Gasdynamics: Fundamentals and Applications
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Lecture 24
Shock Tube Relations

So we are looking at the shock tube problem and we have looked at waves in this context. The motion of waves inside a shock tube. This particularly important to the driver section where expansion waves move in decreasing the pressure. While on the right hand side that is towards the driven section, you have a shock moving in at supersonic speeds compressing the driven gases to form a slug of gas which moves at speed u_2.

In between these 2 there is a contact discontinuity that separates the driver gas from the driven gas. And we are looking at the right hand side is the moving shock problem which has been already discussed. We are looking at solving the left hand and then assembling them together. Till now we have discussed the various waves problems infinitesimal waves finite waves. And in the finite waves we came to the method of characteristic solutions of the equations.

And we said that the way these waves move these finite waves move is that they move along lines $\frac{dx}{dt} = u + a$ and $\frac{dx}{dt} = u - a$. These are the C^+ and C^- characteristics respectively. And along C^+ characteristics there is a constant, it is a Riemann constant J^+ which is

$$J^+ = u + \frac{2a}{\gamma - 1}$$

and along C^- characteristic there is another Riemann constant J^- which is

$$J^- = u - \frac{2a}{\gamma - 1}$$

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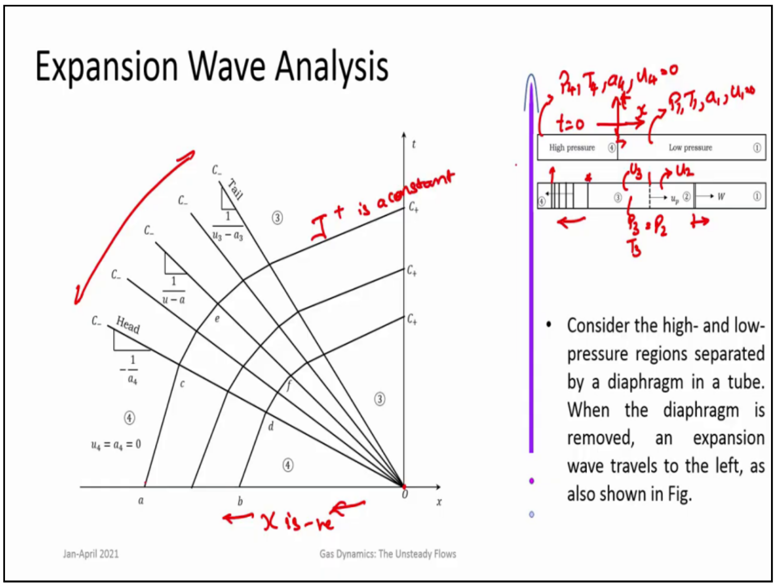

And now if we want to solve any wave flow or wave features flow feature then we have to see the total of both C^+ and C^- characteristic. Towards the end of the class we had discussed that an important behaviour is that when you have a region of non uniform flow between 2 uniform flows then the one set of characteristics become straight lines while the other set remains curved and this is very much the case for the expansion wave.

So you look at this expansion wave. So this is what is = 0, u is 0 here so this is P_4, T_4, a_4, u_4 =0 similarly this is P_1, T_1, a_1 and u_1 = 0. The moment the interface ruptures diaphragm ruptures you get a shock moving in the driven section, expansion waves moving in the driver section. And you have the contact discontinuity which separates the driver side from the driven side.

Shock causes a mass motion which is u_2 and across the contact discontinuity you need these 2 are the same P_3 equal to P_2, u_3 equal to u_2. So velocities are the same so here region 3 is a uniform region of u_3, P_3, T_3. Similarly region 4 that is here is a uniform region which is quiescent P_4, T_4, u_4 = 0. Now this particular wave which travels first into the region 4 is the head of the expansion. So this is the head of the expansion wave.

And the wave that travels the last is the tail of the expansion wave and if you plot now the characteristics for this starting from this was the initial starting point t = 0. So you have this now this is you should understand, now we are dealing with; so this is the convention this is x and t is going right up here. So x is positive going right from the diaphragm station.

So this part of the x is negative. So these are C^{-} characteristics so this region 3 uniform flow region 4 uniform flow. Now from our previous understanding you have the C^{-} characteristics which are straight lines that you can see over here they are spread out like a fan and C^{+} characteristics are the curves here. So they become curved moment they enter the uniform region they become straight again.

So this is the understanding, now we have to put things together. So we have been doing the things in parts. Now we will put it together along with the shock tube problem. So how do we solve along a particular C^{+} curve. So this is a C^{+} curve along this curve this joins u_4 region 4 and region 3 J^{+} is a constant. So along this J^{+} is a constant. So we use this particular approach , J^{+} is constant.

