**Gasdynamics: Fundamentals.. And Applications**
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**Lecture 46**
**1D Flows with Heat Addition Rayleigh Flows - II**

So we are looking at 1D flow with heat addition which is Rayleigh flow. The previous class we looked at the Rayleigh line or Rayleigh curves. So, in a P $v$ diagram it is a straight line while in a Ts diagram it has a curve.

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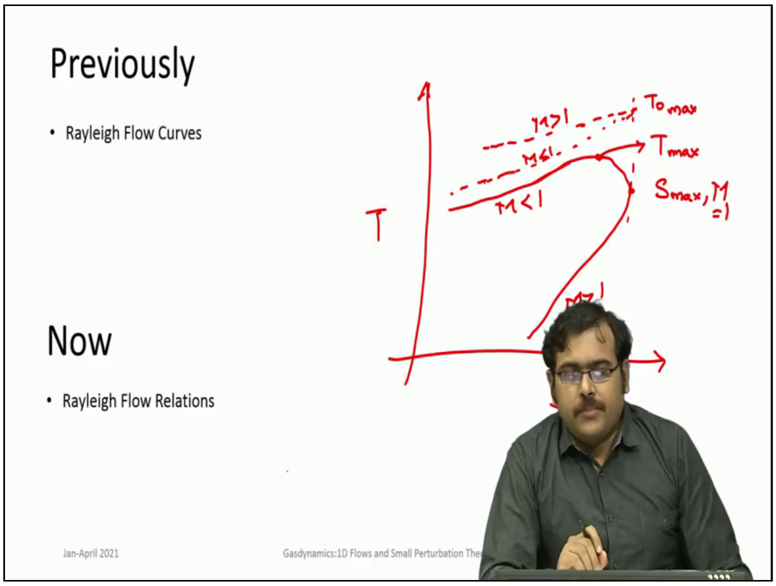


And it is very good to know the nature of this curve, it has 2 branches and 2 critical points. So, this point is where temperature is maximum and this is a point where entropy is maximum. So, this corresponds to a Rayleigh curve and this is for static conditions you can also draw the corresponding stagnation conditions and at the point of maximum entropy you have maximum heat addition also.

These are stagnation lines $T_0$ lines, $T_0$ each is a maximum. So, the maximum amount of heat that can be added corresponds to the location of s maximum that is the entropy maximum and this corresponds to Mach number equal to 1. So, all these upper branch cases are of subsonic conditions while lower branch corresponds to supersonic conditions. So, when heat is getting added in a subsonic flow the velocity increases, Mach number increases it drives it towards sonic conditions.

This is the subsonic branch for the stagnation conditions and supersonic branch for the stagnation conditions. Now in the subsonic flow you also have the maximum enthalpy point or maximum temperature point. So, the temperature initially increases until $T_{max}$ and thereafter it decreases to $s_{max}$ that is to the max critical point. While in supersonic flow the temperature keeps increasing, pressure keeps increasing, velocity and Mach number decrease in the supersonic flow.

So, being able to draw these curves effectively will be a nice way to look at problems in Rayleigh flow and then also understand them properly.

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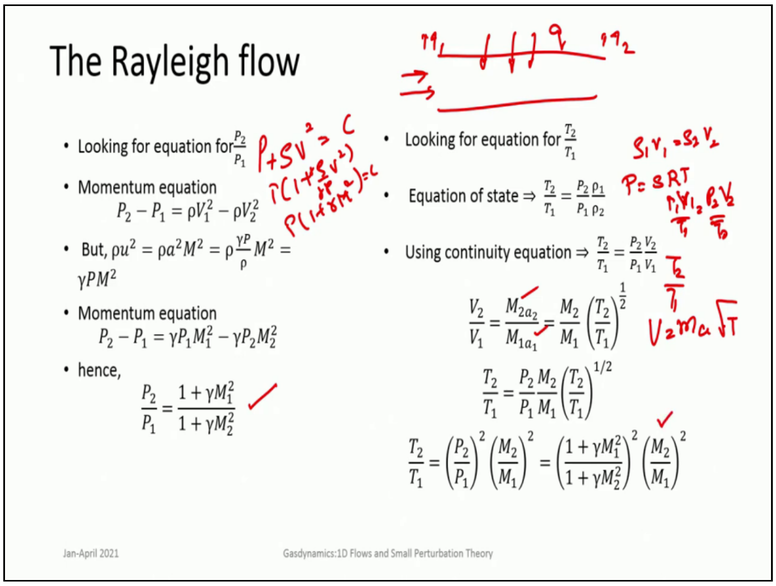


