Gasdynamics: Fundamentals. And Applications
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Lecture 42
1D Flow with Friction - Fanno Flow- II

So, we are looking at friction flow compressible flow with friction it is also known as the Fanno flow. And in the previous class we looked at the thermodynamics of this flow with the help of Tminuss diagrams. And it is very useful to look at it because it tells you the entire story. In this flow, it is an adiabatic flow therefore $T_0 =$ constant. And you have a curve which is of this particular shape qualitatively where you have 2 branches.

And the upper branch is the flow where Mach number is less than 1 it is a subsonic flow. While the lower branch the Mach number is greater than 1 or is a supersonic flow. And the critical point at which entropy is maximum, s is maximum is the point where Mach number equal to 1. So, the flow proceeds in the duct if you start from a particular point here one then the flow proceeds along the Fanno curve.

And the flow can happen. So, essentially you are varying the length of the duct. So, you can have different points on this flow at the exit of the duct. So, this is say point 2. So, in a subsonic flow velocity increases, Mach number increases, pressure, temperature decrease, entropy increases correspondingly stagnation pressure decreases. But in a supersonic flow the Mach number decrease, velocity decreases, pressure, temperature will increase. And stagnation pressure of course will decrease because even here the entropy increases.

And since it is an adiabatic flow and entropy cannot decrease there is only one direction in which this flow can occur you can only start from an initial subsonic flow continues to be subsonic at the maximum it can reach Mach number equal to 1. Similarly if you start with the supersonic flow then Mach number decreases, the Fanno curve or Fanno flow drives it towards Mach number 1 and maximum it can achieve is Mach number = 1. So, we have understood this qualitative picture and how variables are interrelated.

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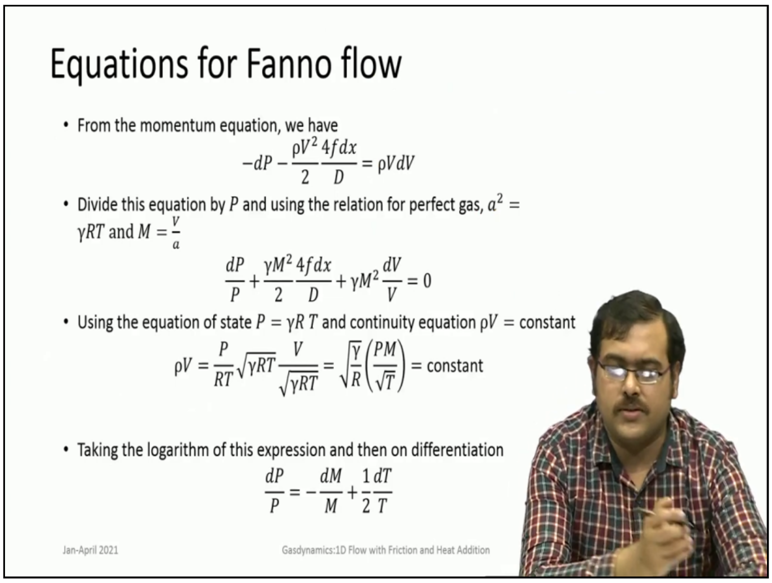

Now let us get a little bit more quantitative and get to the flow equations. Towards the end of the lecture we had come to this particular expression,

$$\mathrm{dp} + \rho V\ dV\ + \frac{\rho V^2}{2}\ \frac{4f'}{D} dx\ = 0$$

This is the differential equation for a small volume dx in the long pipe, for this small volume. Now what we can do over here is now we want to express everything in terms of Mach number.

So, we will try to do that. And for this what we do is divide the entire equation by P. So, it is divided by P. And we can use the fact that $a^2 = \frac{\gamma P}{\rho}$. So, we can use that fact. So, here if at this point you multiply and divide by V, $\frac{\gamma \rho V^2}{\gamma P V} dV = \gamma M^2 \frac{dV}{V}$.

