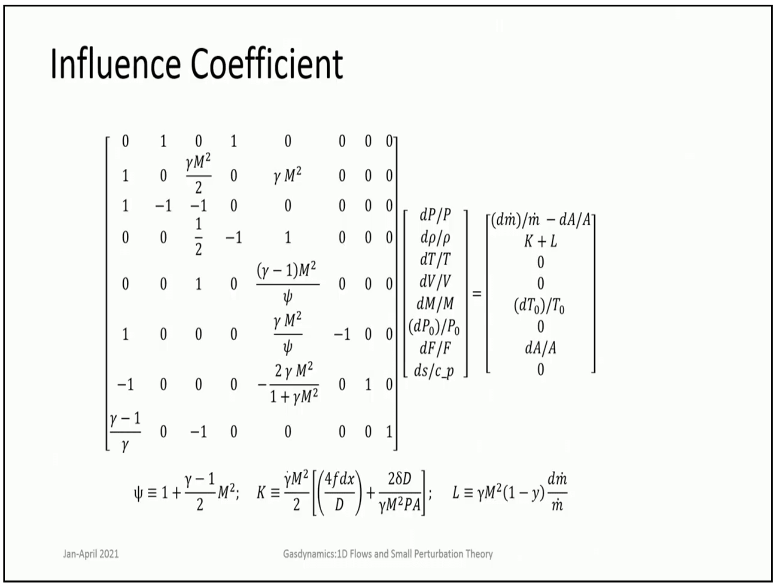

So, you get all these different equations. So, all of them can be taken together. So, what are the unknown pressures, density, temperature, velocity, Mach number, stagnation pressure, impulse function and entropy. And what are the various driving changes that drive these different parameters they are mass addition, change in area, this K + L is due to drag forces. So, that is momentum equation ,in the momentum equation you have friction drag and momentum deficit due to mass addition.

So, that is over here and $\frac{dT_0}{T_0}$ which is change in stagnation temperature and of course $\frac{dA}{A}$ is just the change in area. So, each of those equations can be written in a matrix form. For example $\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$

this can be expressed in completely matrix form over here. So, that can be done if you do the proper multiplications here then you will find that equation.

Similarly if you take the first equation which is the mass flow equation which was,

$$\frac{dV}{V} + \frac{d\rho}{\rho} = \frac{d\dot{m}}{\dot{m}} - \frac{dA}{A}$$

This is from the mass conservation equation that we have done over here from this equation. And that equation can be found by doing the multiplication of the first row with this column corresponds to the first value over here. So, this complete equation basic matrix equation basically represents the set of all equations that we have considered.

Now this can be inverted. So, the way to solve this is you have to express each parameter $\frac{dP}{P}$, $\frac{d\rho}{\rho}$, $\frac{dT}{T}$ in terms of these driving potentials and a influence coefficient associated with them.

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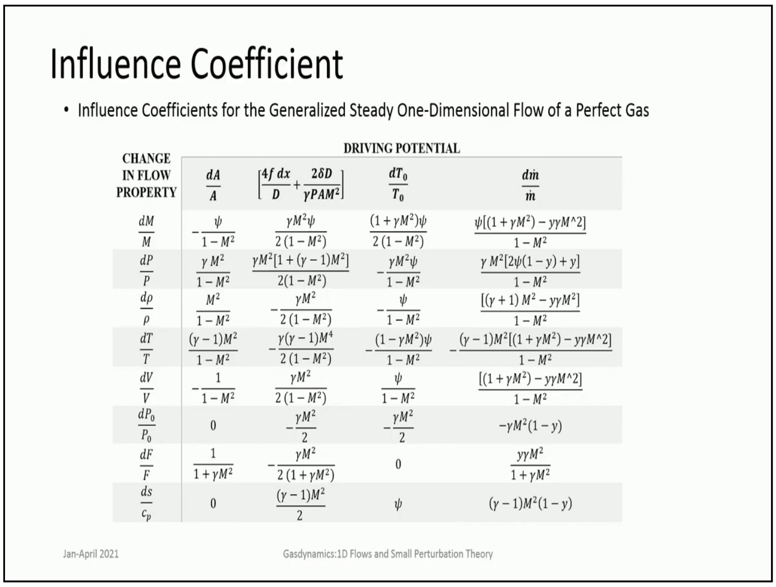