So, now we go to the equations and see how we can get various quantities for a flow. So, if there is a 1 Dimensional flow and the heat is getting added to this at a certain rate and flow is occurring. So, it is having $M_1$ at this point, $M_2$ after a certain point. So, pressure temperature similar. So, just as we did for other flows for Fanno flow we are looking at ratios, $\frac{P_2}{P_1}$ , $\frac{T_2}{T_1}$ , and so on.

In this case we directly go and look at the momentum equation it is an inviscid flow.

So, $P + \rho u^2 = \text{constant}$,

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$\rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma P}{\rho} M^2 = \gamma P M^2$$

Now if you take out P as common. this will be,

$P(1+\gamma M^2) = \text{constant}$

$P_2 - P_1 = \gamma P_1 M_1^2 - \gamma P_2 M_2^2$

From this you can easily get what is $\frac{P_2}{P_1}$

$$\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

So, we get $\frac{P_2}{P_1}$ .Now we are looking once $\frac{P_2}{P_1}$ is got, we can look at $\frac{T_2}{T_1}$ and for $\frac{T_2}{T_1}$ we use the continuity equation and equation of state together. So, because $\rho_1 u_1 = \rho_2 u_2$ and we also use P = ρ RT combine them together. So, that the continuity equation can be expressed in terms of pressure and temperature rho is $\rho = \frac{P}{RT}$ .

$\frac{P_1 v_1}{RT_1} = \frac{P_2 v_2}{RT_2}$

So, from here we can get $\frac{T_2}{T_1}$ as

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{v_2}{v_1} = \frac{u_2}{u_1}$$

$\frac{u_2}{u_1} = \frac{M_2}{M_1}\frac{a_2}{a_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}$

So, we use all these here.

So, finally $\frac{T_2}{T_1}$ can be expressed ,

$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \rightarrow \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^2 \left(\frac{M_2}{M_1}\right)^2 = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2$

So, once we know both $\frac{P_2}{P_1}$ and $\frac{T_2}{T_1}$ , $\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1}\frac{T_1}{T_2}$ can be easily got by equation of state.

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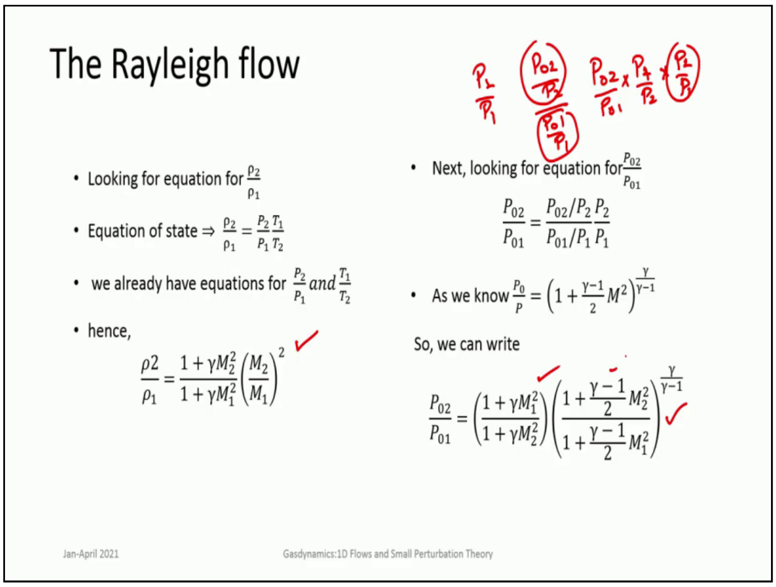


So, you can get $\frac{\rho_2}{\rho_1}$,

$$\frac{\rho_2}{\rho_1} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2}\left(\frac{M_1}{M_2}\right)^2$$

Now what about stagnation quantities ?

$P_{02}$ and $\frac{P_{02}}{P_{01}}$ and $\frac{T_{02}}{T_{01}}$ that you can couple it along with the $\frac{P_2}{P_1}$ .

$$\frac{P_{02}}{P_{01}} = \frac{\frac{P_{02}}{P_2}}{\frac{P_{01}}{P_1}}\frac{P_2}{P_1}$$

So this is what we will we need to get. So, this you will get $\frac{P_{02}}{P_{01}}$ .

$$\frac{P_{02}}{P_{01}} = \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

So that is how we get.