So $\gamma M^2 \frac{dV}{V}$.

$$\frac{dP}{P}. + \frac{\gamma M^2}{2}\ \frac{4f'}{D} dx\ +\ \gamma M^2\ \frac{dV}{V} = 0$$

Now we use the fact that ρV = constant . And V can be expressed in terms of Mach number. So, ρ M a = constant. While $\rho = \frac{P}{RT}$, a = $\sqrt{\gamma RT}$, $\frac{P}{RT}$ M $\sqrt{\gamma RT}$ = constant

So, $\sqrt{\frac{\gamma}{R}}\ \left(\frac{PM}{\sqrt{T}}\right)$ = constant, this constant from here using a logarithmic differentiation that is you take a logarithm.

And differentiate you can get,

$$\frac{dP}{P} = \frac{1}{2}\frac{dT}{T} - \frac{dM}{M}$$

So, which is the expression given here and now there is an expression for $\frac{dP}{P}$ in terms of $\frac{dM}{M}$. And also $\frac{dT}{T}$ occurs here.

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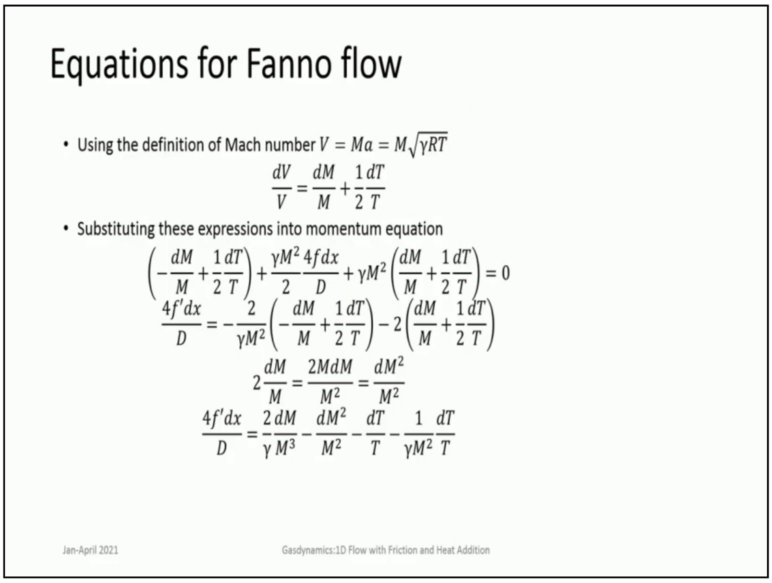

And we can we express $\frac{dT}{T}$ in terms of $\frac{dM}{M}$ only this can be done because we have the definition of Mach number also that is because there is a term $\frac{dV}{V}$ here. So, $\frac{dV}{V}$ can also be expressed in terms of $\frac{dT}{T}$ because V =Ma, a = $\sqrt{\gamma RT}$. So, we get,

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

Substituting these quantities and taking the $\frac{dT}{T}$ term common.

So, that is considered taken common. And on the left hand side we take only the term corresponding to friction $\frac{4f}{D}dx$. And we express all the others now come in terms of Mach number. Here you have $\frac{dM^2}{M^2}$. So, we express the friction term in the left hand side terms relate to Mach number on the right side. And you have a term $\frac{dT}{T}$ over here.

$$\frac{1}{2}\frac{dT}{T} - \frac{dM}{M} + \frac{\gamma M^2}{2}\frac{4f'}{D}dx + \gamma M^2\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right) = 0$$

$$\frac{4f'}{D}dx = -\frac{2}{\gamma M^2}\left(\frac{1}{2}\frac{dT}{T} - \frac{dM}{M}\right) - 2\left(\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}\right)$$

$$2\frac{dM}{M} = \frac{2MdM}{M^2} = \frac{dM^2}{M^2}$$

$$\frac{4f'}{D}dx = \frac{2}{\gamma}\frac{dM}{M^3} - \frac{dM^2}{M^2} - \frac{dT}{T} - \frac{1}{\gamma M^2}\frac{dT}{T}$$

