# Gasdynamics: Fundamentals and Applications
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## Lecture 23
## Waves of Finite Amplitude

So we are looking at the shock tube problem and we are looking at solving the left hand side of the shock tube problem which initially you start off with a large pressure difference between the driver side and the driven side. On the right hand side into the driven side you have a shock wave, a moving shock wave and behind it there is a motion of gas. On the left hand side you have expansion waves these are waves, isentropic waves are they move with without any change in entropy.

And we are trying to address the left hand side of the shock tube problem and they are waves. So we began with waves of very small changes or infinitesimal changes. Now we move on to waves of finite amplitude or when the amplitudes of these waves are quite significant what happens.

So, let us move ahead with the problem. so let us directly go to the equations of motion this for fluid flow.

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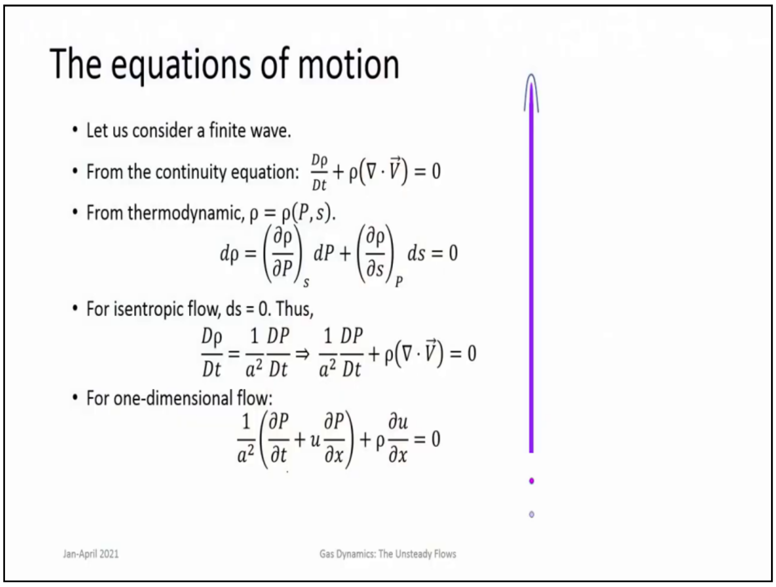


So we go to fluid flow equations and also is compressible in nature so density is a variable. So take the continuity equation, you start with the continuity equation $\frac{D\rho}{Dt}$. This material derivative. $\frac{D\rho}{Dt} + \rho\nabla.\vec{V} = 0$. So we start with this equation and we know that it is a nice isentropic flow. So density is a variable here and it can be function of pressure and entropy.

For an isentropic flow change in density is 0. I mean change in entropy is 0, so this is 0 and you get the term $\frac{\partial\rho}{\partial P}$ . All of course you should know $\left(\frac{\partial P}{\partial \rho}\right)_s = a^2$ . So one can relate $\frac{D\rho}{Dt}$ to $\frac{DP}{Dt}$ that is changing pressure. So they can be related to each other by using the acoustic equation i.e. $\frac{D\rho}{Dt} = \frac{1}{a^2}\frac{DP}{Dt}$ .

So, in the continuity equation the change in density is or the material derivative density is replaced by the material derivative of pressure and why did this will soon become evident. So for a one dimensional flow $\frac{D}{Dt}$ , the material derivative is $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ .where u is the velocity in x direction. So you can write the material derivative of pressure as $\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}$ and $\frac{1}{a^2}\left(\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0$

So this form of the continuity equation where we have used definition of speed of sound and brought in the pressure variable and then it has been put here and this is the converted continuity equation.

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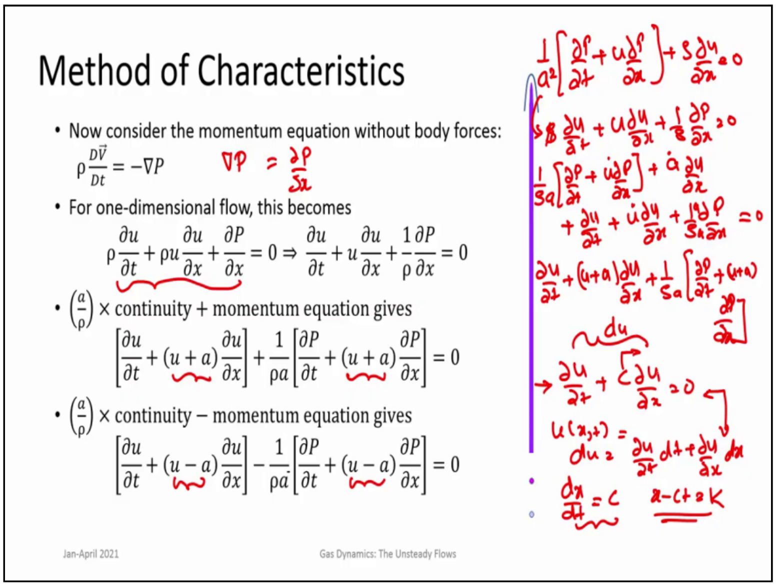


Now we move to the momentum equation without any, body forces or any viscous forces it is a inviscid flow. So again material derivative of velocity and we are considering only one dimension so this becomes,

$$\rho\frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

in one dimension.