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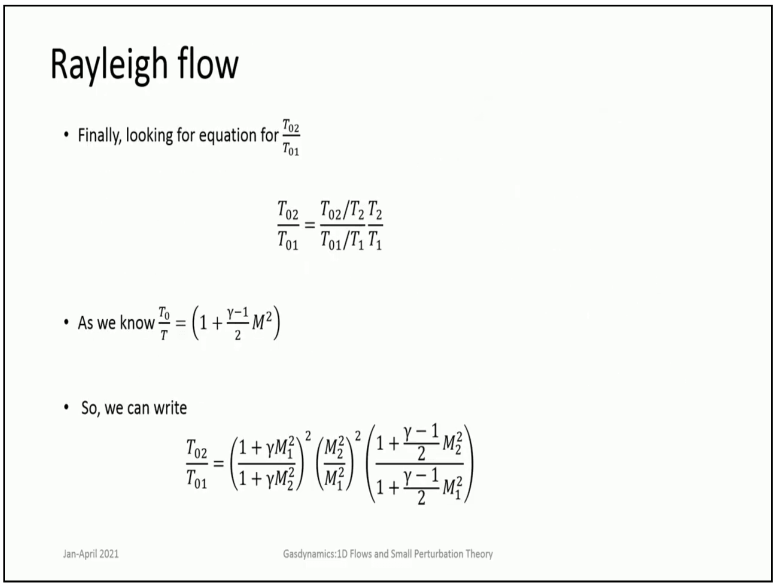


Here, next we look at $\frac{T_{02}}{T_{01}}$ it is a same principle.

$$\frac{T_{02}}{T_{01}} = \frac{\frac{T_{02}}{T_2}}{\frac{T_{01}}{T_1}} \frac{T_2}{T_1} = \frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2} \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2$$

So, we can now write the expression for $\frac{T_{02}}{T_{01}}$. So, using these set of equations we get $\frac{P_2}{P_1}$, $\frac{T_2}{T_1}$, $\frac{\rho_2}{\rho_1}$ and $\frac{P_{02}}{P_{01}}$, $\frac{T_{02}}{T_{01}}$. So, but the main principle here when we look at Rayleigh flows is that the heat added q = Cp($T_{02}$ - $T_{01}$). So, when we solve some problems this connection will become clear.

But now if you look at these equations it must be clear the suppose all the initial values are given you know $M_1$, $T_1$ and you know the amount of heat added q then $T_{02}$ can be found out. Then $\frac{T_{02}}{T_{01}}$ = can be evaluated. $M_1$ is known. So, we need to find $M_2$ but we get an equation which is not easily solvable, is analytically difficult to solve this in directly. So, the way ahead is similar to what was done in Fanno flows.

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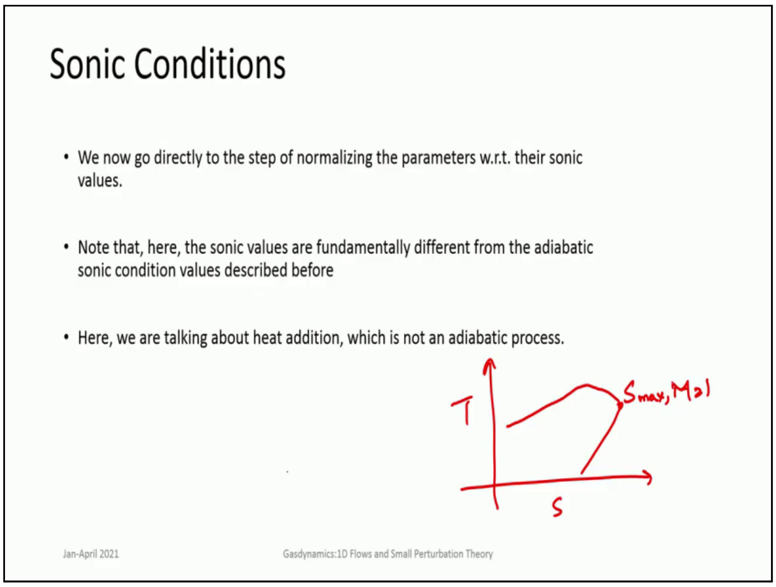


You look for a reference condition and for a Rayleigh flow there exists such a reference condition in the maximum entropy point s max which corresponds to Mach number equal to 1. So, for Rayleigh flow it is possible to take the reference as the condition where the flow goes to supersonic values but what we should really underline very much underline is that the sonic values that is referred to in Rayleigh flow is fundamentally different from adiabatic sonic conditions.