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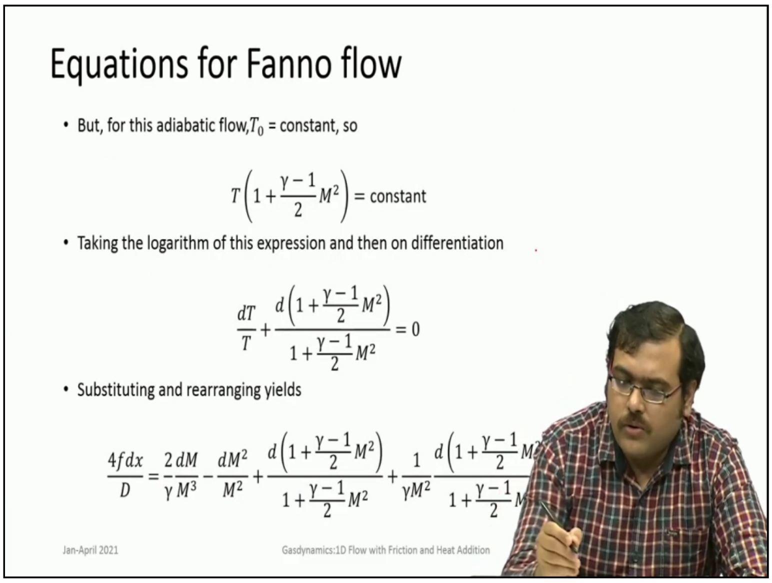

Now $\frac{dT}{T}$ can be expressed in terms of Mach number only because in this flow T_0 is constant,

$T_0 = T(1+ \frac{\gamma-1}{2} M^2)$

This is a constant. Now $\frac{dT}{T}$ again you can take a logarithmic differentiation $\frac{dT}{T}$ can be expressed in terms,

$$\frac{dT}{T} + \frac{d(1+\frac{\gamma-1}{2}M^2)}{1+\frac{\gamma-1}{2}M^2} = 0$$

So, there wherever $\frac{dT}{T}$ was there in the previous expression over here and at this particular point we can substitute this term.

$$\frac{4f'}{D}dx = \frac{2}{\gamma}\frac{dM}{M^3} - \frac{dM^2}{M^2} + \frac{d(1+\frac{\gamma-1}{2}M^2)}{1+\frac{\gamma-1}{2}M^2} + \frac{1}{\gamma M^2}\frac{d(1+\frac{\gamma-1}{2}M^2)}{1+\frac{\gamma-1}{2}M^2}$$

And we will be getting these values and here what we need to do is now, this can this is expressed entirely in terms of Mach number. Of course in this particular formulation the calorically perfect gas assumption is taken. So, we are saying that h = C_P T where Cp is constant. So, that is why we are able to do all these calculation or all these changes and we can get a closed form solution.

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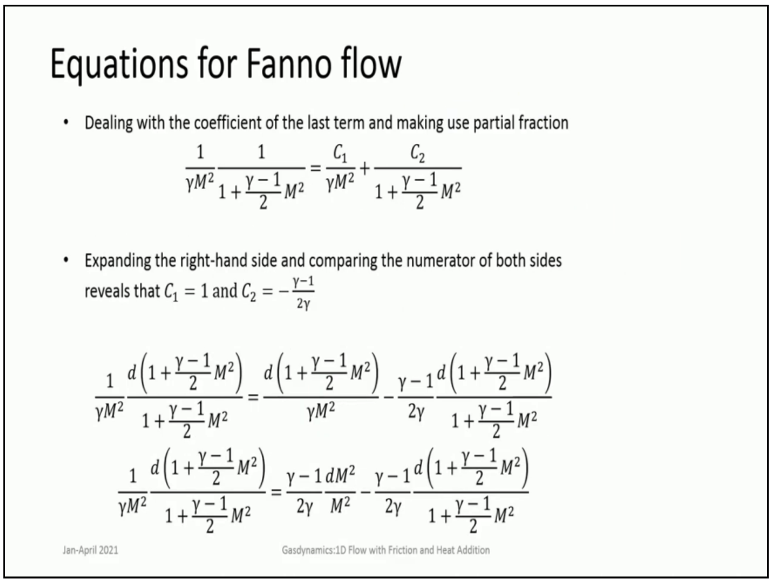

And to get the close form solution we have to express this term $\frac{1}{\gamma M^2}\frac{d(1+\frac{\gamma-1}{2}M^2)}{1+\frac{\gamma-1}{2}M^2}$ terms in terms of partial fractions. And you can do that do the partial fraction expansion of this particular these particular terms plug them in the main equation.

