Gasdynamics: Fundamentals and Applications
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Lecture 27
Expansion waves

The last class we looked at Oblique shock waves, now in a two-dimensional flow and whenever there is a turn of the flow towards itself. It causes compression and Oblique shock forms. The way to analyse Oblique shocks is to look at components of velocity parallel and perpendicular to the Oblique shock, and component parallel to the Oblique shock velocity remains the same across the Oblique shock, while the perpendicular components are related by Normal shock relation. So that was the straightforward understanding of Oblique shocks if you understand that it is very easy to analyse Oblique shocks.

The other main feature is the way the Oblique shock behaves to changes in flow deflection which is given by $M - \theta - \beta$ relationship. So, there are attached Oblique shocks and detached Oblique shocks or detached shocks. Attached shocks are when the angle of deflection is less than the maximum angle for which an attached shock can be solved and those can be solved using the M-θ-β relation.

So, we saw all this in the previous class the counterpart to that is when the flow turns away from the mainstream, then how do we analyse these features or flow features? And then we saw that you get Expansion waves or Expansion fan. So how do we analyse expansion fans.

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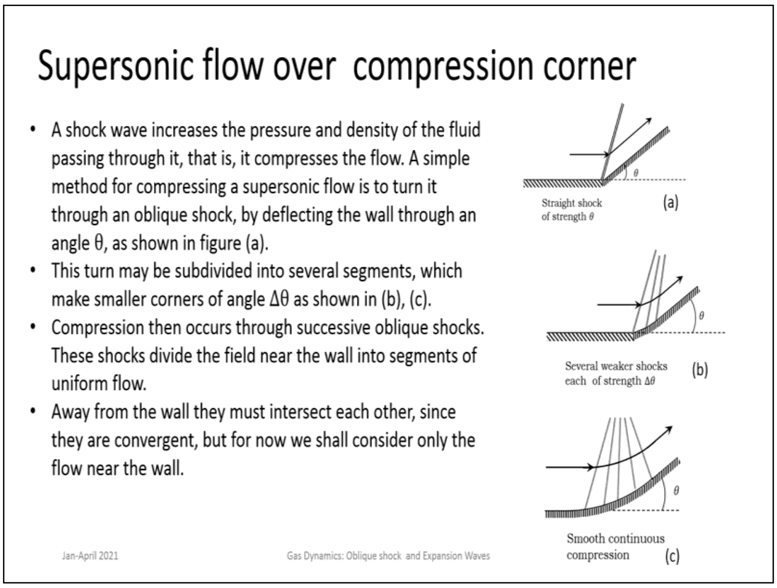

For this we begin initially with the flow undergoing a gradual turn towards itself and understand what happens as the Oblique shock become weak? So as Oblique shocks become weaker and weaker, they turn to be very close to Mach waves that we had discussed very early on and there the entropy changes across the Oblique shock become almost 0. So, this is what we had discussed earlier, which is the Oblique shock when there is a sharp turn by θ and an Oblique shock is formed.

Now if you take limits of this Oblique shock at the maximum limit, Oblique shock can have an angle 90 degree to the flow that is a particular form, that is the Normal shock. The flow deflection in this case is 0 that is absolutely normal, the second one is when the shock can go to very weak values, the minimum that it can go towards is a Mach wave so across a Mach wave also the flow deflection is almost negligible so you can take it to be 0.

The pressure ratio is 1, so the two limits for an Oblique shock are at the lower limit it tends to be a Mach wave and at the very strong limit it goes to a Normal shock. So, both are normal in the Oblique shock system itself. So now instead of considering such a sharp turn the total turn through angle θ can be done through a subsequent or sort of set of small changes small angles.

At each small turn you have an Oblique shock. So, you have an Oblique shock but now since the initial turn is very small the $\Delta\theta$ or small change then the shock wave strength is also small. So, the same turn θ is accomplished by many waves you can keep increasing the number of waves and make the turn more gradual. So, this is a very gradual turn smooth turn across to the angle θ. So now let us consider this problem a gradual turn.

