Gasdynamics: Fundamentals and Applications
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Lecture 32
Varying Area Duct Flows- II

So we are looking at varying area duct flows in the context of Quasi-1D assumption and flow is inviscid and has no other effects only area is changing. As a consequence, the flow within varying area ducts is taken as an isentropic flow is a very good approximation to understand flows through nozzles diffusers and so on. And in the previous class we looked at how the definition whether a particular duct is a nozzle or a diffuser changes as you move from different flow regimes from subsonic to supersonic flow.

In a subsonic flow a duct which decreases area that is a convergent duct is a nozzle it accelerates the flow. While a duct which increases area that is a divergent duct it decreases velocity when the incoming velocity is subsonic it behaves like a diffuser. But the moment you go to a supersonic flow that is the incoming flow is supersonic then a convergent duct that is a duct which decreases area with an incoming supersonic flow the velocity actually decreases.

So, a converged duct behaves like a diffuser. So, a supersonic diffuser implies a convergent duct. Similarly, if you go and increase area with an incoming supersonic flow then you find that velocity increases or it behaves like a nozzle. So, a divergent duct is a supersonic nozzle. So, this is the behaviour one has to always understand and keep in mind when we discuss varying area ducts.

So now if you want to move from subsonic flow to supersonic flow or supersonic flow to subsonic flow you need to combine the ducts in a certain way. And the way it is combined is always a minimum area is produced and at the minimum area the Mach number is 1. So, these kinds of ducts which combine the two principles are convergent divergent ducts. So, you have various kinds of varying area ducts and they behave differently in different flow regimes.

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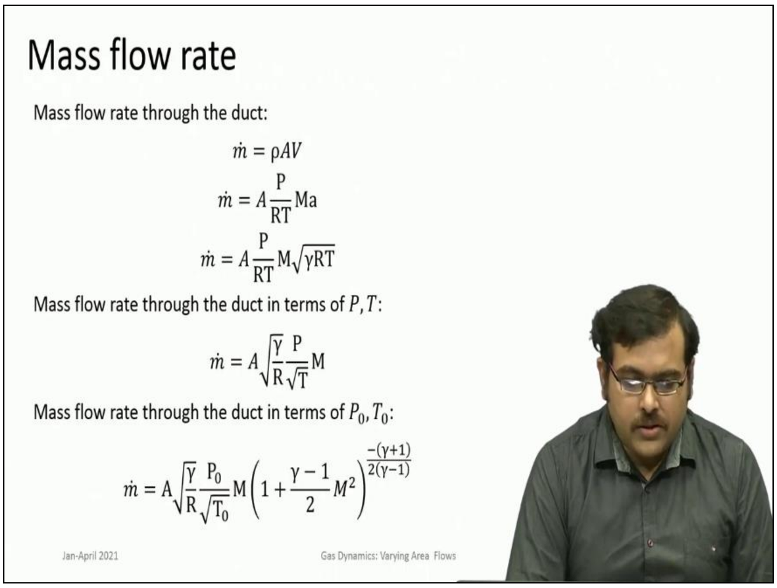

And we were now we had begun looking at quantitative aspects of this understanding and the previous class we had looked at the area ratio relation $\frac{A}{A^*}$ we had looked at that relation and that was done from continuity equations we had derived it. Now we also look at how we can write the mass flow rate through a duct. What is mass flow rate from $\dot{m} = \rho A\,V$. Now depending on various applications, you may need to find the mass flow rate.

Because for example propulsion-based applications thrust produced is dependent on $\dot{m}V_e$ that is mass flow rate multiplied by the exhaust velocity. So mass flow rate is an important term there. Similarly, in applications like wind tunnels you need to know what is the mass flow rate happening to the entire system. So mass the calculation of mass flow rate is quite routine in engineering devices and many different conditions may be known to you. So how do we calculate m dot in different conditions?

So first is of course $\dot{m} = \rho A\,V$ but generally density is not known a priori it is usually a calculated variable something that is generally measured is pressures and temperatures, static pressure, static temperature. So, can we express this in terms of static pressure and temperature and it can be done because $\rho = \frac{P}{RT}$. $\dot{m} = \frac{P}{RT}AV$.