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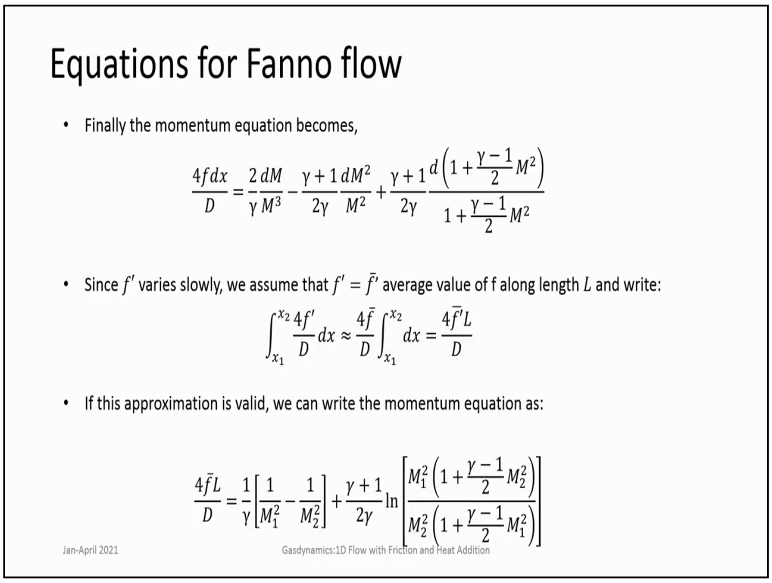

And if you do that you get the final momentum equation becomes,

$$\frac{4f'}{D}dx = \frac{2}{\gamma}\frac{dM}{M^3} - \frac{\gamma+1}{2\gamma}\frac{dM^2}{M^2} + \frac{\gamma+1}{2\gamma}\frac{d(1+\frac{\gamma-1}{2}M^2)}{1+\frac{\gamma-1}{2}M^2}$$

So, you have three terms, they can be completely integrated as you go from. So, you have a duct going from length 1 to 2 having a length L. So, they have to be integrated now from 1 to 2.

So, of course f is a variable here friction factor or friction coefficient. So, for this is friction coefficient for f' here. f' varies very slowly. So, we can take that the average value of f' or friction factor any of them both are constant along the length. But we have to take the average value. So, that can be clubbed together. So, that is given the term $\bar{f}$. So, 4f bar by D this term is actually a constant.

$$\frac{1}{L}\int_{x_1}^{x_2} \frac{4f'}{D}\,dx = \frac{4\bar{f}}{D}\int_{x_1}^{x_2} dx = \frac{4\bar{f}' L}{D}$$

$$\frac{4\bar{f}' L}{D} = \frac{1}{\gamma}\left[\frac{1}{M_1^2} - \frac{1}{M_2^2}\right] + \frac{\gamma+1}{2\gamma}\, ln\left[\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_2^2\right)}{M_2^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}\right]$$

So, you get this closed form solution that expresses change in Mach number as you move from an initial entry condition with Mach number M_1, it passes through a duct having a certain coefficient of friction $\bar{f}'$ of length L and at the end it gets Mach number, it is Mach number becomes M_2 pressures, temperatures everything undergoes a change.

So, this is the final expression in a closed form solution. So, of course; if you look at this expression while you can actually solve if you are given M_1 then you can solve and L. You can solve for M_2 in principle . But it is not an easy equation to solve. So, iterative tools may be needed. But a different approach is used when actually solving these problems which we will discuss in a moment before we go there.

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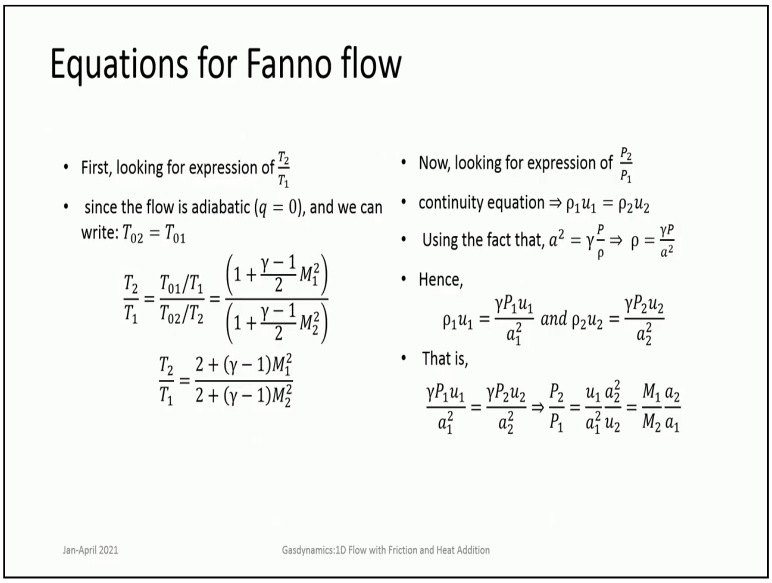