So, this what was earlier referred to if you had taken $\frac{T_0}{T^*}$ for an isentropic flow or an adiabatic flow they would remain the constant because flow is adiabatic there is no heat being added. But if you look at the Rayleigh flow this $T^*$ is not a constant it keeps varying it is different at different conditions. So, it is different from the adiabatic case but what you should understand here is that for a given Rayleigh curve or for a given Rayleigh flow which implies that G is a constant which is true with a 1D flow, steady 1D flow the mass flux will be constant.

And $P + \rho u^2 = \text{constant}$. So, these are the 2 conditions which goes in evaluating this Rayleigh curve. So, for a given Rayleigh curve, this point is a unique point. So, that should be understood. So, if a flow process in a duct falls in a Rayleigh curve then it will have a unique start point or a sonic point.

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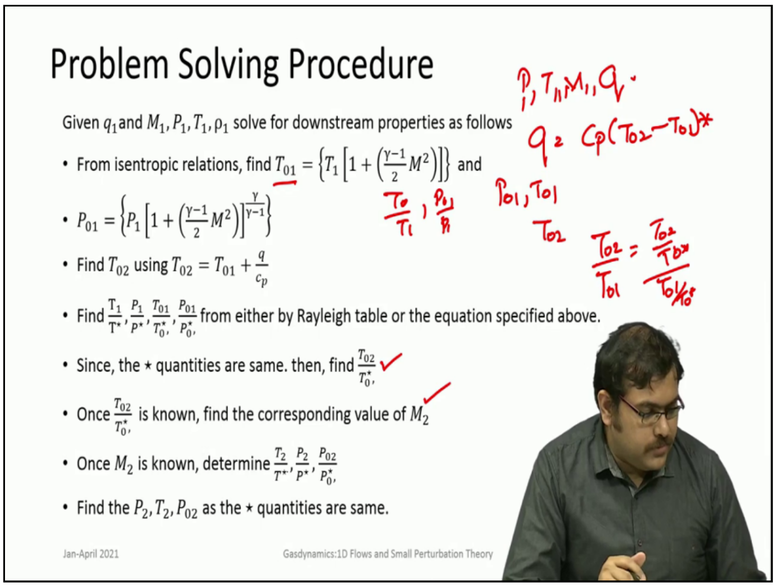


So, how do we evaluate it take the Rayleigh flow equations that you already have found out which is $\frac{P_2}{P_1}$ , $\frac{T_2}{T_1}$ , $\frac{\rho_2}{\rho_1}$ and in that you substitute P2 = P, T2 = T and P1 = P* and T1 = T* then we get

$$\frac{P}{P^*} = \frac{1+\gamma}{1+\gamma M^2} \quad ; \quad \frac{T}{T^*} = \frac{1+\gamma}{1+\gamma M^2} \quad \left(\frac{1+\gamma}{1+\gamma M^2}\right)^2 M^2 \; ; \quad \frac{\rho_2}{\rho_1} = \frac{1+\gamma M^2}{1+\gamma}\,\frac{1}{M^2}$$

So, if you want to express $\frac{P_2}{P_1}$ , this can be represented as ,

$$\frac{P_2}{P_1} = \frac{\frac{P_2}{P^*}}{\frac{P_1}{P^*}}$$

where the P* is the same for a Rayleigh curve.

Similarly $\frac{T_2}{T_1}$

$$\frac{T_2}{T_1} = \frac{\frac{T_2}{T^*}}{\frac{T_1}{T^*}}$$

So, once this T* this evaluation is done then we can represent any ratios in terms of the sonic conditions. And the advantage here again we are able to, we plot all these values in terms of tables or they are available through calculators.

So, that is the ease of using these reference conditions. So, always the flow drives the thing towards in Rayleigh flow it flies drives it towards a sonic condition. So, always heat addition achieves sonic conditions. So, how do we go about solving this problem suppose we know the conditions initial conditions pressure $P_1$ , $T_1$ and $M_1$ are known then and heat added is known q is known.