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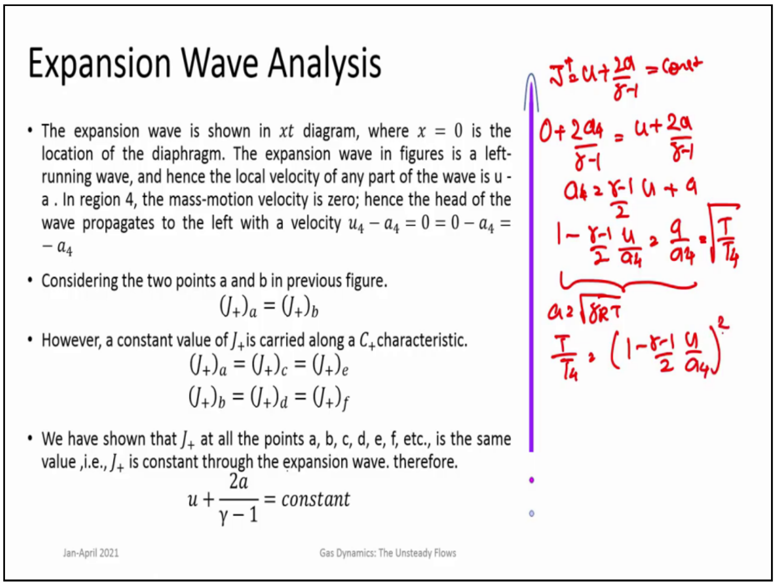

So $(J^{+})_a = (J^{+})_b = (J^{+})_c = (J^{+})_e$, so is carried along a particular J^{+} . So along see a Particular C^{+}

$$J^{+} = u + \frac{2a}{\gamma - 1}$$

. This is constant. Now if you take in region so you go back to this picture and you stake in region 4, u_4 , a_4 is not 0 only u4 is 0. So you are actually looking at, you are looking at this wave $u_4 + a_4$, this is $\frac{dx}{dt} = u_4 + a_4$ that means here it travels only with a_4 .

Similarly C^{-} travels with minus a_4 . or $-\frac{1}{a_4}$. so you have $u_4 = 0$ and a_4 is there so $J^{+} = 0 + \frac{2a_4}{\gamma - 1}$ 0 is equal to in any at any other point along this any other point if you consider any other point then in general this is $J^{+} = 0 + \frac{2a_4}{\gamma-1}$. So you can now relate say a and u how they vary with each other and you can do that so you can take this so $a_4 = \frac{\gamma-1}{2}u + a$

Or $1 - \frac{\gamma-1}{2}\frac{u}{a_4} = \frac{a}{a_4}$

So you get this equation now $\frac{a}{a_4}$ is nothing but $\frac{a}{a_4} = \sqrt{\frac{T}{T_4}}$ because a is of course a = $\sqrt{\gamma RT}$ and we are considering calorically perfect gases so you get,

$$\frac{T}{T_4} = \left(1 - \frac{\gamma-1}{2}\frac{u}{a_4}\right)^2$$

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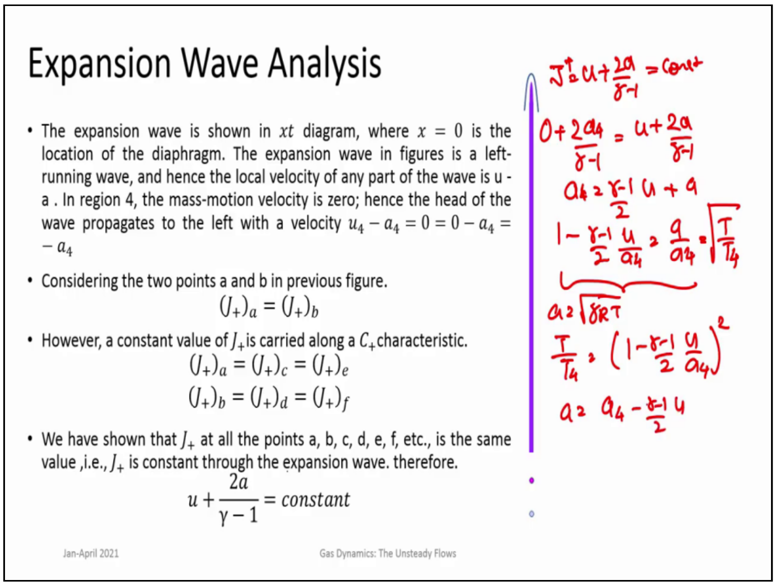

So using the J^{+} characteristics, now we are in a position to relate the flow variables. Pressure along the expansion fan, temperature along these, if you know temperature along the expansion fan, $\frac{T}{T_4} = \left(1 - \frac{\gamma-1}{2}\frac{u}{a_4}\right)^2$. Now $\frac{P}{P_4} = \left(\frac{T}{T_4}\right)^{\frac{\gamma}{\gamma-1}}$ because this is an isentropic process. So that is how you get these relations $\frac{P}{P_4}$ is $\frac{P}{P_4} = \left(1 - \frac{\gamma-1}{2}\frac{u}{a_4}\right)^{\frac{2\gamma}{\gamma-1}}$,