It is also useful to express velocity in terms of Mach number, $\frac{P}{RT}$ A multiplied by Mach number multiplied by acoustic speed a. So you get $\dot{m} = \frac{P}{RT}AMa$. And a is $\sqrt{\gamma RT}$, so $\dot{m} = \frac{P}{\sqrt{RT}}AM$. So this is the equation if you know what is the static pressure, temperature and Mach number at a particular location in any duct, We can find the mass flow rate.

But generally static pressure temperature may not be known; something that is known is the reservoir conditions which is pressure & temperature measured in a condition where the velocities are very small, velocities are approaching 0. The velocities may be a few meters per second, but they may not be very high, and flow need not be compressible in this region. That kind of a system is the reservoir you can look at various such applications where you can have a huge tank.

And then the flow is taking place from the tank and then goes to a set of pipes and some varying area ducts. Then the velocity in the huge tank will generally be very small. Then if pressure of tank and temperature of tank is known then the pressure and temperature at such low speeds will approximate will be almost equal to the stagnation quantities P_0 and T_0. So because of this expressing mass flow rate in terms of P_0 and $T_0$is also very useful and so you can convert this.

So you can use the relations $\frac{P_0}{P} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}}$ is and $\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2$. So while we are actually looking at $\sqrt{\frac{T}{T_0}}$. So this has a $\frac{1}{2}$ attached to it. So, using this if you convert this particular equation in terms of P_0 and T_0 then the equation $\dot{m} = A\sqrt{\frac{\gamma}{R}} \frac{P_0}{\sqrt{T_0}} M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$.

So here we have expressed $\dot{m}$ as a function of γ, P_0, T_0, area and Mach number. So there is another form in which you can write the mass flow rate starting from $\rho A\,V$, you can always go back to $\rho A\,V$. But these are various forms in this in which the mass flow rate can be written.

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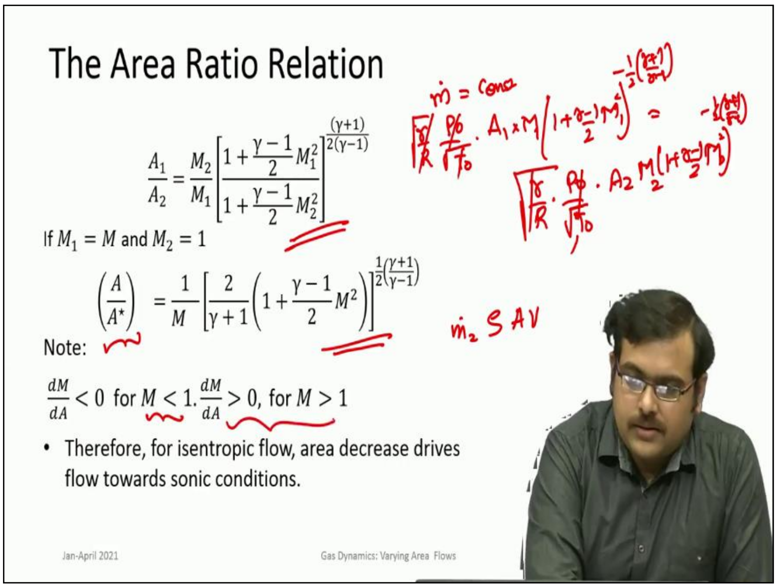

$$\frac{A_1}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{(\gamma+1)}{2\gamma-1}}$$

And we will see there are other forms also. Since mass flow rate is constant $\dot{m} = constant$. Now we can relate. So the two areas so we get

$$A_1\sqrt{\frac{\gamma}{R}}\frac{P_0}{\sqrt{T_0}}M_1\left(1+\frac{\gamma-1}{2}M_1^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} = A_2\sqrt{\frac{\gamma}{R}}\frac{P_0}{\sqrt{T_0}}M_2\left(1+\frac{\gamma-1}{2}M_2^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}.$$