CHANGE IN FLOW PROPERTY	DRIVING POTENTIAL $\frac{dA}{A}$	$\left[\frac{4f\,dx}{D}+\frac{2\delta D}{\gamma PAM^2}\right]$	$\frac{dT_0}{T_0}$	$\frac{d\dot{m}}{\dot{m}}$
$\frac{dM}{M}$	$-\frac{\psi}{1-M^2}$	$\frac{\gamma M^2\psi}{2(1-M^2)}$	$\frac{(1+\gamma M^2)\psi}{2(1-M^2)}$	$\frac{\psi[(1+\gamma M^2)-y\gamma M\text{^}2]}{1-M^2}$
$\frac{dP}{P}$	$\frac{\gamma M^2}{1-M^2}$	$\frac{\gamma M^2[1+(\gamma-1)M^2]}{2(1-M^2)}$	$-\frac{\gamma M^2\psi}{1-M^2}$	$\frac{\gamma M^2[2\psi(1-y)+y]}{1-M^2}$
$\frac{d\rho}{\rho}$	$\frac{M^2}{1-M^2}$	$-\frac{\gamma M^2}{2(1-M^2)}$	$-\frac{\psi}{1-M^2}$	$\frac{[(\gamma+1)M^2-y\gamma M^2]}{1-M^2}$
$\frac{dT}{T}$	$\frac{(\gamma-1)M^2}{1-M^2}$	$-\frac{\gamma(\gamma-1)M^4}{2(1-M^2)}$	$-\frac{(1-\gamma M^2)\psi}{1-M^2}$	$-\frac{(\gamma-1)M^2[(1+\gamma M^2)-y\gamma M\text{^}2]}{1-M^2}$
$\frac{dV}{V}$	$-\frac{1}{1-M^2}$	$\frac{\gamma M^2}{2(1-M^2)}$	$\frac{\psi}{1-M^2}$	$\frac{[(1+\gamma M^2)-y\gamma M\text{^}2]}{1-M^2}$
$\frac{dP_0}{P_0}$	0	$-\frac{\gamma M^2}{2}$	$-\frac{\gamma M^2}{2}$	$-\gamma M^2(1-y)$
$\frac{dF}{F}$	$\frac{1}{1+\gamma M^2}$	$-\frac{\gamma M^2}{2(1+\gamma M^2)}$	0	$\frac{y\gamma M^2}{1+\gamma M^2}$
$\frac{ds}{c_p}$	0	$\frac{(\gamma-1)M^2}{2}$	ψ	$(\gamma-1)M^2(1-y)$

This can be solved analytically, you can use nowadays there are good analytical solvers like Maple, Mathematica even in Matlab it is possible. You can represent $\frac{d\dot{m}}{\dot{m}}$ using Cramer's rule you can represent all these different flow property to a driving potential which is changes in area due to friction, due to heat processes which is change in stagnation temperature and mass addition.

So, if you consider say $\frac{dM}{M}$ how does Mach number change? It has an influence coefficient associated with it for a driving potential $\frac{dA}{A}$ along with that an influence coefficient due to frictional forces , an influence coefficient due to $\frac{dT_0}{T_0}$. So, for every flow parameter we can flow property we can write this influence coefficient for that corresponding driving potential. So, it is possible to do it.

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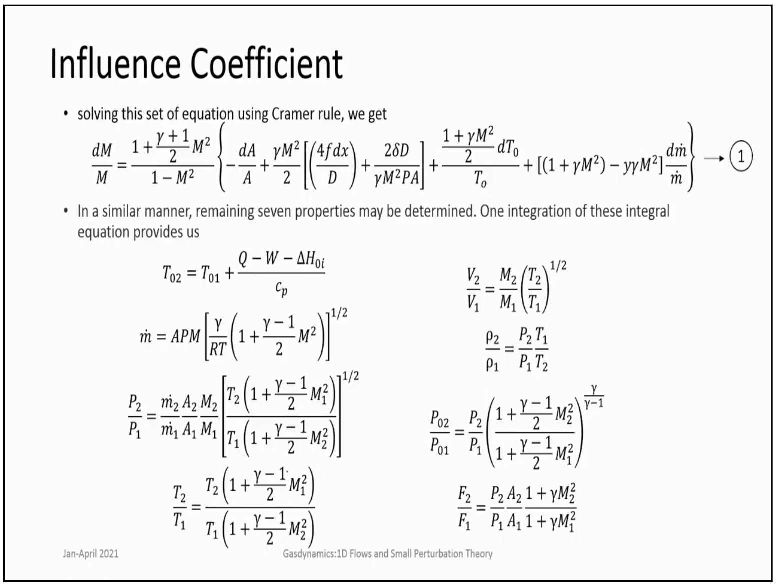