Similarly you get $\frac{\rho}{\rho_4}$. $\frac{\rho}{\rho_4} = \left(1 - \frac{\gamma-1}{2}\frac{u}{a_4}\right)^{\frac{2}{\gamma-1}}$. So, now we have got the solution of pressures and temperatures in terms of u and a_4 is a constant here. So a_4 is the initial condition. Now how do we relate u? So for that you go to the C^{-} characteristics are $\frac{dx}{dt} = u - a$ so you get x = $(u - a)t$. But if you look at how these characteristics are the C^{-} characteristics they originate from 0 that means they start from x= 0, t= 0.

So while integrating this, $\frac{dx}{dt} = u - a$, x = $(u - a)t$ +constant , this is 0 only then it satisfies that condition. So you x = $(u - a)t$. So from here we can express a, u in terms of $\frac{x}{t}$. That is $\frac{x}{t} = u - a$ or $u = \frac{x}{t} + a$ and what is a? a is can be related because we just did the relation here where $a = a_4 - \frac{\gamma-1}{2}u$ .You can do the substitution here.

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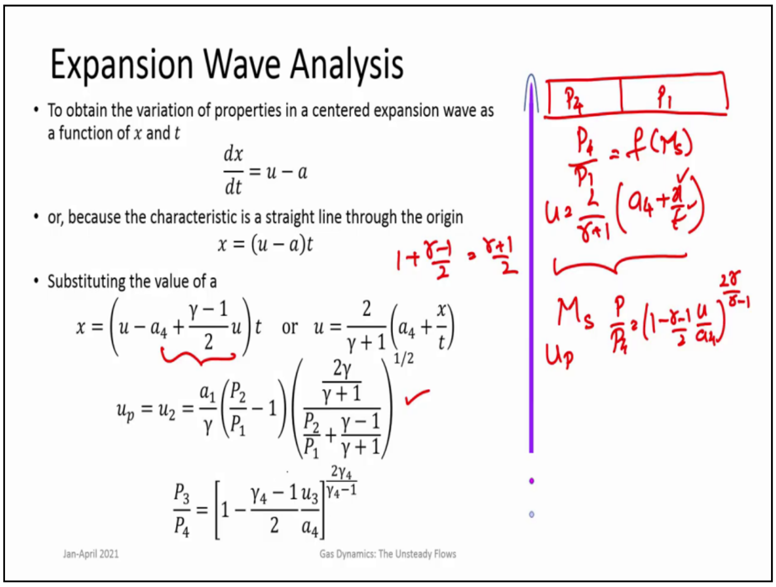

So you can do the substitution here. So that is what is done over here x = $(u - a)t$. So a is substituted by $a = a_4 - \frac{\gamma-1}{2}u$. x = $(u - a_4 - \frac{\gamma-1}{2}u)t$ So here you have $1 - \frac{\gamma-1}{2} = \frac{\gamma+1}{2}$ and then solve for u, you get , $u = \frac{2}{\gamma+1}(a_4 + \frac{x}{t})$. So now u can be completely represented in terms of the space and type coordinates. So you can look at the evolution of the expansion waves starting from the point 0 when that particular large pressure differential was imposed between the driver section and driven section.

So this gives the evolution of pressure, temperature, velocity where here x and t are the variables that come into picture. Now our core understanding that we want from here is if you are going to use a shock tube for certain applications then we would like to know what should be the conditions that need to be given so as to achieve a particular shock Mach number. So that is because that is the gas that we are going to use for any of the experiments or any applications.

So if I know that a certain shock mach number has to be produced, Ms. Then what should be the pressure ratio that I should give across the shock wave shock tube. So this is P_4 and P_1. So how is P_4 and P_1 or I can write $\frac{P_4}{P_1}$ how is it related to shock mach number ? This is what we need to do. So now we have done the moving shock analysis. We have done the expansion wave analysis.

And we know that shock tube consists of all these regions. They are connected by the contact discontinuity with the boundary condition that pressures are the same and velocities are the same across the contact discontinuity. If you put all of them together then we can get an expression for $\frac{P_4}{P_1}$ in terms of original shock strength or the shock strain that is required Ms. So how do we go about doing that.

We need to know what is the mass motion of gas behind the shock u_p? u_p is equal to u_2 . $u_p = u_2 = \frac{a_1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left(\frac{\frac{2\gamma}{\gamma+1}}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right)$. This comes from our moving shock analysis that we did in the previous module. So we have taken that result directly. Now from the expansion wave analysis we know that $\frac{P}{P_4}$, $\frac{P}{P_4} = \left(1 - \frac{\gamma-1}{2}\frac{u}{a_4}\right)^{\frac{2\gamma}{\gamma-1}}$. This is known. Now if we connect the region 3 and region 4 of the shock tube.

