Gasdynamics: Fundamentals and Applications
Prof. Srisha Rao M V
Aerospace Engineering
Indian Institute of Science – Bangalore

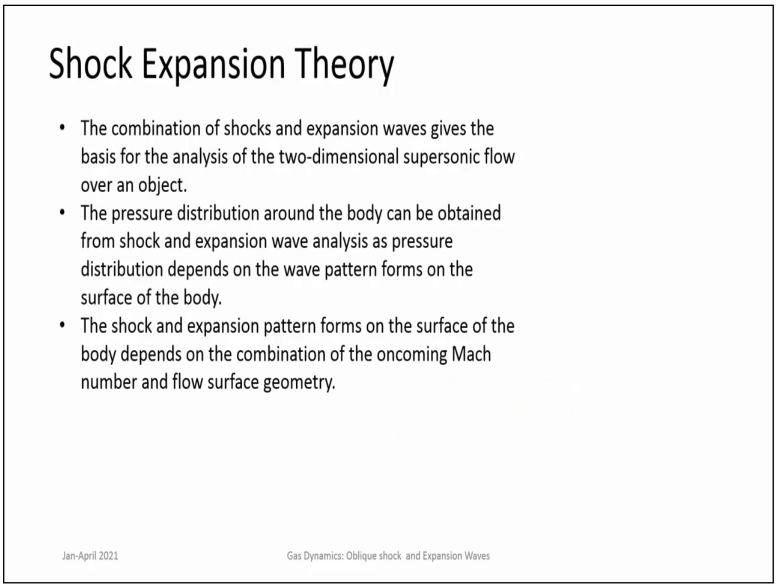
Lecture 28
Shock Expansion Method

So, in the previous classes we have looked at Oblique shocks and Expansion waves. Oblique shocks are compression shocks they are at an angle to the flow. They turn the flow towards itself. While expansion waves are more gradual, they are isentropic, and they turn the flow away from itself. So, if you take any shape, that is in a supersonic flow then it can have various changes in angles.

The flow may turn towards the flow or away from the flow and consequently you can have Oblique shocks and Expansion fans a series of them, and across the Oblique shocks and expansion fans the flow is uniform, only at the Oblique shock or across through the expansion fan the flow is changing. So, this gives us a method to calculate the pressures that is going to be acting on the surfaces of bodies in supersonic flow.

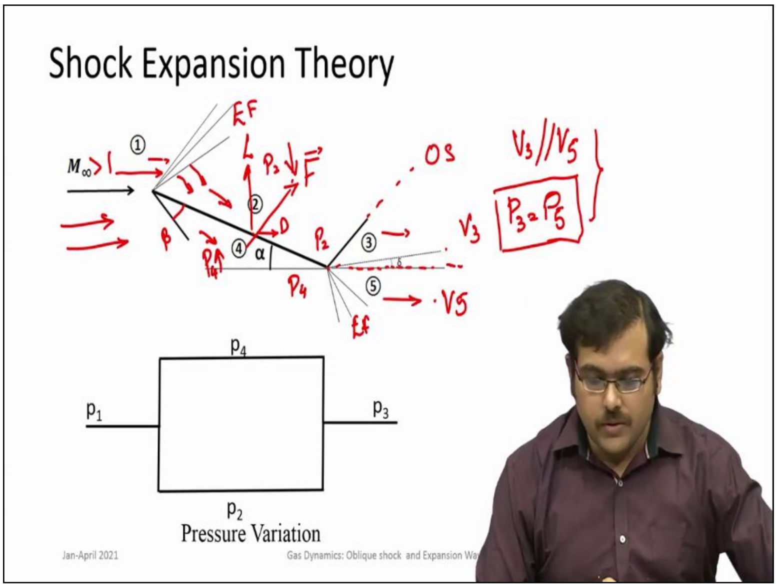
From there if you know the pressures you can always calculate the forces, that is the lift and the pressure drag forces of bodies.

(Refer Slide Time: 01:43)



So that is what we are going to do in this lecture, we will take an example. So, this becomes clear to all of you. So, what we are really looking is how the wave patterns form over bodies in supersonic flow.

(Refer Slide Time: 02:07)



So, if you look at this, a simple example, if you take a flat plate, a plate and put it in a supersonic flow. So, $M_\infty > 1$ and at an angle of attack (α) and then for all the streams that are coming on lower half of this flat plate this body is turning the flow towards the flow. So, a shock wave forms at this point, this shock would have an angle β.