So in varying area duct its isentropicP_0 and T_0 are constant so they just cancel off on both the sides you can get a the expression is over here $\frac{A_1}{A_2} = \frac{M_2}{M_1}\left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{\frac{(\gamma+1)}{2\gamma-1}}$. And a particular reference area that we discussed even last time is when Mach number goes to 1. So $M_2 = 1$. So that particular area ratio is $\frac{A}{A^*}$ and $\frac{A_1}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{(\gamma+1)}{2\gamma-1}}$ this was derived, same principles expressed in different ways.

This is something that you have to understand depending on the variables that you know in a particular problem. The same quantity can be expressed in different ways you know static pressure, temperature and Mach number then you can calculate mass flow rate using a certain relation. If you know the stagnation pressure and temperature or what are known as reservoir conditions, then you can use another expression.

Or you can always go back calculate the basic variables $\rho A\, V$ and mass flow rate can be calculated $\dot{m} = \rho A\, V$. So now how does $\frac{A}{A^*}$ behave? So for this we plot $\frac{A}{A^*}$ but immediately we can do a differentiation also and understand. What we are looking is how does, Mach number change as area changes $\frac{dM}{A}$. So $\frac{dM}{A} < 0$ for subsonic flows Mach number < 1.

And $\frac{dM}{A} > 0$ for Mach numbers > 1. So, this is what is known if you can differentiate.

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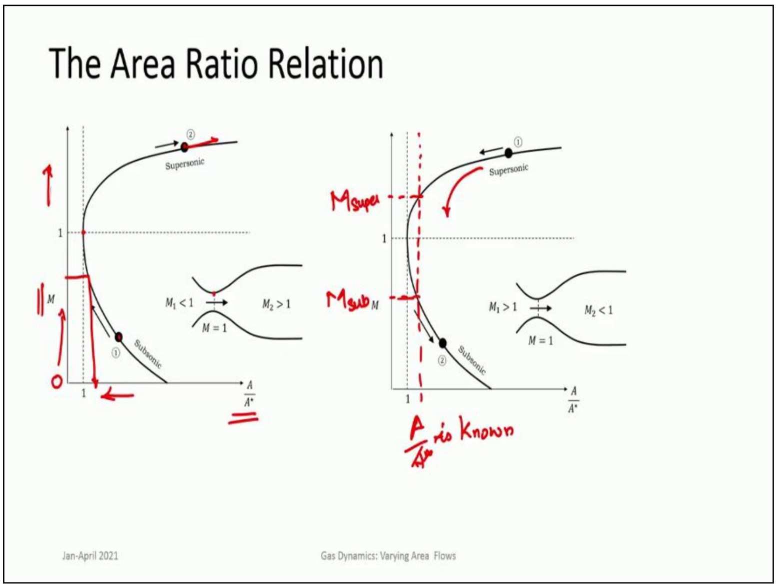

But easier way is let us just plotting it. So if you plot $\frac{A}{A^*}$, so $\frac{A}{A^*}$ is here versus Mach number. So here it is plotted M versus $\frac{A}{A^*}$ then you will find that as Mach number increases from subsonic flow. So this is 0 this is from very low Mach numbers it is going up to Mach 1 area ratio that is $\frac{A}{A^*}$ that is often refers to as star area ratio or the area ratio sometimes used just like area ratio. So, area ratio decreases.

After Mach 1 if you further increase Mach number then area ratio increases. So, the area ratio has a minimum point at Mach number equal to 1. So, if you consider a varying area duct in general then the minimum point where the area reaches a minimum point that is where you expect Mach number to be equal to 1. Expectation is that Mach number reaches one at the minimum area point. Now this figure also gives us an idea on how to understand the behaviour of flows.

So, if you are in the subsonic domain and the area ratio decreases. So, area ratio decreases and Mach number increases. So, you can understand from this graph easily that and in supersonic flow the Mach number will increase when you increase area ratio. So, area ratio increases supersonic flow Mach number increases. So, these are the important points and a convergent divergent duct. So, where there is a convergent and divergence included then there what can happen is you can have the flow going all the way from subsonic to supersonic passing through Mach number 1 at minimum area.