So, the basic equation is q = Cp($T_{02}$ - $T_{01}$).. Then if we know the pressure, temperature and Mach number we can find the stagnation properties $T_0$ using the expression $\frac{T_0}{T_1}$ or $\frac{P_0}{P_1}$ Similarly you can find stagnation conditions at initial. So, $P_{01}$ and $T_{01}$ can be found. Then $T_{02}$ can be found by using this particular equation relating heat added to the change in total enthalpy.

Then $\frac{T_{02}}{T_{01}}$ can be expressed. And this can be,

$$\frac{T_{02}}{T_{01}} = \frac{\frac{T0_2}{T^*}}{\frac{T_{01}}{T^*}}$$

and once we are able to find $\frac{T0_2}{T^*}$ ,we can use tables or calculators to find value of $M_2$. So, once $M_2$ is known all other quantities can be determined. So, that is how the algorithm or the recipe to solve these problems.

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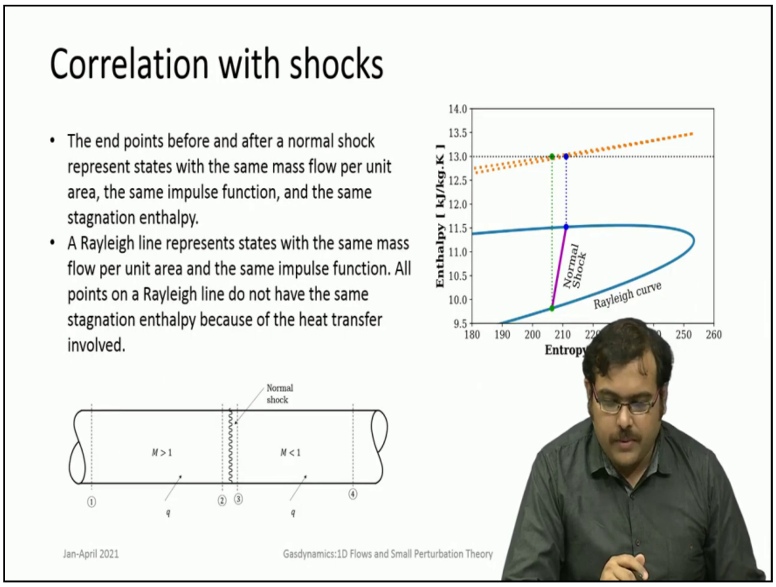


Now we go to the discussions where we look at it is a 1D flow similar to Fanno flow in Fanno flow we looked at if the inlet flow is supersonic. Then is there a possibility that shocks can occur in the context of Fanno flow similarly we look at the point whether shocks can occur in a Rayleigh flow and the analysis is also very similar .

Shock equations are,

$$\rho_1 u_1 = \rho_2 u_2 = G$$

And

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

So, this is G and this is a constant this is true for Rayleigh flow also and of course $h_0$ is constant for a shock.

So,

$$h + \frac{u^2}{2} = constant$$

but can this be located on a Rayleigh curve ? Yes its all the first 2 it satisfies the first 2 equations which are the ones with which actually plot the Rayleigh curve. So, it will satisfy the Rayleigh curve. Now what one has to do if we have look at locating a shock on a Rayleigh curve is to look at not just the static conditions but also the stagnation curves.

Here you can see the stagnation curves there is a lower branch and an upper branch. The scale makes it look almost similar but they are slightly apart and what we look at is draw a constant h curve because that is what satisfies a shock ,constant h. So, one constant h curve is drawn it cuts the lower and upper branches at particular points on the stagnation lines. So, they have the same stagnation temperatures, stagnation enthalpy and look at the corresponding points on the Rayleigh curve.

So, they will correspond to the shock points on a Rayleigh curve. So, this is point before the shock that is one and this is point after the shock. So, if you correspond to this one this is point 2 and this is point 3 . So, that is the point. So, you can locate a shock in a Rayleigh curve. This also brings about an interesting point in Fanno curve, we have seen that in case a shock appeared in a Fanno curve then it had an effect on the maximum length of the duct that can we had for achieving sonic conditions and that was achieved.