While on the top surface, if you look at the top surface all the streamlines coming on top surface have turned away from the flow. So, you have an Expansion fan or an Expansion waves are formed here. So, pressure on the top surface P_2 decreases because of these expansions while the shock compresses and P_4, that is in this region 4, increases.

So, if you would plot schematically what would be the pressures like in P_4 is higher, P_2 is lower. So consequently, you will get a force in this direction which can be decomposed into the lift and drag forces. So, the lift force is here, the drag force much smaller of course it comes here, drag force is here. So, the lift and drag forces can be put in this way.

So, now this is the way the flow behaves at the flat plate. Now the flow is going parallel to the flat plate after these expansion fans. But when it comes back at the trailing edge now at the trailing edge the flows come back together. Now they have completely different pressures, it is P_4 down here and it is P_2 up here. Of course, when it goes back the flow goes at the end of the trailing edge, they cannot have any pressure difference because there is no longer a solid surface to sustain a pressure difference.

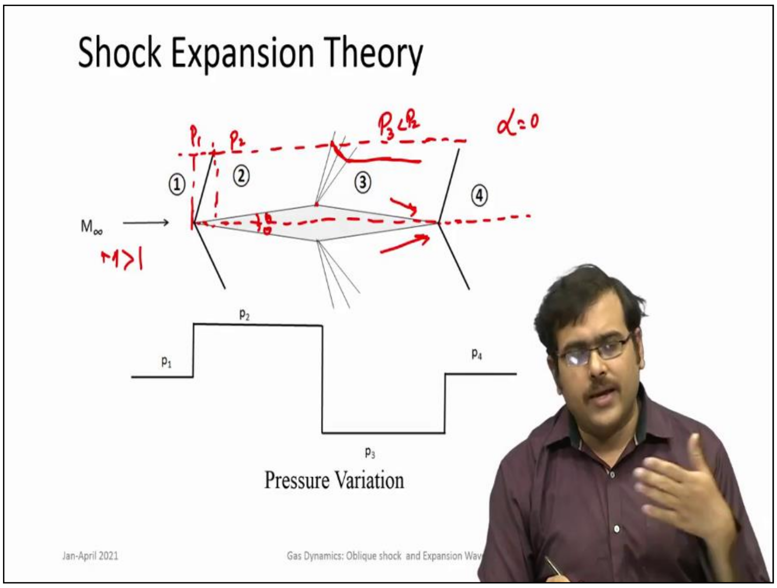
So pressures must get equalized, as a consequence of that you get additional wave forms at the trailing edge. You get a shock here (on the top surface) because the pressure is lower it has to get compressed there is a shock that gets formed here. So, this you get an Oblique shock and at the bottom surface you have higher pressures, and the pressure has to be relieved or it has to get expanded, you get Expansion fans here.

You get a slip stream, essentially a surface which demarcates the boundary between the trailing edge between the upper surface and lower surface. The idea here is that it is not always necessary that these 2 velocities should be the same. But the velocities need to be parallel to each other. So, if you take V_5 and V_3 they need to be parallel.

So, V_3 should be parallel to V_5 and P_3 should be equal to P_5. So, this is all the conditions that we can apply at the trailing edge. So, in solving the trailing edge of such bodies involves applying these conditions that the angle of the flow should remain the same and the pressure should remain the same. Velocities can be different so that is why you get a shear or slip stream because we are considering an inviscid flow.

If you consider the specifics its essentially a shear layer there. So, but it is an inviscid flow so it is a slip stream a discontinuity of velocities, also you can have discontinuity in temperatures, and densities pressures must be the same. So, you can now plot these kind of wave diagrams for several of these kinds of configurations. What is plotted here pressure variation is at the surface. So, this is at the surface at this surface.

(Refer Slide Time: 07:12)



So similarly, you can plot the pressure variations at the surface, these kind of airfoils are called Diamond shaped airfoils. You can see the shape is that of a diamond and when it is facing a Mach number greater than 1. This is a wedge so a very small angle. So, this is symmetric about this chord line theta. So, this is turning the flow towards itself it forms Oblique shocks, but at this point it turns away from the flow. So, Expansion waves are formed pressure decreases at 3 so P_3 is less than P_2 ($P_3 < P_2$).