Let us look at the variables of the flow we are looking for what happens to temperatures $\frac{T_2}{T_1}$, $\frac{P_2}{P_1}$. So we look at $\frac{T_2}{T_1}$ first. Now because flow is adiabatic T_{02} is equal to T_{01} therefore you can express $\frac{T_2}{T_1}$ in terms of $\frac{T_{02}}{T_{01}}$. So, $\frac{T_2}{T_1}$ can be written as,

$$\frac{T_2}{T_1} = \frac{\frac{T_{01}}{T_1}}{\frac{T_{02}}{T_2}} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)} = \frac{\left(2+(\gamma-1)M_1^2\right)}{\left(2+(\gamma-1)M_2^2\right)}$$

What about $\frac{P_2}{P_1}$? Here we cannot use isentropic relations it is not an isentropic flow. But we know that ρu = constant. And you can express u =M a. And a can be expressed as $a^2 = \frac{\gamma P}{\rho}$ And also you can use the equation of state P = ρ RT.

So, if you use all of them together you get that you get to the fact that,

$$\frac{P_2}{P_1} = \frac{M_1}{M_2}\frac{a_2}{a_1}$$

Now, $\frac{a_2}{a_1} = \sqrt{\frac{T_2}{T_1}}$

$$\frac{P_2}{P_1} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right]^{\frac{1}{2}}$$

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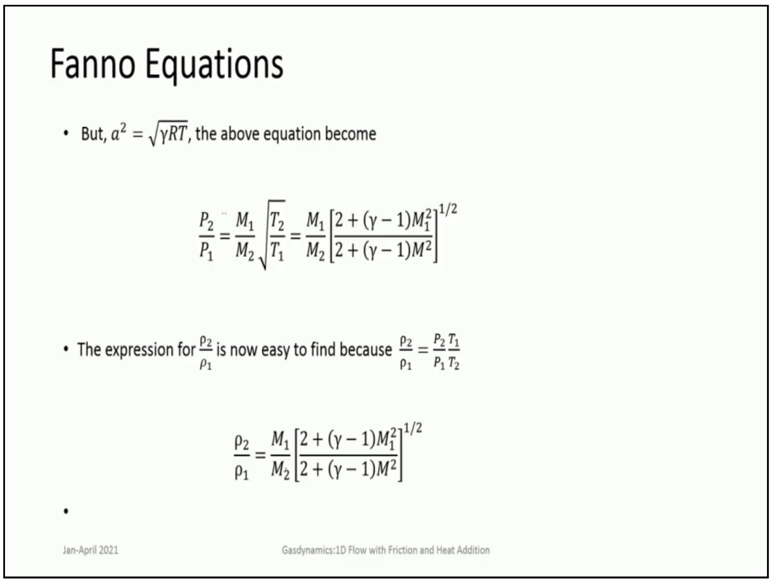

And you get $\frac{P_2}{P_1}$ and $\frac{\rho_2}{\rho_1}$ is now just use the ideal gas equation of state, P = ρRT. So, or ρ is. So, $\rho = \frac{P}{RT}$.