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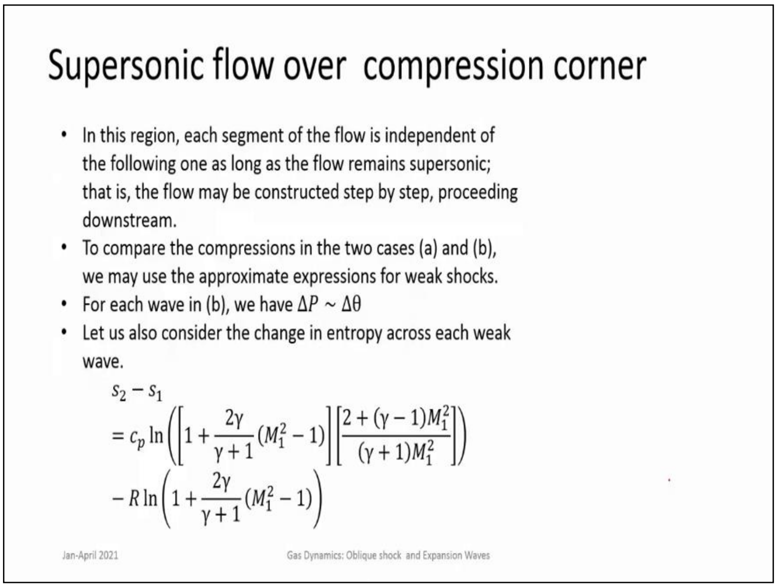

So, across each small turn there is an incremental change in pressure. So that is how it happens. There is a entropy change across the Oblique shock and the entropy change is the same as that for a Normal shock for normal component. So, that is what is written over here Δs for the Oblique shock, and this is for a Normal shock and here if you substitute instead our M_1 you substitute M_{n1} then you get it for an Oblique shock.

In the Normal shock discussions, we also saw what happens when the strength of Normal shock goes weaker and weaker which becomes very small it is called the weak shock limit.

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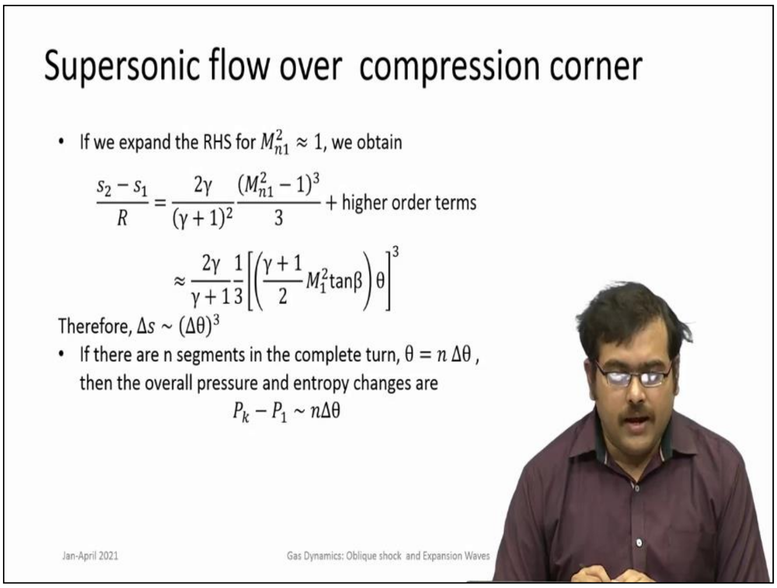

In weak shock limit we saw that the entropy goes extremely small and the way it goes is to the power cube. So, $(M_{n1} - 1)^3$ this was discussed in the context of Normal shocks. Now in the context of Oblique shocks, if you look at it now the same angle θ is divided into small turns of very, very small angle you consider many, many such turns.