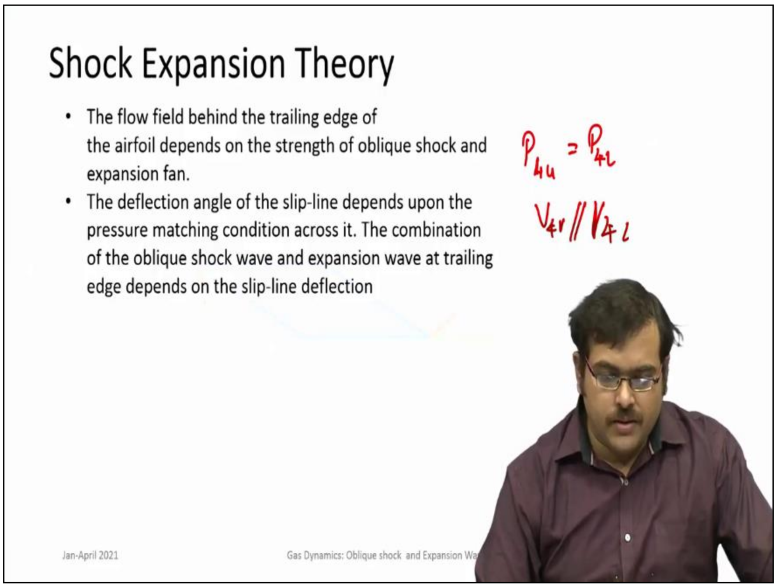
Again, when it comes back to the trailing edge you need to give matching conditions for the slip line over here the flow is parallel to these surfaces. So now you can see that the flow has to turn into itself in order to become parallel again and therefore another set of shocks are formed. So, angle of attack here is zero, you can consider various angles of attacks and you can draw such wave diagrams.

This pressure variation is on the surface. So, on the surface the pressure is varying in this fashion. You should always understand that in supersonic flow what we had discussed earlier in terms of Mach waves and how you have zones of silence and zones of influence. So, unless a Mach wave passes a particular point, pressure does not change. So, if you consider a line here in the free stream.

The pressure here remains P_1 until this point that is well behind the leading edge. So, you can see that leading edge would have passed into the flow, but the pressure will remain P_1 until it encounters the Oblique shock. Only after that the pressure will change to P_2. Similarly, if you go right here until it meets the Mach wave pressure will remain P_2 and gradually it will decrease slowly to P_3 until the last Mach wave.

Once the last Mach wave leaves then it becomes P_3 it becomes a constant again. So, one has to understand as you take different sections in the flow the pressure variation will be different in the flow according to how the waves move across. So, the wave nature of the flow and how the waves go in a supersonic flow is important. And this will come again as we move on in the course.

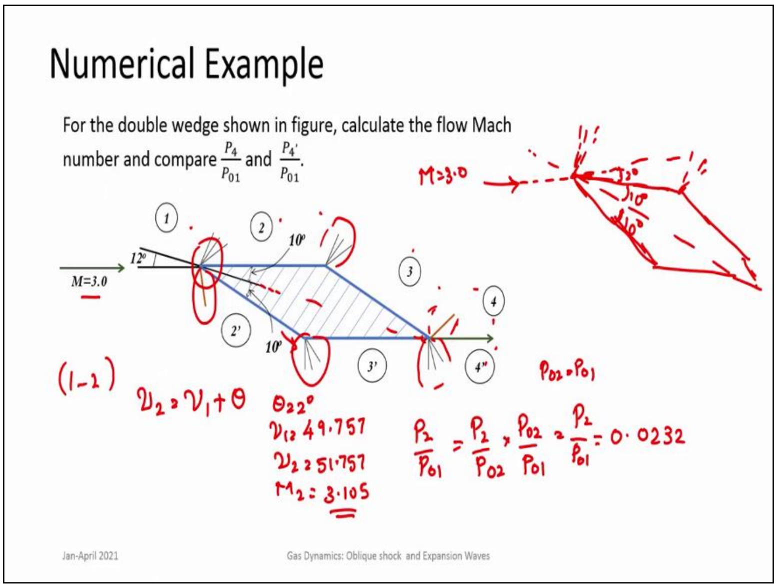
(Refer Slide Time: 10:12)



So here you should understand, the front portion of these kind of bodies you can always solve using Oblique shocks and expansion fans. But at the trailing edge you should be very careful you should apply the conditions that P_1 that is at the trailing edge here the nomenclature is 4 in the region 4. So, P_{4u}, upper, is equal to P_{4l}, lower, they should be the same, these 2 pressures should be the same ($P_{4u} = P_{4l}$).