So, there in the Rayleigh curve, what you were in the Fanno curve the appearance of a shock actually increased the maximum length . So, if you sort of look at that and remember it. So, it will be that is wrong. So, T s for a Fanno is like this and if a shock appears then the maximum length it included a supersonic length and a subsonic length and that was total length could be higher.

But what about in the case of a Rayleigh curve will it change the maximum heat that can be added which corresponds to this particular point. So, if you take any point from a supersonic solution say point 1 and this corresponds to point 4 and this is the maximum heat that can be added. If a shock appears anywhere in between, will it change the maximum heat that can be added and that undergoes no change because a shock does not affect the enthalpy, the enthalpy is constant across the shock.

So, you take at any location it just changes the flow from supersonic condition to subsonic. So, it moves like this. So, it goes up and goes over like that but it does not change the total heat that is getting added.

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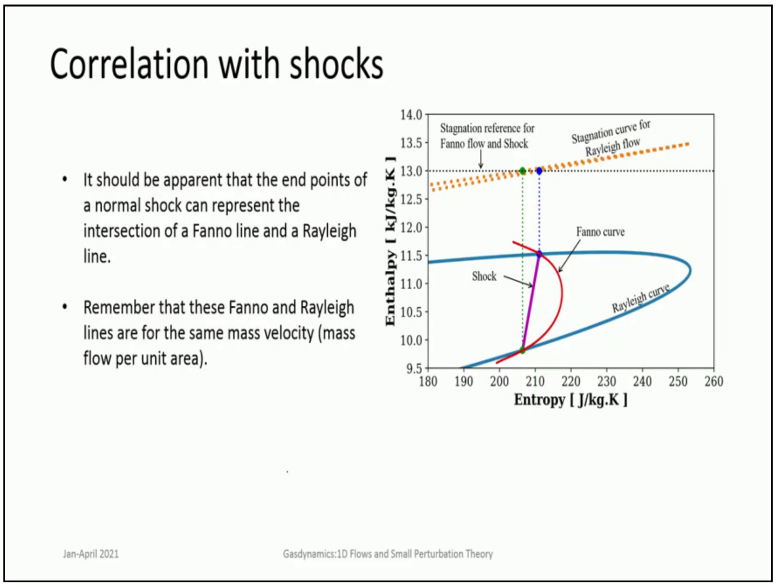


So, now if you look at the point that if you consider both a Fanno occur and a Rayleigh curve Fanno curve also has shocks associated with it and Rayleigh curve also has a shock associated with it, is there any correlation between these 2 is there any connection between 2? If you look at Fanno curve what it considers is $\rho_1 u_1 = \rho_2 u_2$ and adiabatic flow

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

This is Fanno flow.

While Rayleigh considers $\rho_1 u_1 = \rho_2 u_2$ and $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$

Now if you take them together both satisfy $\rho_1 u_1 = \rho_2 u_2$ and also it satisfies total enthalpy remaining constant and momentum also remaining constant, $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$ So, if you take all of them together it satisfies all the conditions of the shock.

So, an intersection of a Fanno curve with a Rayleigh curve is nothing but the shock. So, you can locate a shock at intersections of Fanno curve and the Rayleigh curve. And for a Fanno curve the stagnation temperature remains a constant. So, the stagnation temperature reference for both the Fanno and the shock are the same. While in the Rayleigh it actually cuts across the 2 branches. So, if you consider Fanno curve, Rayleigh curve and the shock all of them can appear as intersections of each other.

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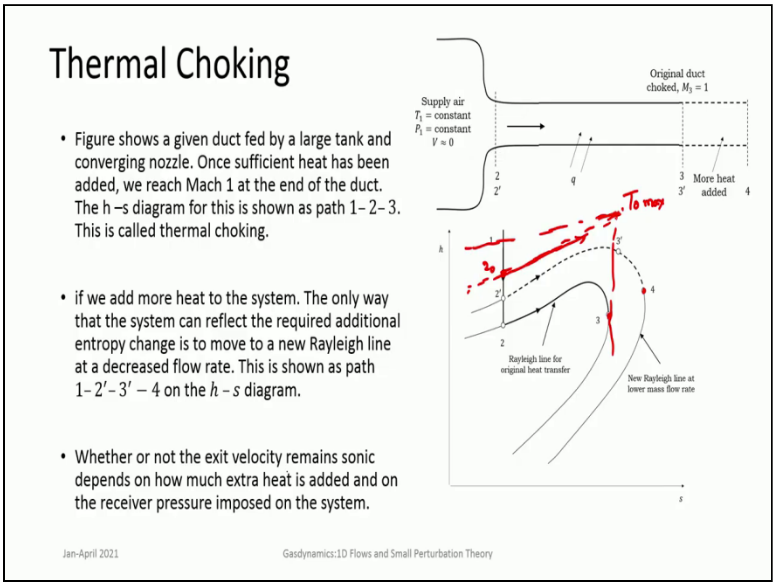