So this part is important. The same thing will occur if you go from supersonic to subsonic all the way. Then another important point that you should observe in this graph is that if you take any area ratio in particular. So, let me draw a curve here. So, if I know $\frac{A}{A^*}$. So $\frac{A}{A^*}$ is known is known then there are two solutions to Mach number. So you have a subsonic solution and a supersonic solution.

So, what do you observe whether you observe subsonic flow or a supersonic flow at a particular area has to depend on other factors these are pressures and pressure ratios. So, we will come to it soon. But this has to be understood for a given Mach number you can always find an area ratio. So, given a Mach number you will get an area ratio that is given here. But for a given area ratio you can you always get two solutions one can have one can be subsonic the other can be supersonic.

So this has to be clearly understood and when you solve problems you should always look for whether the solution is meaningful in the context of the problem which one should be used. Whether it should be subsonic or it should be supersonic we will soon come to these fine details.

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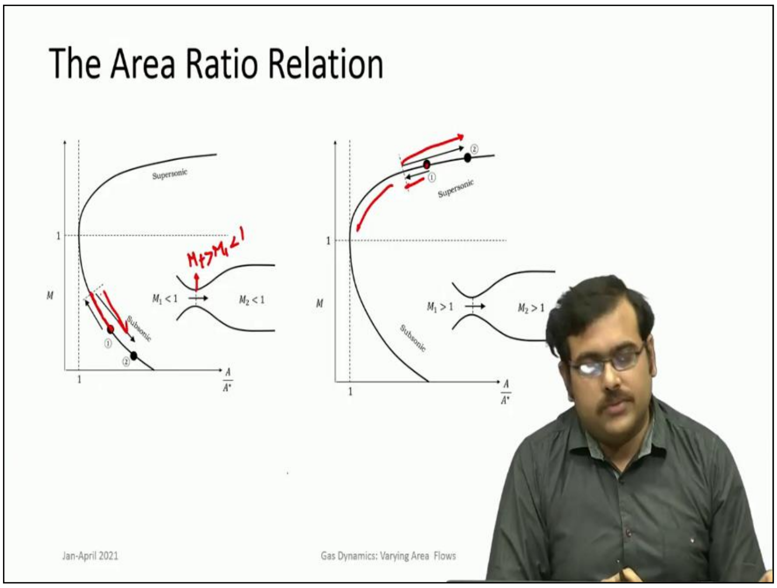

So it is not always necessary that at the minimum area Mach number should go to 1. If you are moving from a if during the motion through the convergent divergent varying area duct the flow passes from a subsonic region to a supersonic region then it has to go through a Mach number equal to 1 at minimum area. This is for a completely shock free flow when shocks are present things change a little bit.

We will come to that also soon. But you could if the pressure ratios across the duct do not support such a flow behaviour then you can always have changes like first there is you started off with the subsonic flow point one. There was a convergence, so the flow accelerated increased its Mach number. But then the pressure ratios across the duct did not support its further acceleration.

So it reached a certain increased Mach number at this point. So M, I will call it as the M_{throat} usually this small area is called as a throat and M_{throat}. M_{throat} is greater than M_1 but it is less than one. So, in that case if now if further area is increased then the flow will not go supersonic because it is still subsonic at the minimum area. So, it will decrease its velocity. So, it will become subsonic again.

So, you see that various combinations can happen. Similarly, in supersonic flows in this case for example you started with the same duct you start with the supersonic flow convergent duct Mach number decreases but it does not decrease all the way to 1. And then further the area is increased then you see it again increases the Mach number. So, looking at varying area ducts you have to be careful and understand these concepts very carefully.