Deflection angles the velocities have to be parallel to each other V_{4u} is parallel to the lower surface these 2 are the conditions that you can apply at the trailing edge.

(Refer Slide Time: 11:12)



So, with this let us try to solve 1 problem we have a diamond shaped airfoil here and the angle of this is the symmetric line the angle is 10 degrees for this diamond, and it is facing a flow of Mach 3 with an angle of attack of 12 degrees.

What we are interested in this is if the wave structure is so, it has to be like this because there is a flow turn away from the flow direction here by 2 degrees, so you have expansion fans here.

And this flow again turns the flow away another set of expansion fans. Here you have an Oblique shock but at this point flow turns away from this direction. So, an expansion fan now they follow different sets of wave structures. So here you get an expansion fan at the bottom half and an Oblique shock at the top half and finally the flow is turned again to the free stream direction the idea is what is P_4/P_{01} and compare P_4/P_{01} and P_4'/ P_{01} what the pressures at these 2 sides is?

If the wave structure is given is so, how do we solve this problem? So, you have to; now put in place together all the concepts you learnt in Oblique shocks and expansion fans together. So, we will go and good thing with supersonic flows is you can go step by step from region 1 to region 2 and solve from region 2 to region 3, 3 to 4 and similarly on the bottom half because there is a specific directionality involved in supersonic flows.

So let us go start from 1 and go to 2, 1 to 2 region 1 to 2 is an expansion fan region the way to solve expansion fan flows is to look at the Prandtl-Meyer angle. So, the key equation here is

ν_2 is ν_1 + $\Delta\theta$. We know θ here is 2^o and we know ν_1 is known because you know the Mach number upstream which Mach number is 3.

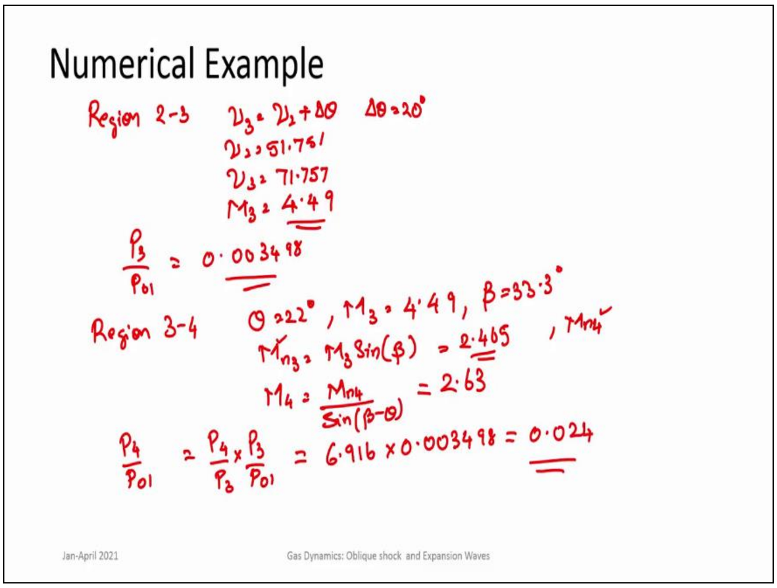
So, ν_1 is 49.757 so therefore you know ν_2 which is 51.757. So now you have to get back the Mach number. Now this is direct plot the Prandtl-Meyer function versus Mach number you can look at the charts or you can use the online calculator and you can get what should be the Mach number if the Prandtl-Meyer angle is ν_2 which is 51.757, it comes out to be 3.105.

So, you see that Mach number has increased due to a flow deflection away from the flow of 2^o. Now here when you use charts and tables you may be off in these numbers by few degrees 1 or 2 degrees therefore the answers that you get might be slightly different from between people. So, this sort of small errors is expected because it involves sometimes interpolation sometimes reading from a graph and these kinds of operations have small errors.

Or you can use online calculators where you can get accurate numbers. Even so there if you round off you get round off errors. So small changes in numbers are expected when we do these kinds of numerical so that should be borne in mind. So, what are we interested always we should go back to P_4/P_{01} so what is P_2/P_{01} = P_2/P_{02} * P_{02}/P_{01}, but P_{02}/P_{01} is an isentropic flow there is no change in stagnation pressure.