If we connect them then this becomes $\frac{P_3}{P_4}$ and here u becomes u_3, $\frac{u_3}{a_4}$ and the relevant γ here is of the region 4 which is γ_4. So that is how this particular term comes about,

$$\frac{P_3}{P_4} = \left(1 - \frac{\gamma_4-1}{2}\frac{u_3}{a_4}\right)^{\frac{2\gamma_4}{\gamma_4-1}} .$$

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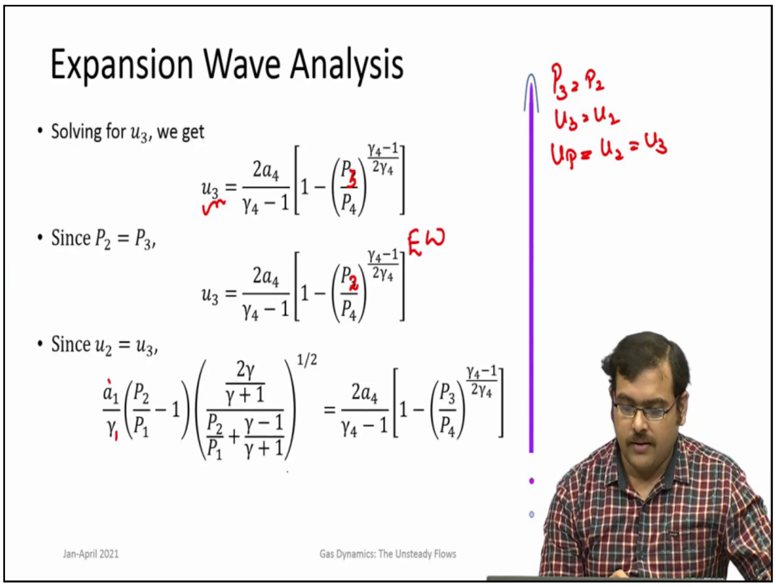

Now what is the boundary condition across the contact surface? Across the contact surface we have P_3 = P $_2$ and $u_3 = u_2$. Now whatever we just now described u_p is actually u_2 the same as u_3. So that is what is given Now from $\frac{P}{P_4}$ from this particular expression from this expression we can solve for u_3 . What is the velocity u_3, $u_3 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{P_2}{P_4}\right)^{\frac{\gamma_4-1}{2\gamma_4}}\right]$

It is actually $\frac{P_3}{P_4}$ but 2 and 3 are the same. So you can use either P_2 or P_3. So we can give this to be $\frac{P_2}{P_4}$ So u_3 you got from the expansion wave analysis. This comes from expansion wave analysis. While the u_2 that you got here this comes from moving shock analysis moving shock analysis. So this is u_p or u_2 but $u_2 = u_3$. So these 2 parts must be equal.

So you put one to be equal to the other and now you get a relationship where you have P_4, P_3, P_2, P_1, a_4 and a_1. So you have all this and you have γ_1 and γ_4. This has to be sort of algebraically rearranged so that you can write $\frac{P_4}{P_1}$ on left hand side. This can be done, it can be rearranged.

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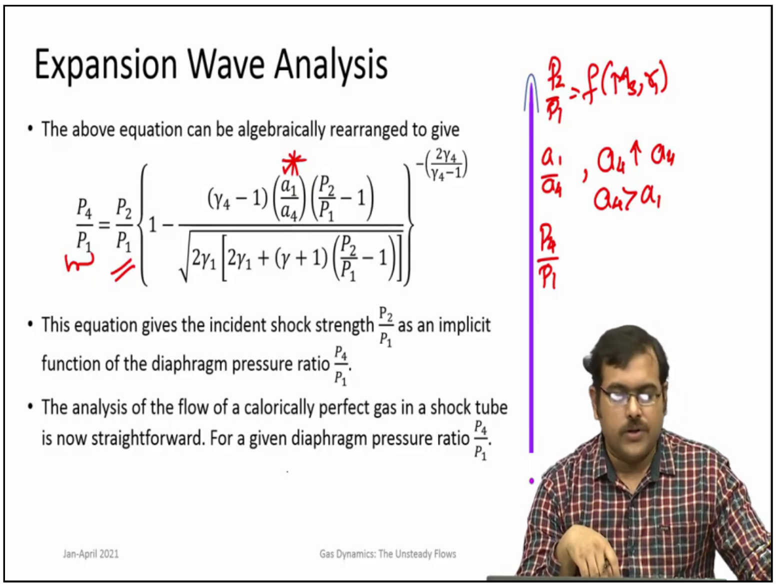

And $\frac{P_4}{P_1}$ comes on the left hand side and right hand side is all functions of $\frac{P_2}{P_1}$ which is a strength of shock. So $\frac{P_2}{P_1}$ is a function of shock Mach number and γ only. This is γ_1 .