So, now we come to another important point which is that there is there . So, there is a location of maximum temperature or maximum heat added which is maximum heat added can be located along the stagnation curves. So, it will correspond to this particular point of maximum entropy , corresponds to that. So, we are considering. So, let us take a subsonic flow in the subsonic flow that is the point here 2,0 that is the stagnation point and heat is being added.

So, as heat gets added it is driven towards sonic conditions and if it gets it continues to be added and reaches the maximum heat addition point which is $T_0$ maximum then you cannot add any more heat beyond this point. So, that is known as thermal choking. So, we have discussed mass flow rate choking in the context of variable area ducts and friction choking which relates to the length of the pipe that in frictional flows or Fanno flows.

Now we come to in the case of thermal Rayleigh flows you have thermal choking where flow achieves Mach number 1 due to heat being added and that is the maximum amount of heat that can be added. Suppose you want to add more in the case of a subsonic flow. Then exactly similar to what happens in a Fanno flow condition you can have 2 cases one is that if you want the flow to remain on the same Rayleigh curve.

Then the point 2 should be shifted to some other location $2''$ where now there is scope available for adding more heat but this implies greater pressures, so, more pumping and so on. The other solution that can occur is simply the flow just shifts to another Rayleigh line where G is smaller, G decreases that means mass flow decreases. So, this is a $2'$ here where mass flow rate has decreased but now it can allow larger heat to be added.

So, basically when choking happens if you try to change anything at a choking point it always affects the upstream.

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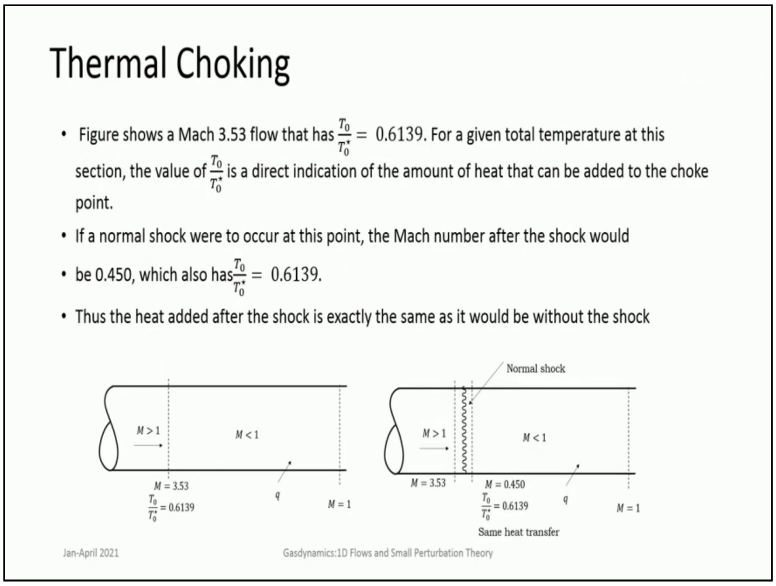


Now what happens if it happens in a supersonic case, then in the supersonic case can a shock change the location of the maximum heat transfer, it cannot change because shock is a constant total enthalpy process or constant stagnation enthalpy process. So, that would not change, location of a shock would not change the maximum heat being added it still has to change upstream conditions in order to change the total amount of heat that is being added.

But the appearance of a shock allows upstream propagation of information. So, that can change things. So, once the subsonic flow appears then it can change things regarding pressures and how they are felt upstream. So, that can happen .