Then across each very weak Oblique shock the change in entropy is given by this term basically this term or where this is now $\Delta\theta$. So, Δs goes approximately as $\Delta\theta^3$. So, you make θ smaller and smaller Δs is going to be smaller and smaller. Now what is $\Delta\theta$, so if you divid this turn into n number of turns then, θ is $\Delta\theta = \theta/\text{n}$.

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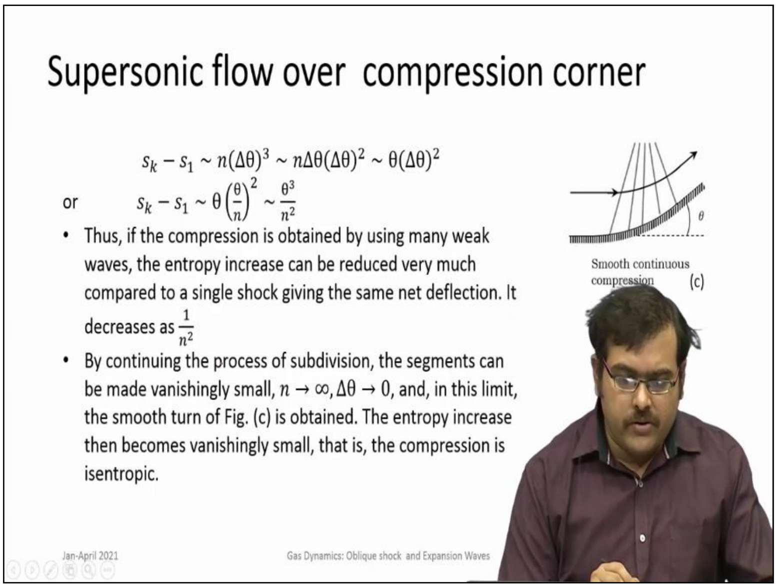

So, you can substitute that, and you get that the change in entropy Δs essentially goes as θ^3/n^2 for a gradual turn. Now we apply the state condition that n goes to infinity if you want to make it a smooth and continuous turn. As n goes to infinity Δs goes to 0 or you are getting an isentropic flow. So, a gradual turn very close to the wall if you look at it an isentropic flow and each wave is actually a very, very weak Oblique shock in the limit of weak Oblique shock its nothing but a Mach wave.

So, you have several Mach waves. So, this gives us the idea how to go ahead and analyse Expansion waves because you cannot have an Expansion shock. So, when you have to turn the flow away from the wall the flow has to accelerate. But pressure, density, and temperature will decrease but that cannot be accomplished by a sharp discontinuity like the Oblique shock. But this small analysis on gradual turns gives you an idea that if the chain flow happens to change gradually then the flow is essentially isentropic.

Then there is no problem with violation of second law of thermodynamics and so on. Therefore, you can turn the shock or they turn the flow away from itself and this is accomplished.

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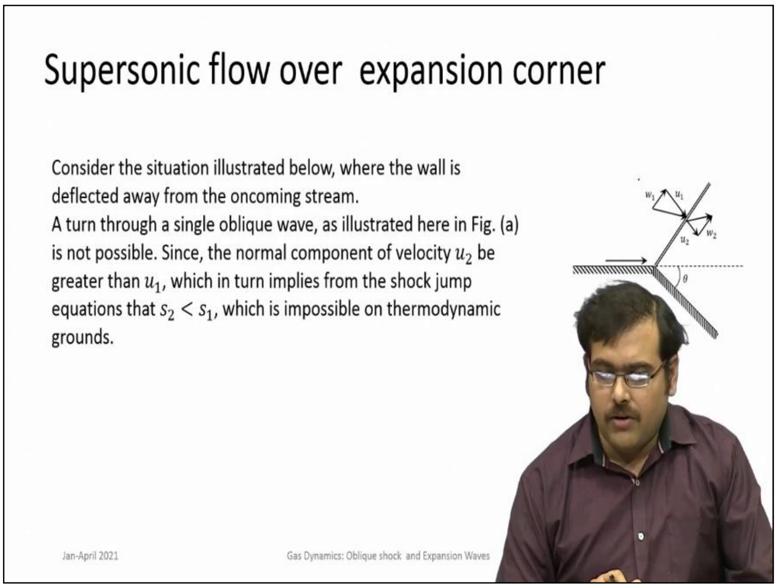