You have, $\frac{P_4}{P_1} = \frac{P_2}{P_1}\left[1 - \frac{(\gamma_4-1)\left(\frac{a_1}{a_4}\right)\left(\frac{P_2}{P_1}-1\right)}{\sqrt{2\gamma_1\left[2\gamma_1+(\gamma_1+1)\left(\frac{P_2}{P_1}-1\right)\right]}}\right]^{\left(\frac{-2\gamma_4}{\gamma_4-1}\right)}$

So if you want shock Mach numbers to be very high then $\frac{a_1}{a_4}$, this term should be quite small or in this term if it has to be small then a_4 should be larger than a_1. a_4 should be greater than a_1. And this is a guiding principle when we look at shock tube applications as well as experimentation. We always like to keep the driver side the speed of sound higher than the driven side speed of sound.

And this is accomplished by many means you can get higher a_4 by using a different gas, a lighter gas and usually helium is used to get stronger shock waves with the given pressure ratios. You can only achieve certain $\frac{P_4}{P_1}$ which is limited by the kind of pressure vessels that one has in so or you need other kind of approaches which is heating the driver gas. The heating can be done a priory or it can be done in situ.

That is as the experiment is being done some kinds of heating by adiabatically compressing a gas through a piston such kind of shock tubes are called free piston shock tubes or you can you even heat up the gas by combusting certain inflammable materials. So you have a combination of gas and oxygen and they can burn together and heat up the gas that is the driver gas though those kinds of shock tubes are known as combustion driven shock tubes.

So these are some ways by which this ratio $\left(\frac{a_1}{a_4}\right)$ can be modified and hence one can get higher strength of $\frac{P_2}{P_1}$ for given $\frac{P_4}{P_1}$. So this gives you ,so now this is what we want basically a relationship that tells us if we know the strength of shock that we require for our application what should be the pressure ratio to be given across the shock tube.

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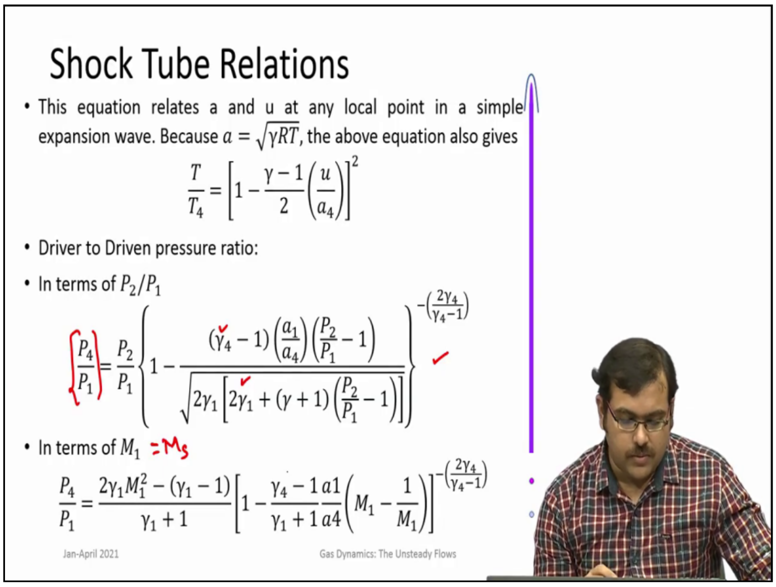

So how do we go about now this entire shock tube analysis? So we have to combine all things together now. So if we know Ms we can calculate the shock Mach number if you know you can calculate $\frac{P_2}{P_1}$ and from $\frac{P_2}{P_1}$ you can calculate $\frac{P_4}{P_1}$. If you know $\frac{P_2}{P_1}$ you also can calculate what is the speed of the shock, temperature ratio, density ratio as well as the u2 or contact surface speed or speed of gas behind it, u 2. $\frac{P_3}{P_4}$, P_2 and P_1.

Now P_2 and P_3 are the same. So $\frac{P_3}{P_4}$ is the pressure ratio across the expansion fan. So along the expansion fan the process is isentropic. So you can derive the expressions for the variation of flow variables across the expansion fan. And local properties are related to $\frac{x}{t}$ by the simple expression. So you know u is a function of $\frac{x}{t}$. So that can be used and we can use that to find $\frac{T}{T_4}$. Once $\frac{T}{T_4}$ is known we can get pressure ratios and temperature ratios across the expansion fan.