And therefore we can write for example if you consider $\frac{dM}{M}$. So, for this you can write down the entire solution,

$$\frac{dM}{M} = \frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\left\{-\frac{dA}{A}+\frac{\gamma M^2}{2}\left[\left(\frac{4fdx}{D}\right)+\frac{2\delta D}{\gamma M^2 PA}\right]+\frac{1+\gamma M^2}{2}\frac{dT_0}{T_0}+[(1+\gamma M^2)-\gamma y M^2]\frac{d\dot{m}}{\dot{m}}\right\}$$

So, this considers all the different properties and how is T_{02} related to T_{01} it is related by the energy equation where you can consider heat, work done ,heat added work done by the system and this is the difference between injectant mass, total enthalpy and the enthalpy of the main flow. So, if you consider mass flow rate mass flow rate is nothing but $\dot{m} = \rho AV$.

If you consider 2, particular points or at a particular point you can relate it to pressure, Mach number. This was done in previous classes from here we can get to $\frac{P_2}{P_1}$. And in since there is an injection of mass also available m_2 and m_1 need not be the same that should be understood over here. So, if you can solve for say Mach number and to do that you have to integrate this, this can be integrated using any numerical tool there is no analytical way this can be done.

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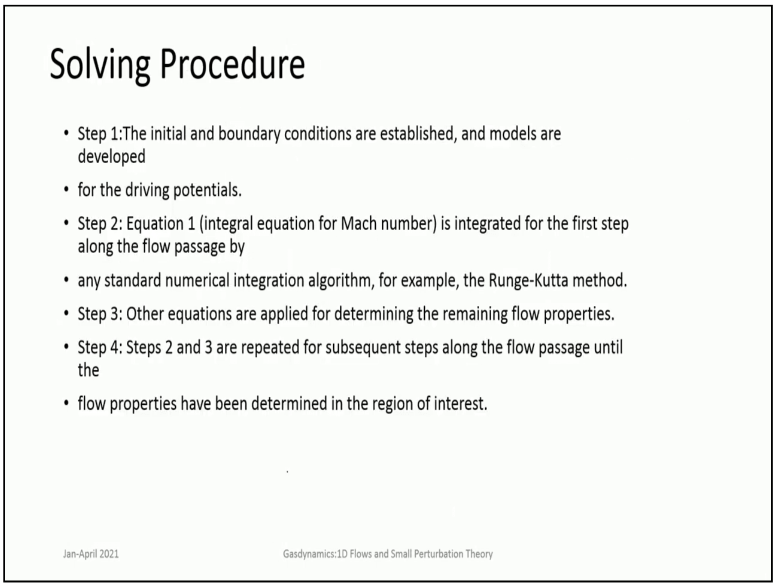

But some qualitative aspects of how we can know what is happening can be understood by just clubbing all of them together into a particular function λ. Before we go there the way to solve these problems is write down any parameter for example here we have written it in Mach number $\frac{dM}{M}$ can be written as a function of all the driving potentials and Mach number and we take points 1 and 2 we should know the initial condition at one and it can be integrated to 2.

It can be integrated to 2 you can use any numerical integration algorithm. Once you know properties at 1 and 2, M_1 and M_2 then all other parameters can be done.

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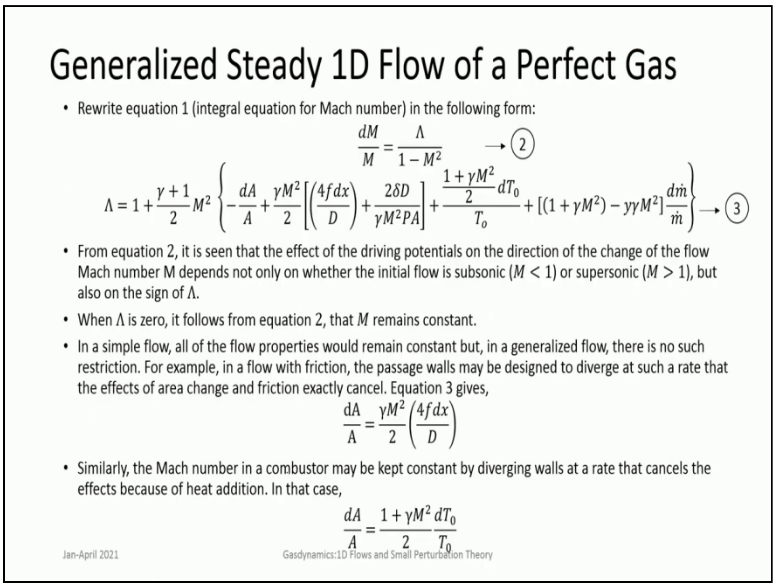