Always the flow turns away from the flow, that always happens by means of a gradual turn. So, a sharp discontinuity of this sense is not possible, that is an Expansion shock is not possible because always you should have positive change in entropy.

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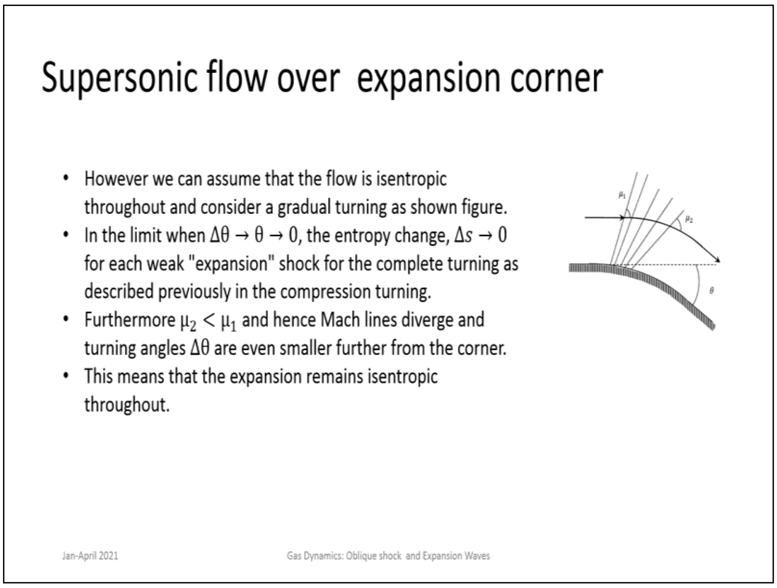

So, what is possible is a gradual turn or gradual change in flow. This is gradual turn away from the flow and the flow remains isentropic and it is bound by Mach waves. Now these are Mach waves. So, this is starting Mach wave ν_1 this is at the end ν_2 and since they are related υ is

$\frac{1}{\sin(\frac{1}{M})}$. So, Mach number is increasing across the turn, therefore you have ν_2 is always less than a new one.

Now in the limit what is considered is this gradual turn this corner can be sharp. So, it can be a sharp corner but the turn actually happens very gradually in as a series of waves, series of Mach waves, do the turn to the flow. Only at this corner you have where all the points are starting.

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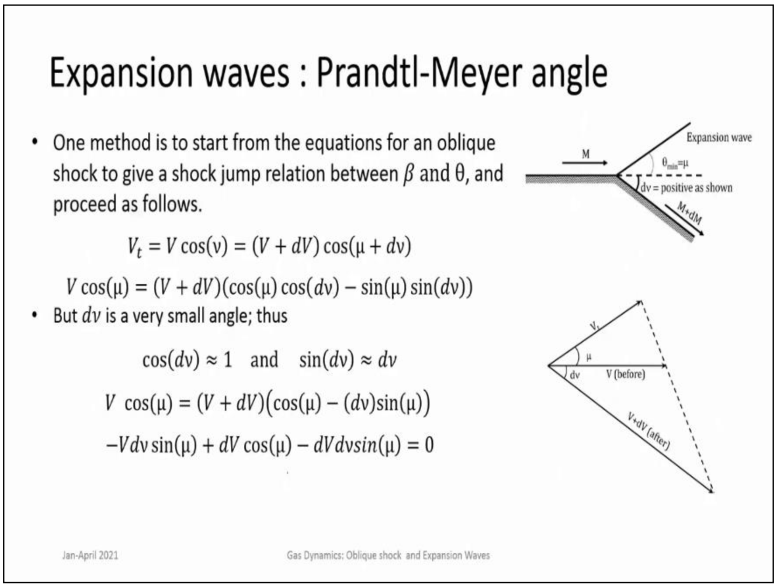