So if you look at all the shock tube relations they can be expressed either in terms of $\frac{P_2}{P_1}$ or in terms of the primary shock Mach number that is M_1 or it is also known as Ms sometimes called as Ms. So all the equations can be represented now in terms of $\frac{P_2}{P_1}$ or Ms. So this is the pressure ratio the initial pressure ratio across the shock tube expressed in terms of $\frac{P_2}{P_1}$.

And the same thing expressed in terms of Ms.

$$\frac{P_4}{P_1} = \frac{2\gamma_1 M_1^2 - (\gamma_1 - 1)}{(\gamma_1 + 1)}\left[1 - \frac{(\gamma_4 - 1)}{(\gamma_1 + 1)} \frac{a_1}{a_4}\left(M_1 - \frac{1}{M_1}\right)\right]^{\left(\frac{-2\gamma_4}{\gamma_4 - 1}\right)}$$

So one has to be very careful with what γ have to be put where because in general the 2 sides of the shock tube need not have the same gas. So need not have the same γ and need not have the same acoustic speeds also. So one has to be careful when look using these equations.

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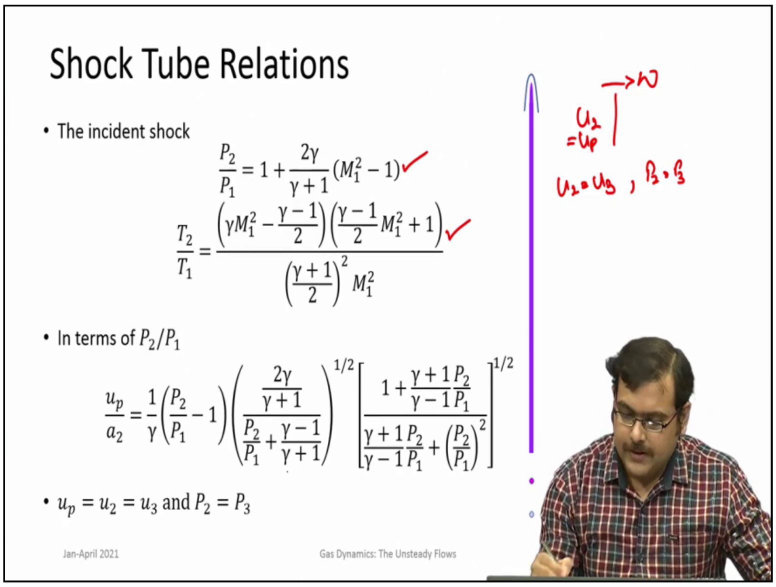

So incident shock the pressure ratio across the incident shock is this is the same result from a normal shock analysis. Similarly the temperature ratio this is because you can solve the incident shock by moving along with the shock and in the frame of the shock the shock equations become stationary and therefore you can solve them. And another important variable is the mass motion behind the shock.

So mass motion behind the shock as this goes with W you have u_2 which is also termed as u_P. And u_P can be written in terms of $\frac{P_2}{P_1}$ and uP for a shock tube u_2 is also is same as u_3 and P_2 is same as P_3. So this is the equation for $\frac{u_p}{a_2}$. So the Mach number now u_2, M_2 is actually $\frac{u_2}{a_2}$.

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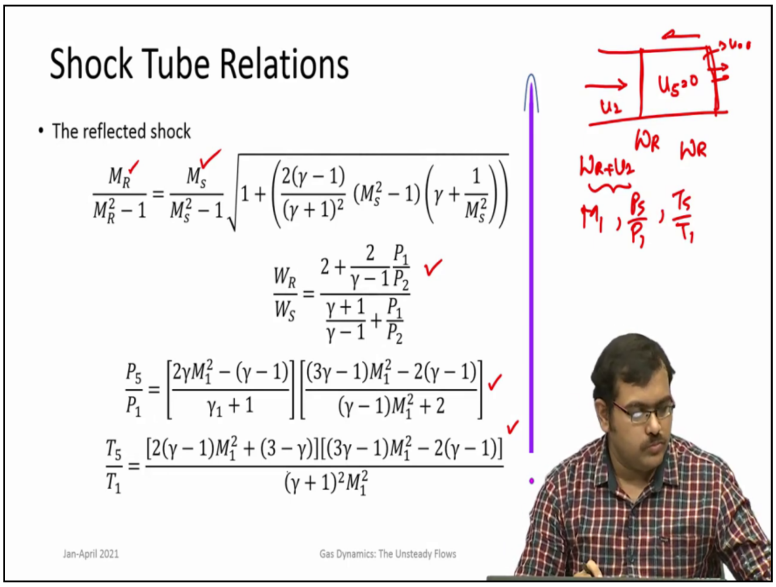

Now the other case that we have to consider is there is a possibility that the shock can go all the way to the end of the wall and there at the end of the wall suddenly abruptly the velocity has to be brought to 0 which is the no penetration condition , it is an end wall. So there is no penetration inside. So u has to go to 0 immediately when the shock comes and hits the end wall. And this is accomplished by reflected shock WR where u_2 is what is coming into the reflected shock and u_5 is 0.