So we can club all these together, the entire thing can be clubbed together as λ. So, this complete expression is λ. λ now has influence of all various parameters. So, now if you see that your $\frac{dM}{M} = \frac{\lambda}{1-M^2}$ .So, we can look at what will happen if λ has different signs similar to what we were looking for say area change or friction or heat addition separately then what will happen to this particular value.

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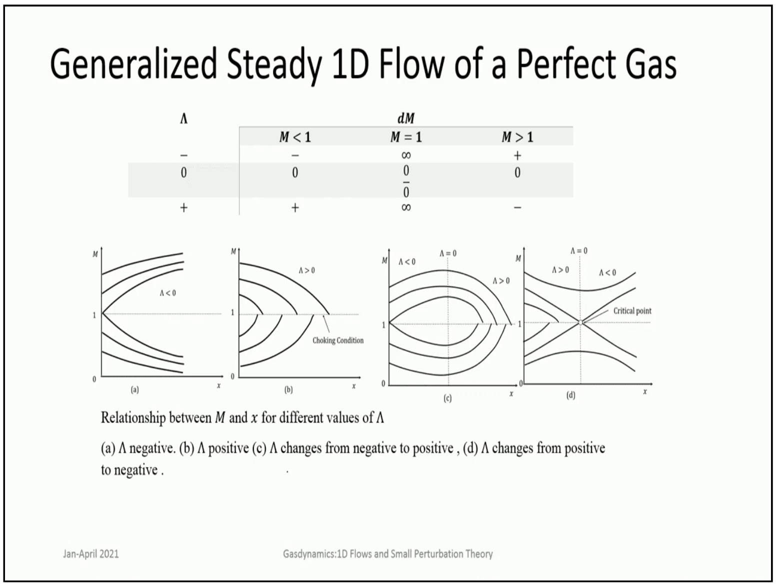

So, if you take λ, it can have initially it can have a negative sign or it can have positive sign or it can be zero. So, you have this equation $\frac{dM}{M} = \frac{\lambda}{1-M^2}$. So, depending on the sign of lambda you have various descriptions of the flow. So, initially suppose we take that λ is negative and you consider that λ is less than 0 and here you have in x direction. So, this is the flow that is varying along x quasi 1D flow and on y axis it is Mach number. So, if you begin with say a subsonic Mach number that is Mach numbers less than 1 and that λ is negative.

So, λ is negative if λ is negative. So, $\frac{\lambda}{1-M^2} < 1$. So, this quantity is going to be positive. So, $\frac{dM}{M}$ will be negative that means Mach number will continue to reduce. So, this is the direction of how the flow would vary with λ being 0 there is no change in Mach number that is a critical point over here. But if λ is initially negative, a subsonic flow continues to reduce, supersonic flow will continue to increase.

Because this becomes negative and the λ is negative, negative by negative is positive. Initially if λ is positive then the directions reverse and initially subsonic flow will accelerate, you will have increase in Mach number 2.1. Similarly you have a supersonic flow its Mach number will reduce but what if λ switches sign in between. So, that can happen λ can be initially negative it can switch sign at some point become equal to 0 and then the sign can change and become it can become positive.

So, if you consider such an effect then initially if λ is negative a subsonic flow will decelerate then λ will become 0 and after that it will accelerate again. Similarly a supersonic flow first will accelerate it will increase Mach number λ will become zero and then it will decelerate. So, if it was if it had started from one it accelerates decelerates and comes back to a 1.

But if you consider a case where you have initially λ is positive then a subsonic flow will accelerate then it will reach λ equal to 0 which is a critical point when Mach number is one and then further it can increase when λ switches sign and λ becomes less than 0. All the cases of variable area ducts must remind you of this kind of an approach. From a initial subsonic case it goes to supersonic flow or from an initial supersonic flow it goes to a subsonic flow.

If you are not achieving the critical point which is λ equal to 0 at Mach number equal to 1 then you will just increase in a subsonic flow, first it will increase then achieve the maximum point then again it will decrease. Similarly for supersonic flow first it will decrease achieve a minimum point and then increase the Mach number. So, you can see that these different curves for different cases of lambda have the elements of various aspects that we have already discussed separately in different cases whether it be variable area ducts or it be Fanno flow or Rayleigh flow.