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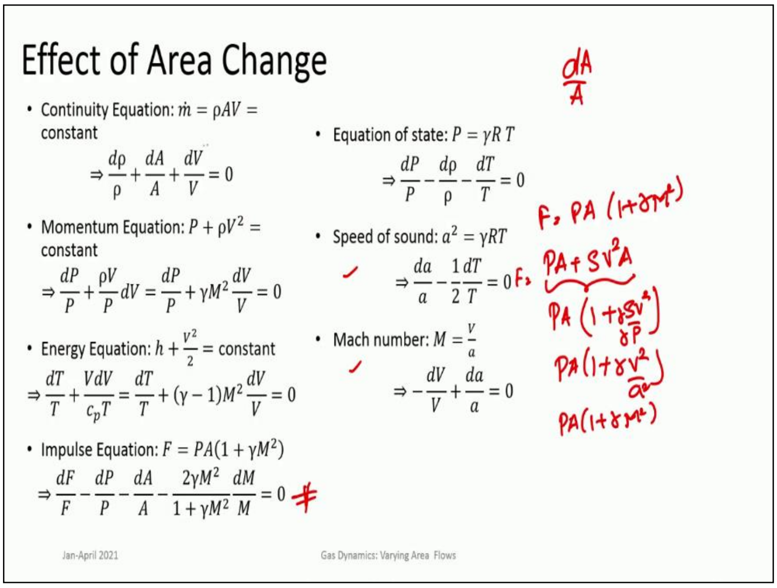

Now let us look at the sum total of all the things in a varying area duct. There is changes in velocity there is change in pressure temperature density. So, we can look at all the equations together and with $\frac{dA}{A}$ basically the area change as a forcing function it is like a forcing. You are changing area and you want to know what happens to all these quantities. Pressure, temperature, velocity, Mach number and another parameter is the impulse equation.

Impulse equation F is $PA(1 + \gamma M^2)$ this comes from $PA + \rho V^2 A$ which is nothing but and it at the particular section quasi 1D flow you integrate the momentum equation. Then at any particular section you can define this quantity $PA + \rho V^2 A$ it comes from the momentum equation. And this is nothing but the impulse that is imparted there.

It is often used in discussions of propulsion systems. So PA is you can take out PA this becomes $PA\left(1 + \frac{\rho V^2}{P}\right)$ and $\frac{\gamma P}{\rho} = a^2$. So this is $PA(1 + \gamma M^2)$. So that is the impulse function this is also a parameter. So, this way what if we look at all these changes together. So, speed of sound Mach number we write these equations in terms of $\frac{dA}{A}$ as a variable as a change.

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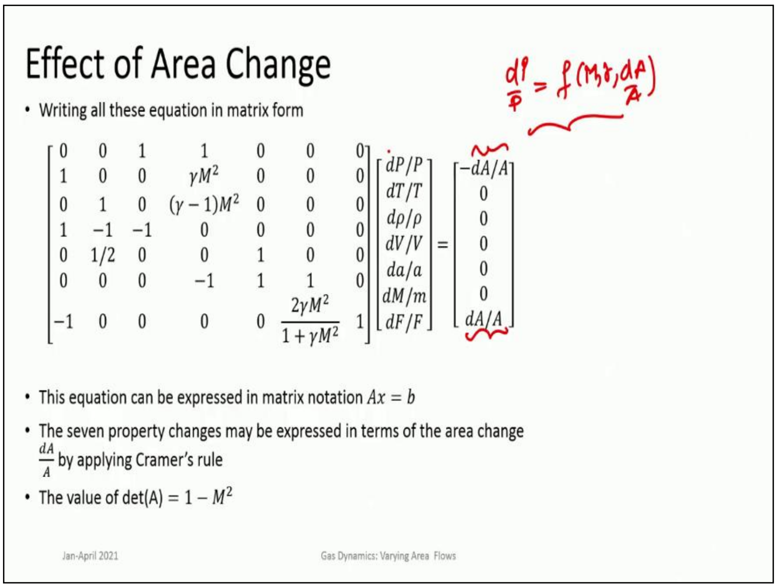

And collect all the equations together and write it in a matrix form with $\frac{dA}{A}$ as a forcing function. Then we will know how each quantity is getting affected by $\frac{dA}{A}$. So this is nothing but you are expressing all the equations that have written over here for various quantities in terms of a matrix. And then this can be solved this matrix can be solved you can express $\frac{dP}{P}$ solely as a function of Mach number, gamma and $\frac{dA}{A}$.