So this is again a moving shock this moving shock can be analyzed. W_R + u_2 will be the relative velocity. So and here you will have W_R as the relative velocity and you can do the analysis for this shock and this was done the previous module. And we take the results directly from the previous module the reflected shock Mach number can be represented in terms of the incident shock or the primary shock.

And also it can be represented in terms of $\frac{P_1}{P_2}$ and the pressure ratio similarly can be represented in terms of the incident shock. So here also the incident shock is what plays the role. So M_1 if you know M you can get $\frac{P_5}{P_1}$ and $\frac{T_5}{T_1}$.

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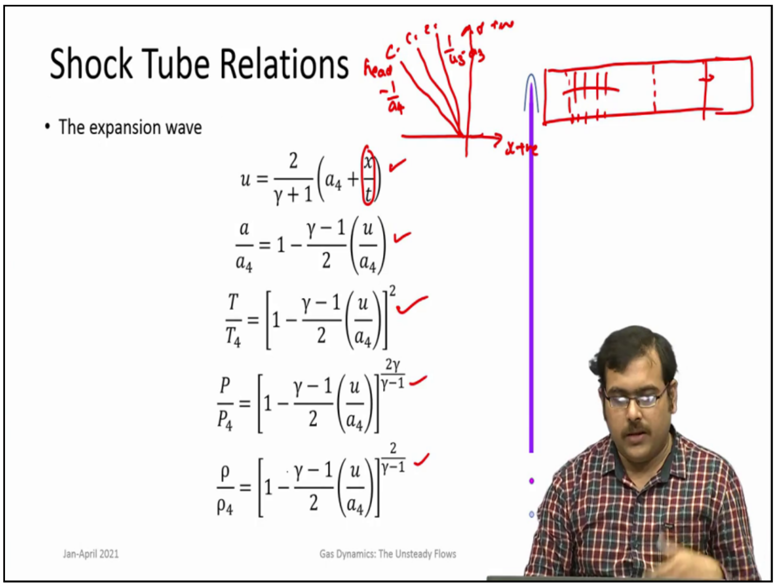

So, to put; now if you look at the entire shock tube. So you have you got the equations for shock wave you got the equations for the contact surface. Now we have the equations for the expansion fans which come here. So the velocity in the expansion fan it varies gradually that is a function of $\frac{x}{t}$. You have to bear in mind that the xt diagram for an expansion fan is like this.

This is x is positive t positive and the expansion fan appear like C^- characteristics. So this is the head the head of the expansion fan goes at a speed $\frac{-1}{a_4}$, tail of the expansion fan goes at a speed one by u minus a_3 is the slope or it goes at $\frac{-1}{u_3 - a_3}$. This is the tail of the expanded fan. So and you can relate a_4 and a with respect to u. So once u is known all these variables pressure, temperature, density can be recovered.

So now let us put all things together now we have solved the entire shock tube problem and let us see how things go if we look at the shock tube.

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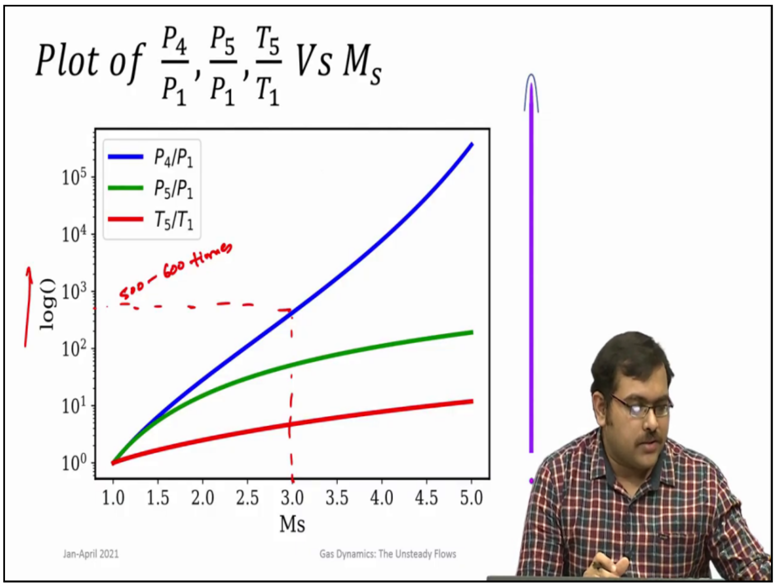

So this is a plot of $\frac{P_4}{P_1}$, $\frac{P_5}{P_1}$ and $\frac{T_5}{T_1}$ in terms of shock Mach number this is what is desired. So, we need if you need a certain $\frac{P_5}{P_1}$ or $\frac{T_5}{T_1}$ then what should be the shock Mach number that needs to be given and correspondingly what should be the pressure ratio across the shock tube that needs to be given. So if you do find and calculate that you can see that the scale is in log.