So that is this kind of an analysis is done by Prandtl and Meyer, so this is known as Prandtl-Meyer function or Prandtl-Meyer angle. How do we do this analysis? So, we again draw the velocity triangles across one single expansion wave across a wave very similar to the analysis that was done earlier for the Oblique shock. We look at the components of these velocities.

So, before that is the upstream of the expansion wave you have the velocity V that is drawn over here. Corresponding to this V or Mach number M you have a Mach wave, so this is the Mach wave at an angle 'υ'. Now with respect to this Mach wave you draw all the others. So as the float goes through the microwave it undergoes a very small change in angle that is $d\upsilon$.

A small change in velocity V+ dV, but again the concepts of Oblique shock remain that is the tangential component is remaining conserved. So, V_t is the same what is V_t? V_t is V cos υ, so that is what is V cos mu is the same as V + dV cos $(\mu + d\upsilon)$. So, $\mu + d\upsilon$ that is what is written over here conservation of the tangential component.

Now you know that ν is very, very small is almost tending to 0 it is very small. So, cos ν is 1 and sine $d\nu$ tends to $d\nu$, expand cos $(\mu + d\nu)$ using trigonometric relations and you get $'\cos\mu\cos d\nu - \sin\mu\sin d\nu'$ and use these relations. So, you get (P+dv)cos $\mu - d\nu$ sin μ, so you get these terms. From here there is a V cos μ term here and a V cos μ term that comes over here.

So, they cancel each other, and you have this term dV dν sin μ both are small changes. So, this is a very high order term small changes if you multiply together, it becomes very small. So, this term is also neglected.

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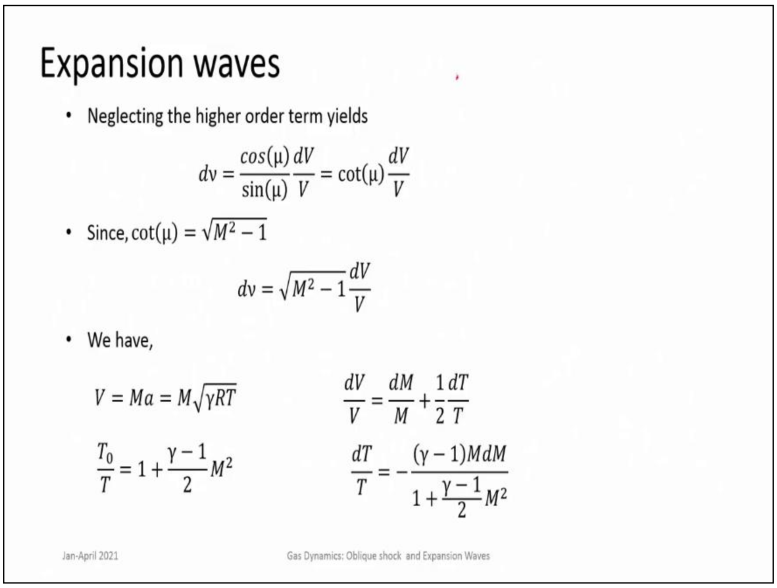

So, if you then write down this equation what you get is d nu is equal to cot μ (dV/ V). So, this is the equation. So, what is cot μ? You can draw the angles, so this is what is μ, μ is $\frac{1}{\sin(\frac{1}{M})}$.

So, this side is $M^2 - 1$, so cot μ is $\sqrt{M^2 - 1}$. So that is given here. So now this expression relates $d\nu$, which is known as nu is the Prandtl-Meyer function to the change in velocity dV/V.