So, P_{02} is the same as P_{01} so I can write this as P_2/P_{01} and I know what the Mach number is. Which is 3.105 and for this Mach number what is P_2/P_{01}. I can get from the charts 0.232. So now moving on from this region we go to from region 2 we go to region 3.

(Refer Slide Time: 17:22)



From between region 2 and region 3 it is again another expansion fan. So same principle holds good ν_3 is $\nu_2 + \Delta\theta$, what is the change in angle if you use trigonometry this is 10, it is symmetric so this angle has to be 20^o, is from trigonometry you can find that out. So, $\Delta\theta = 20^o$. So, we know what is ν_2, we have just found out 51.757.

So, ν_3 turns out to be 71.757 corresponding Mach number M_3 is 4.49. So now we need P_3/P_{01}, again P_3/P_{01} is the stagnation pressure has not changed either in region 2 or in region 3 isentropic flow. So directly you can find P_3/P_{01} for 4.49, which is 0.003498. Now between region 3 and region 4 now that region there is an Oblique shock. So, it is an Oblique shock. Now the flow totally has deflected by 2 degrees and 20 degrees.

So, at the trailing edge the flow returns to the angle 0. So, the free stream angle so the flow must turn towards itself by 22 degrees. So, region 3 to 4 the flow deflection is θ is 22 degrees towards itself. So, it forms an Oblique shock Mach number is 4.49. So now use the $M-\beta-\theta$ charts find out what should be beta and it is 33.3 degrees. So now from this now what we must do we know beta.

So, we find M_{n3} is $M_3 \sin\beta$ and apply normal shock relations for M_3 which is 2.465 and also that M you will get M_{n4} from here M_{n4} and from M_{n4} you can get M_4,

$$M_4 = \frac{M_{n4}}{\sin(\beta-\theta)}$$

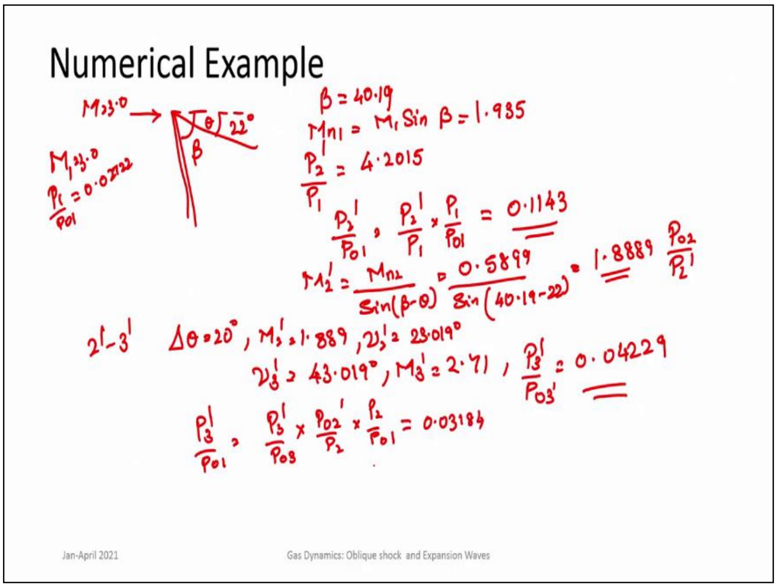
all of this from normal shock relations you will get M_{n4} if you know M_{n3} and from here we can get M_4, M_4 is 2.63. So obviously the Mach number has reduced.

So now what do we need here we need P_4/P_{01},

$$\frac{P_4}{P_{01}} = \frac{P_4}{P_3} \times \frac{P_3}{P_{01}}$$

P_4/P_3 is the pressure ratio across the shock, which you know here M_{n3} you know from this you can find out what should be the pressure ratio across the shock 6.916 multiplied by, $\frac{P_3}{P_{01}}$, 0.003498 and this turns out to be 0.024. So, bear in mind 0.024. Now we look at the lower side what happen in the lower side.

(Refer Slide Time: 21:47)



So lower side you have started off with the shock. So, it is something like this nature there is a shock here and the flow from the initial has deflected by 22 degrees. Oncoming flow Mach number is 3.0 and so beta for this shock wave angle beta so theta is known beta is 40.19. So, this angle is 40.99. So now we know beta so we can find out what should be

$$M_{n1} = M_1 \sin \beta$$

Mn1 this is 1.935, all properties of the shock like P_2/P_1 can be found out it is 4.2015 by using normal shock relations normal shock charts or calculators.