That is because these values really take off exponentially very, very quickly they rise. So if you see the Mach number has just risen from of course the shock is always supersonic. So starting from just beyond Mach 1 even when you reach values of say Mach 3 the kind of pressure ratio that you have to provide to achieve a shock Mach number is almost of the value of around; so it is Mach 3 and it is around say 500 to 600 bar or 600 times.

So it is quite huge pressure ratio that you have to provide in order to give a certain shock strength. Of course this is drawn for air air with driver and driven as air. But you can always in actual shock tubes when people use it for experimentation they use lot of methods like either heating or using free piston or combustion or using very easily you can do it by using helium as a driver.

And one more aspect to shock tubes is that you get significant increase of temperature with pressure. If you take the same pressure ratio and you consider a compressor which is more or less it functions very close to an isentropic process, compared to a shock tube because a shock produces large increase in entropy. So efficiency wise the shock tube is not a very efficient device because it produces lot of entropy in the shock.

But if you want to very quickly produce high pressures and high temperatures then shock tube is a good device. So it has a very effective in producing very high pressures and high temperatures. For the same pressure ratio if you look at the temperature ratio produced by an isentropic compression the temperature ratio will be less. This is particularly useful for certain aerodynamic testing at very high enthalpies and high Mach numbers typical to hypersonic flows. So that is where this is quite useful.

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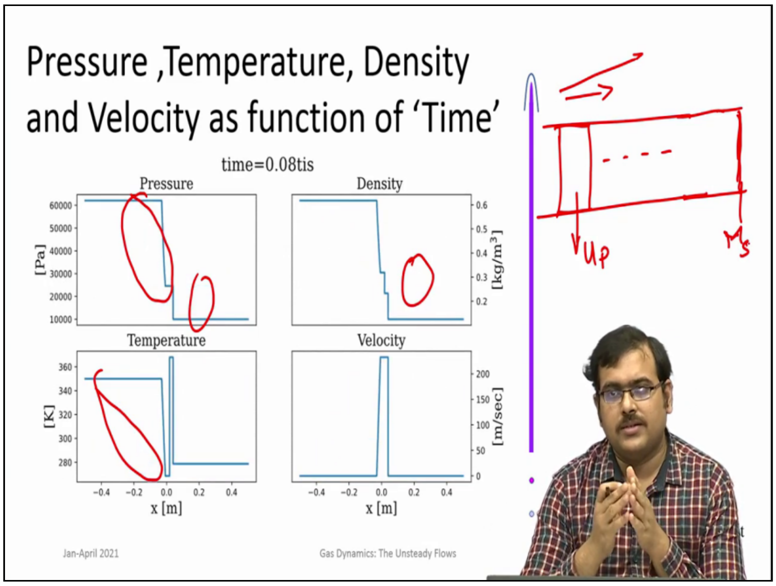

So let us look at how the all the parameters change during the shock. So I will pause here for a while and look at the analytical solutions, exact solutions that we have and as the shock propagate how these variables pressure, density, temperature and velocity change. Of course this includes a small portion where there is a reflection of the shock also. So and time as it goes you can observe the shock is moving you have the contact discontinuity here and then the reflected shock.

This is the primary shock, this is contact discontinuity and reflected shock. And you should also observe that the expansion fans are very gradual. So they gradually change the pressure and temperature. So with this we have looked at the shock tube problem in great detail. It is a template problem for unsteady gas dynamic flow. Several concepts that is important here.

Moving shock, reflected shock, the boundary conditions across a contact discontinuity. Then the analysis of isentropic waves like the expansion wave and you could also get isentropic compression waves in particular problems. For example, in the initial portion of having a piston, a piston moving in a long tube, if you have a piston moving in a long tube and initially when this is gradually accelerated then the piston actually compresses the gas in front of it.

So later if this is accelerated to higher and higher speeds then these compression waves catch up and later on form a shock which was the piston analogy that we discussed during the formation of shock waves. But in the initial time the compression waves how do they move can be solved using the finite waves that we had have discussed just now. Now these are isentropic compression waves similar to the expansion waves.

But you should understand that these compression waves if you have the forcing for quite long a time then they can actually coalesce together and form shock waves. So that is where you get this velocity of the piston, u_p to create a certain shock with the Mach number Ms. So if the piston is suddenly accelerated then it will create a shock wave quite quickly. So these are also some important situations where you can look at these unsteady problems.

And we will solve a few problems shock tube problem and something on the unsteady nature in the next class.