When you have combinations of them you have to look at λ where you have a complete combination.

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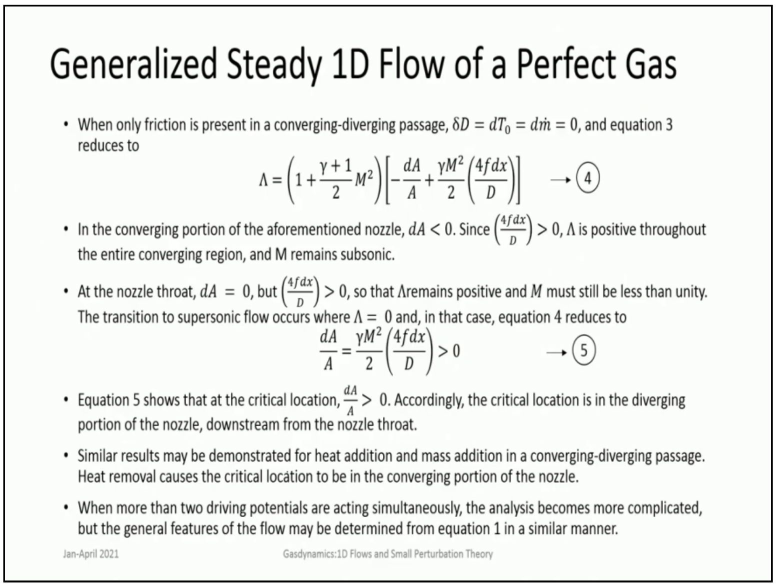

For example if we consider a case consisting only of friction and area change which can be a flow through a nozzle converging divergent passage with friction. Then what you will see all of the drag and change there is no heat addition or mass addition. Then you have a term here

$$\lambda = \left(1 + \frac{\gamma-1}{2} M^2\right)\left(- \frac{dA}{A} + \frac{\gamma M^2}{2}\left[\left(\frac{4fdx}{D}\right) + \frac{2\delta D}{\gamma M^2 P A}\right]\right)$$

If there was no friction at all then its only due to changes in area and we know that the critical point occurs at the minimum area which is at the throat.

But if there is friction also added along with that then critical point should occur when $\lambda = 0$ So, if you get the minimum point minimum point is dA = 0 that is at the throat but still if you consider friction $\left(\frac{4fdx}{D}\right)$ is greater than 0. So, λ remains greater than 0. So, λ remains positive. So, Mach number should be still it should be subsonic.

So, $\lambda = 0$ achieved at M = 1 That is the critical point and this you can you will be able to achieve when $\frac{dA}{A} = \frac{\gamma M^2}{2}\left[\left(\frac{4fdx}{D}\right)\right]$. Now this term is greater than 0 that means $\frac{dA}{A} > 0$ that means the choking ,when you consider a both area and friction choking will happen in the divergent passage, slightly downstream of the throat.

So, this can be this is a result that comes out of considering multiple parameters. Similarly we can consider heat addition or mass addition. So, the idea here was to introduce the topic of generalized quasi 1D and show that you can actually do the generalized solution where different driving potentials can be considered together and we can look at solutions of them.

These are useful for doing initial engineering calculations before going on to more say CFD kind of approach or an experimental kind of approach.

You want to quickly know what will happen when you have both friction, heat addition and area change happening which is typical to say propulsion devices or nozzles or diffusers in real flows. An idea of how they would behave can be understood with the help of this generalized 1D flow and further studies can be carried out later by using CFD approaches or experiments. So, with this we come to the close of discussion of quasi 1D kind of approach.

We have had very elaborate discussions on this approach and they provide you the basic understanding of gasdynamic flows which is important. But when we are considering more practical problems, we want to know what is happening to the flow field what are the details of flow structures and so on. For that we have to use the differential equations and one particular way is looking at the first we always look an approach towards the inviscid flow.

So, that is what we will be dealing from the next class onwards. In this particular course we will not go into high fidelity models like CFD or and so on. But people have been looking at approximate methods, potential flows and such approaches. We will look at them and see; what are the important results that come from such an approach in the coming classes. so, there will be looking at complete flow field instead of such approximation that flow is uniform across a particular cross section. So, with that we end the quasi 1D discussion. Thank you.