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1}\frac{T_1}{T_2}$$

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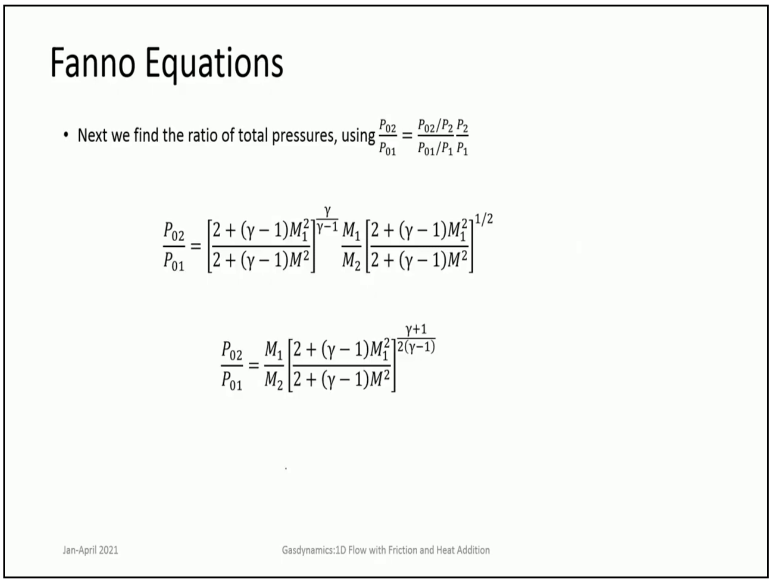

Now what about total pressures, change in total pressure or ratio of total pressures $\frac{P_{02}}{P_{01}}$? That can be expressed,

$$\frac{P_{02}}{P_{01}} = \frac{\frac{P_{02}}{P_2}}{\frac{P_{01}}{P_1}}\frac{P_2}{P_1}$$

Locally at that particular point, the definition of stagnation conditions is always through an isentropic process. So, $\frac{P_{02}}{P_2}$ is an isentropic process that is. And $\frac{P_{01}}{P_1}$ is again an isentropic process. So, those can be expressed in terms of isentropic processes. But $\frac{P_2}{P_1}$ comes from Fanno flow equations.

So, $\frac{P_{02}}{P_{01}}$ can be expressed in terms of M_1 and M_2

So, now so in these equations we have expressed every variable say $\frac{P_2}{P_1}$, $\frac{T_2}{T_1}$, $\frac{P_{02}}{P_{01}}$ all in terms of the Mach numbers across the Fanno flow through a duct of length L.

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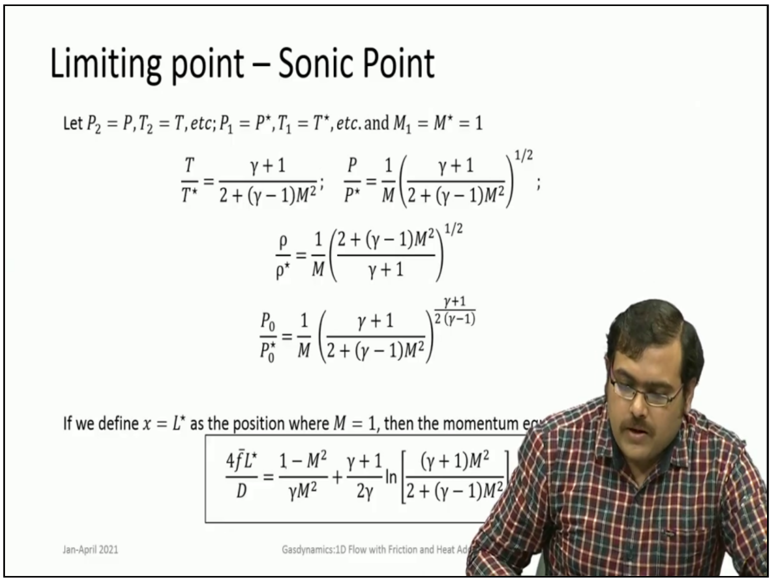

So, but what you would realize from all these is even if you know say M_1 and you want to find M2 then dealing with these equations is not a straight forward and closed form sort of getting solutions from them is difficult. So, what is done is a simplifying procedure is done that is makes use of the fact that if you consider a Fanno flow then there is a critical point which is this particular point star point for the fan of flow.

So, this star point the Fanno flow that is a flow through the duct drives whether it is in subsonic flow or supersonic flow it always drives towards this particular point where Mach number equal to 1. So, that can be taken as the reference point for a given Fanno curve there will be one particular one unique reference point which is the star. So, for that what we do in this is take that reference point as M_1, M_1 is equal to $M^* = 1$. And the other point can be the any other point on this Fanno curve.