So, this can be expressed. So, you get an expression for this can be done by means of by applying Cramer's rule. You can use nowadays you have Mathematica and Maple kind of symbolic mathematical operators to do these operations and you can get this quite readily easily.

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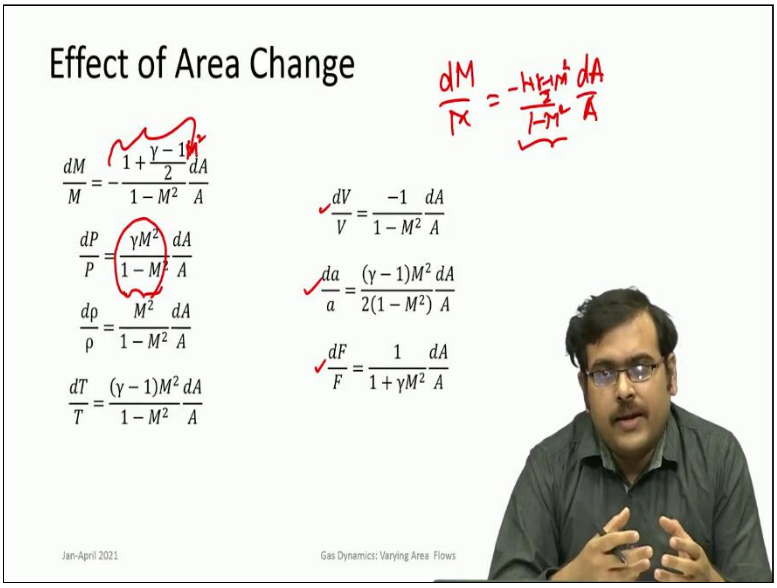

So you can do this use Cramer's rule solve the equations and you can get $\frac{dM}{M}$ as a function of only Mach number. So as a function of only $\frac{dA}{A}$, $\frac{dA}{A}$ is the forcing gamma and Mach number. So like this we can write for all the parameters $\frac{dM}{M}$, $\frac{dP}{P}$, $\frac{d\rho}{\rho}$, $\frac{dT}{T}$ change in velocity, change in speed of sound, change in impulse function.

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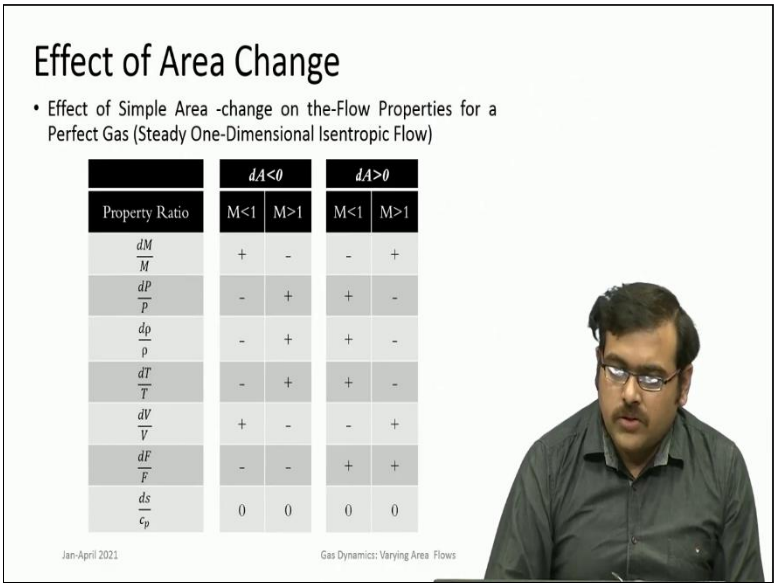

So how are they affected by area change? So, if you look at that this is for a perfect gas calorically perfect gas $dA < 0$ that means convergent duct. But then the flow behaviour changes between subsonic and supersonic flows. So, if the flow is subsonic then Mach number increases, pressure decreases, density decreases, temperature decreases, velocity increases the impulse function decreases there is no change in entropy.