Now the idea is can we express dV/ V in terms of Mach number dM, can this be done? we can do it because V is Mach number multiplied by speed of sound and that is V is $\sqrt{\gamma RT}$ these are constants, if you take logarithm and differentiate dV/V is dM/M + dT/2T.

Now can we express dT/ T in terms of dM/ M we know that this flow is isentropic. So, for an isentropic flow $T_0/T = 1 + \frac{\gamma-1}{2}M^2$, this we know from first principles from the earlier classes. Again, you can use the same kind of logarithm differentiation T_0 is constant. So, this is essentially $dT_0/T_0 - dT/T$.

So you get dT/ T this is 0 so dT/ T is minus gamma minus 1. So, you can put this relationship into there.

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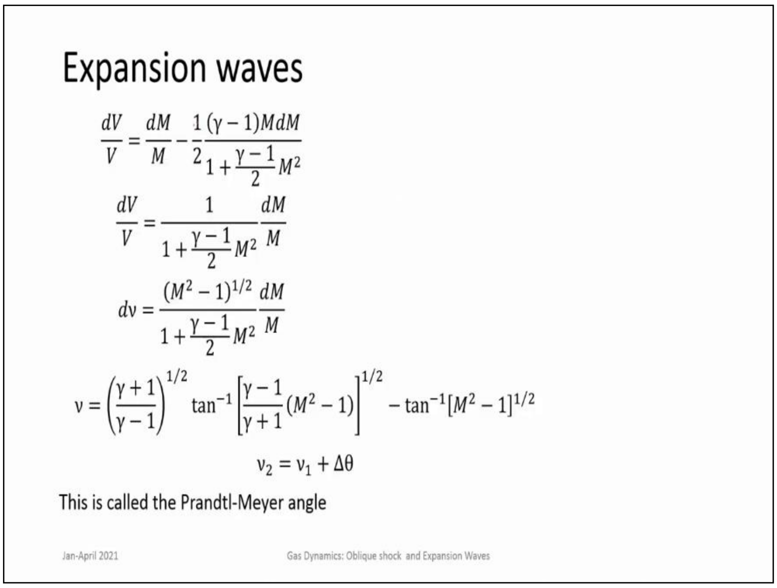

You get the M/(M-1/2) and then do the algebraic manipulations you get dV/ V is $(1 + \frac{\gamma-1}{2}M^2)\frac{dM}{M}$ and this is substituted and so you get expression for the Prandtl-Meyer function explicitly in terms of gamma and M this is a calorically perfect gas gamma is constant this can be integrated, and this is the integration for. So, this is the Prandtl-Meyer angle.

Once you integrate this it is the Prandtl-Meyer angle. Here we take so of course you will have a constant the constant is taken that nu equal to 0 at Mach number equal to 1. So, when Mach number is one the Prandtl-Meyer angle is 0. So once that is taken you get this relationship. So, this is given in any charts you can get this, and this is a Prandtl-Meyer angle. Now how is Prandtl-Meyer angle related to the change in angle during a flow turn.

So, the flow turns by an angle delta θ. So, before this flow turns the Prandtl-Meyer angle is ν_1 and this is ν_2 and ν_2 is $\nu_1 + \Delta\theta$. During a float on away from the upstream flow Prandtl-Meyer angle increases. So that is what you must consider.