So when you express that it becomes $\frac{T}{T^*}$. So, any expression say $\frac{P_2}{P_1}$, $\frac{\rho_2}{\rho_1}$ if you take the reference point as the star point which for a Fanno curve there is that unique particular point where Mach number will be equal to 1. Then you can express all the parameters in terms of this particular point that is the sonic point here. So, $\frac{T}{T^*}$, $\frac{P}{P}$, $\frac{\rho}{\rho^*}$,. $\frac{P_0}{P^*_0}$

Now these are functions only of γ and M. similarly if you look at the term $\frac{4f' L}{D}$ if you take this L such that if you begin from a point it always goes if you take a certain length of duct such that at the end of the duct you get Mach number equal to 1 then this particular length of duct is known as the star length ,L* So, this given here L*. x is equal to L*. Then in the momentum equation what you get as $\frac{4L^*}{D}$ that is what we had derived earlier can be written in terms of γ ,L and M.

Now these are functions of M only. And they are tabulated behind textbooks or there are calculators online which can give you values of these star quantities these star quantities when you give an input of Mach number. So, now they become charts or tables which we can refer to.

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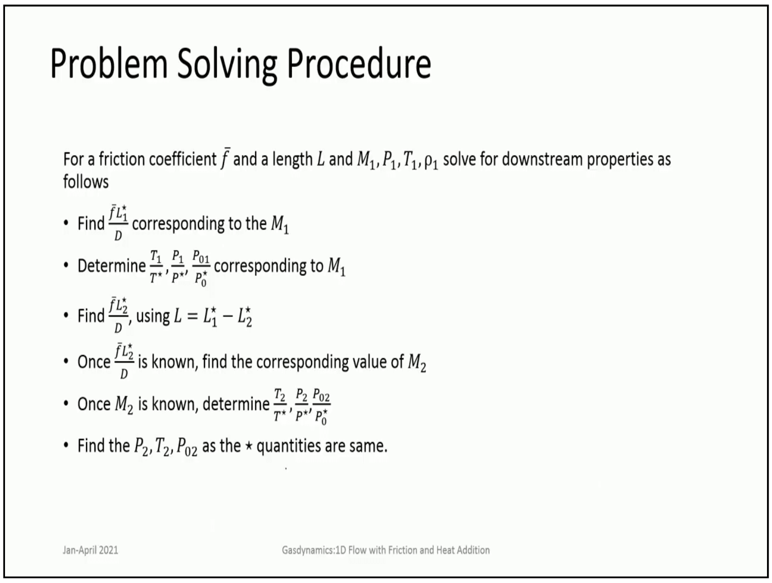

So, how will we solve equations or solve problems when we are doing them actually. When we solve them that is usually what would be given is friction coefficient will be known say length of the duct is known. And the initial conditions are given at the start of the duct which is the duct is of length L then point 1 and 2 what we know is M_1, P_1, T_1. So, we will know ρ_1 and L is given.

So, the way to go about solving this problem is if you view this in a Ts diagram it will be very useful. So, it will of this kind. So this is the maximum entropy point. You start with the point 1. Now this is a unique reference point. So, this reference point L* is unique. So, if I take for point 1 this is L1 * and as it flows through a duct of length L it reaches point 2. For point 2, the L* is over here this is L, L2 *.

$$\frac{4f' L}{D}$$

So, what is L? So, L is actually because we are taking f' that is f' or f' that is a constant it is the average friction .

So, $\frac{4f' L}{D} = \frac{4f' L1^*}{D} - \frac{4f' L2^*}{D}$

So, with this equation basically, so, L is equal to L1 * - L2 *. So with this particular equation you can connect to the two L* or the $\frac{4f' L^*}{D}$ this particular expression.

And this will be tabulated and from there we can extract. So, if you know this particular value which is known because M1 is known since L is known this value is known therefore you can find out $\frac{4f' 1^*}{D}$ for the point 2. once you know that you can back calculate or you look at the charts and get what is M_2? Once M_2 and M_1 are known then it is easy to find all the remaining quantities.

So, that is how the problems are solved. And we look at problem solution very soon. Before we look at that in the next class we look at an important concept which is known as choking due to friction. So, we had seen choking, mass flow choking due to changes in area varying area. Now here if you look at frictional flows also there is a choking involved and we will look at that in the next class.