Entropy is constant. Similarly if you go and look at convergent ducts and supersonic flow then you know that Mach number decreases, pressure increases, density increases temperature increases, velocity decreases, impulse function decreases. So in one short you can get how all these variables change this kind of representation is called and where these coefficients are called as influence coefficients.

So $\frac{dM}{M}$ is influenced by $\frac{dA}{A}$ by a coefficient which is $\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}$. So, these are influence coefficients they are very useful to look at. So similarly $\frac{dP}{P}$ this is the influence coefficient for $\frac{dP}{P}$ they are very useful. Now we are looking at each individual forcing function. Now we are looking only at area change.

In later classes we may look at friction and we may look at heat addition as separate influencing parameters on a duct flow. But in general, these are not always separate they are in some combined fashion usually there will be both area change and friction or they can be both area change friction and heat addition occurring together. So, if you want to analyse such problems then the influence coefficient method is a very useful method to look at such problems.

So, this idea is being introduced very early on but later we will see how they can be combined together in a special lecture on combined effects of these problems.

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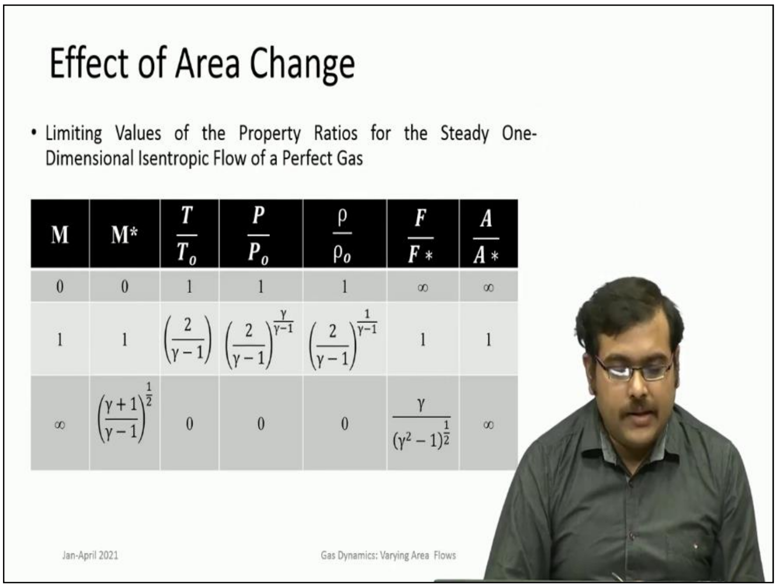

So in the limit as various limits go that you have very small Mach numbers, very large Mach numbers and at Mach number equal to 1. What happens to various quantities like $M^*, \frac{T}{T_0}, \frac{P}{P_0}, \frac{\rho}{\rho_0}$ a varying area ratios. So, it is an isentropic flow. So in an isentropic flow these are the expressions for the limiting cases. So with this now, we will look at still now what we have been looking at is?

We have been looking at there is an area change what happens to velocity or Mach number when there is an area change? What and how do they behave? But will they actually behave in such a way you have a certain area convergent. For example, a convergent nozzle let us say. So you are providing a certain flow to it and you expect there should be an increase in velocity which should happen.

But then for the flow to happen there should be some pressure that needs to be provided. So always there is you should not be considering these varying area ducts only in context of only areas and Mach numbers you should also look at pressures. What happens if we do not provide pressure ratios which are consistent with the Mach number that needs to be produced according to area ratio?

So these are the kind of problems. So it is not just only area ratio and velocity here there is a change in pressure also. How are they mutually dependent on each other this is the focus of our attention in the coming classes? They will be different for nozzles and diffusers so we will treat them separately nozzles and diffusers. So, next class will start looking at convergent nozzles.