You can find these numbers and what we need again is P_2'/P_{01} is

$$\frac{P_2'}{P_{01}} = \frac{P_2{}'}{P_1} \times \frac{P_1}{P_{01}}$$

$\frac{P_1}{P_{01}}$ is nothing but for the upstream Mach number M equal to 3.0 what is $\frac{P_1}{P_{01}}$ this is known 0.02722. So, this value is 0.1143 and what is the Mach number. So, you got

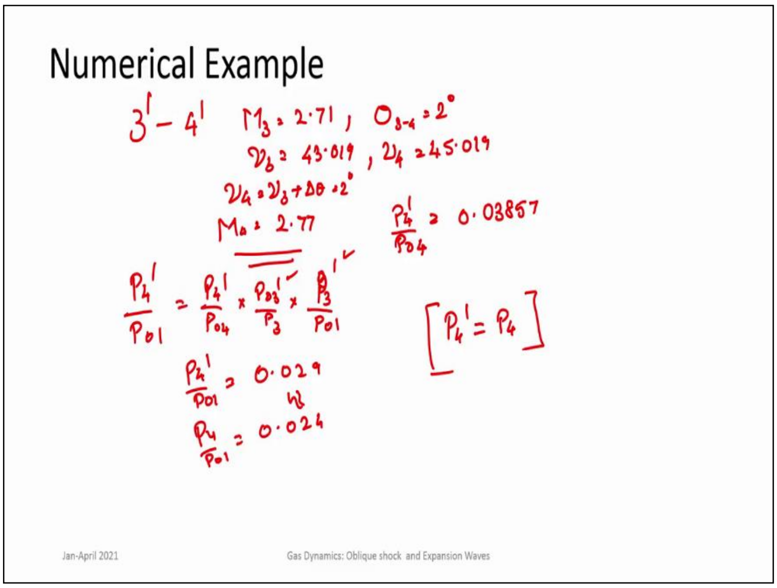
$$M_2' = \frac{M_{n2}}{\sin(\beta - \theta)}$$

5899 by sine of beta 40.19 minus 22 this is turns out to be 1.8889.

So now we go from region 2 prime to 3 primes on the lower side here it is an expansion fan delta theta is 20 degrees you start off with M_2' of 1.889 and here ν_2' is 23.019 degrees. So ν_3' then is 43.019 degrees. So M_3' turns out to be 2.71. So, we can find out what is P_3/P_{03} your stagnation pressure is going to be constant across the expansion fan.

And we need $P_3'/P_{01} = P_3'/P_{03} * P_{02}'/P_2 * P_2/P_{01}$ we have got P_2/P_{01} here and we know Mach number is 1.889 so we can find out what is P_{02}/P_2' that can be found out $P_{02} = P_{03}$. So P_3 prime by P_{03} can be known from it is given here can do the multiplication this turns out to be 0.03184.

(Refer Slide Time: 25:50)



Now at the final point between 3' prime and 4' it is an expansion fan. So, M_3 is 2.71 from the calculations and here the flow turns back by turns of a by 2 degrees. So, it turns away by 2 degree. So ν_3 = 43.019 so ν_4 = 45.019 which is $\nu_4 = \nu_3 + \Delta\theta, \theta$ is 2 degree here. So M_4 is 2.77 and we need P_4 prime by P_{01} this is P_4 prime by P_{04} P_{03} dash by P_3, P_3 dash by P_{01}.

So, these were calculated in the earlier steps P_4/ P_{04} you know 2.77. So for this Mach number P_4/ P_{04} is 0.03857 if you do the multiplication of all these numbers you get P_4 prime by P_{01} 0.029, P_4/ P_{01} previous calculation was 0.024 they are very close to each other we need so the idea is that pressures P_4' should be equal to P_4 we have of course the round off errors slightly but even considering them these are very, very close to each other.

Now if you know the pressures at each surface you can always calculate what is the force pressure multiplied by area is a force. So, you know pressure you have found out pressure at each surface now and from here you can calculate what is the force? So, force acting can be calculated at each surface and then they can be resolved into parallel and perpendicular components.

And therefore, you get lift and drag. So, this way of doing this kind of sort of estimating the forces is known as the shock expansion method I hope this elaborate example has given you an idea of how to use shock waves and expansion wave relations to solve certain problems. We will solve more problems down the line in the next few classes.