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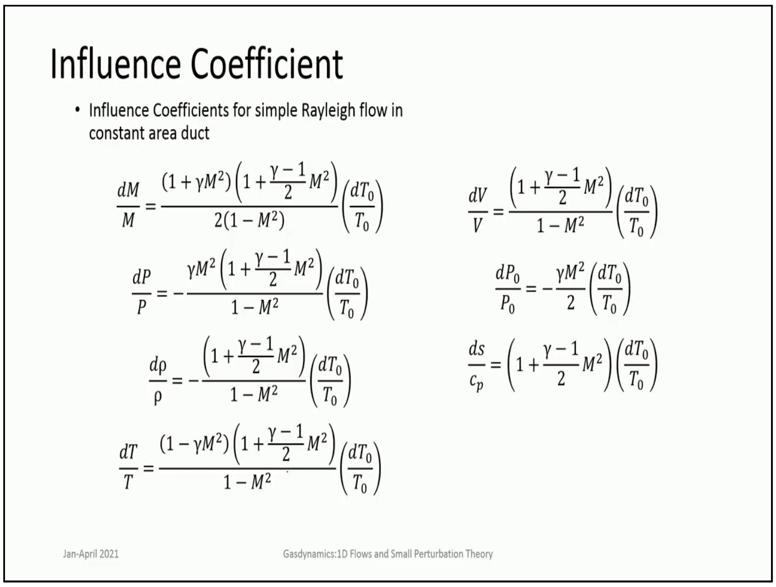


So, now again we can look at influence coefficients here the driver is change in stagnation temperature $dT_0$ which is related to function of the amount of heat added. So, $\frac{dT_0}{T_0}$ is the driver here based on this we can look at what happens to $\frac{dM}{M}$, $\frac{dP}{P}$, $\frac{d\rho}{\rho}$ and the procedure for getting these influence coefficients is the same you write down all the equations corresponding to the conservation of equations as well as the ideal gas law $P = \rho RT$.

What is Mach number? $M = \frac{V}{a}$ and so on. And then club all of them together take them simultaneously in a matrix form and from there you can express every other function in the with respect to only $\frac{dT_0}{T_0}$ as the driving potential or driver.

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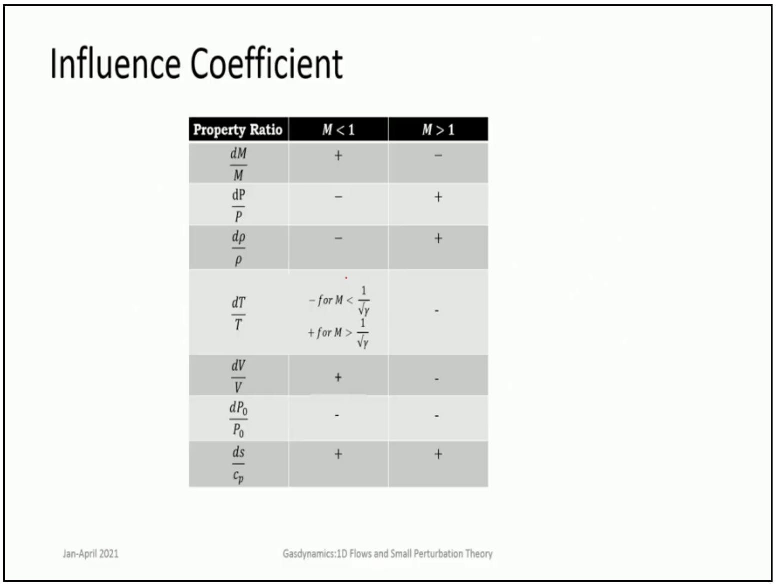

So, if you look at that then how do, these properties change? We have already gone through them. If heat is getting added in subsonic, flow Mach number increases in a supersonic flow Mach number decreases ,pressure decreases in a subsonic flow temperature will increase as long as Mach number is less than $\frac{1}{\sqrt{\gamma}}$ .

But then after $\frac{1}{\sqrt{\gamma}}$ , for Mach numbers greater than $\frac{1}{\sqrt{\gamma}}$ temperature will decrease but in the case of a supersonic flow the temperature will always increase. So, it will continuously increase, pressures will increase and velocity decreases in a supersonic flow but increases in a subsonic flow. So, and always entropy increases in this when we consider heat addition.

And nothing stops us from removing heat if you remove heat then all the processes get reversed. So, with that we come to an end of the discussions on Rayleigh flow. So, we will do a couple of numericals in order to get a really clear understanding of them and apply these principles. And with that we come towards the end of discussions of 1Dimensional flows. So, next class we will look at a couple of numericals.