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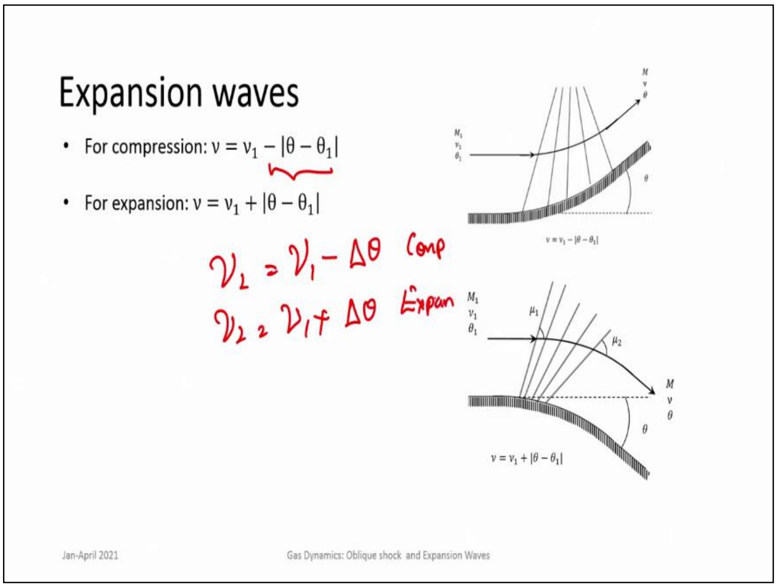

During a turn, a gradual turn so, a compression turns the Prandtl-Meyer angle decreases. So, it takes a negative sign minus delta theta. So, you should always remember these. So, for gradual turns for both compression and expansion you can use the Prandtl-Meyer analysis and you can relate the downstream and upstream conditions. So, the relationship is $\nu_2 = \nu_1 - \Delta\theta$ for compression and $\nu_2 = \nu_1 + \Delta\theta$ for expansion. Expansion or turn away from.

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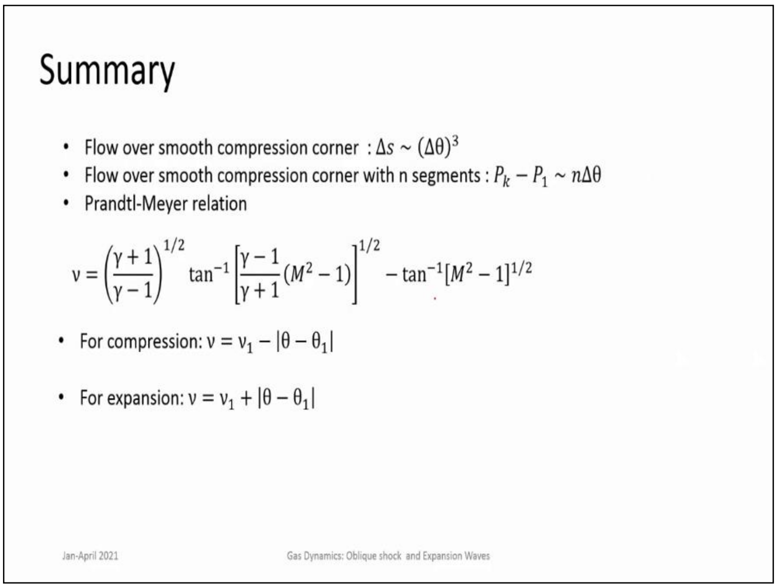

So, what we saw this is here is the analysis of expansion waves and we saw the way to look at this problem is starting from a smooth turn. And show that at the weak limit of Oblique shocks you get Mach waves and entropy change is negligible there. So, you can accomplish the same turn by series of Mach waves which is what is done in case of an which is the case of an expansion fan where the flow turns away from the incoming flow then the Mach number increases pressure temperature density decrease.

How to calculate pressure temperature density? Very simple it is an isentropic flow use isentropic relations. So, from using those concepts we came to the Prandtl-Meyer analysis and the Prandtl-Meyer that relates the Prandtl-Meyer angle and these are the relations for a smooth compression or an expansion. So now we will apply these in certain flow scenarios we will start with we can couple shocks and expansions and look at how pressures behave over bodies.

And then the other point is what happens when the shocks they are present in the flow and they interact with a wall or they interact with virtual surfaces in the flow which have constant pressures. What do happen then? So, these are the questions we will answer in the coming classes.