Now you have the two equations the first equation was from the continuity equation ,

$$\frac{1}{a^2}\left(\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0\,.$$

And the momentum equation is

$$\rho\frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

or it can be divided by $\rho$ also.

So we can do that,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

Now we do this operation we multiply the continuity equation by $\frac{a}{\rho}$ that is the speed of sound divide by density and add the momentum equation to it.

So if you do that you get,

$$\frac{1}{\rho a}\left[\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}\right] + a\frac{\partial u}{\partial x} + \left[\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x}\right] = 0$$

So now if you collect terms or containing pressure and velocity separately you get, So it will be,

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial P}{\partial t} + (u+a)\frac{\partial P}{\partial x}\right] = 0$$

So this is what we get same operation but now what you do is $\frac{a}{\rho}$ multiplied to continuity equation and subtract the momentum equation and you can carry on the same steps now what has been described over here. And you can see that you end up with,

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial P}{\partial t} + (u-a)\frac{\partial P}{\partial x}\right] = 0$$

Now what is the significance you can see that there are two terms $(u+a)$ *and* $(u-a)$. They are quite significant and this is where the application of method of characteristics comes into picture. So what we are looking for is some transformation. So these are partial differential equations. So is there some transformation that we can apply. So as to convert them to more easily solvable differential equations, ordinary differential equations. And for hyperbolic equations these are hyperbolic equations then there is a method to do it and that is the method of characteristics. And to understand this you can go back to our previous classes when we were looking at a general variable and looking at, $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ where c is a constant of propagation. And we were looking how can we convert this to a ordinary differential equation and u being a function of x and t, u =u (x,t) and it can be written du can be written as $du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx$

And comparing these two we sort of arrived at if we consider

$\frac{dx}{dt} = c$ then we can write this as du or an ordinary differential equation ,it can be taken to be du. So along lines $\frac{dx}{dt} = c$. So along these lines which gives out the lines as (x-ct) = k, some constant some constant, (x-ct) = k . So along these lines you can convert the partial first order partial differential equation into an ordinary differential equation.

So this is the basic principle of method of characteristics and you can always come back to the simple equation in order to understand what we are doing. So you can take the same principles here and come back to this problem that we have which is more involved. You have both u and P, both are functions of space and time, x and t, and P is also a function of x and t. But you look at these equations you have similar form you have (u + a) here.

So along the lines $\frac{dx}{dt} = u + a$. So along these lines this particular form gets converted into dP while this will convert to du which is change in velocity. So now du is here d represents total derivative. So here again there are another set of characteristics for this problem which is ( $u-a$ .). So there are two sets one is, ( $u+a$ .). the other one is ( $u-a$ .). so.

$$\frac{dx}{dt} = u - a$$

So this set of characteristics is known as is generally called $C^{+}$ and this set is called $C^{-}$, that is the way it is represented. So if you do make those changes then you get two equations these equations are called as the compatibility equations, so they convert.

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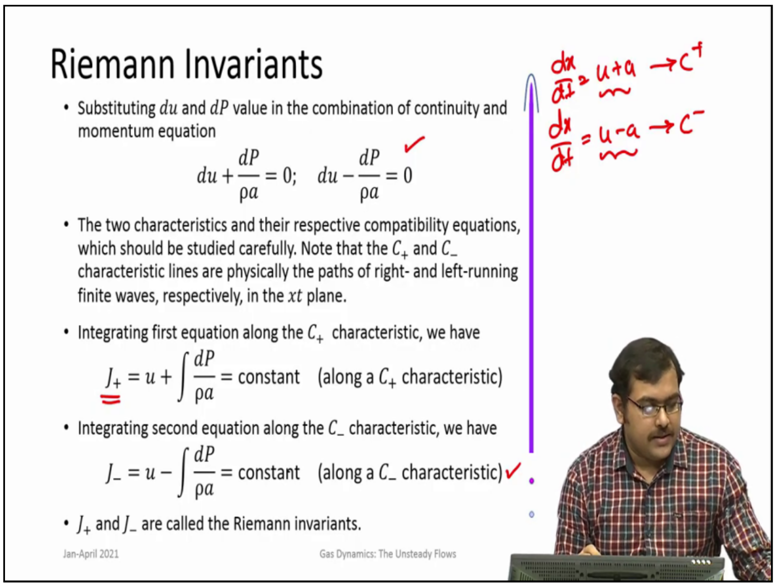


So now by going along certain lines or certain curves in space which are related by $\frac{dx}{dt} = u + a$ this belongs to one set of characteristics which we call as | $C^+$ and the other one is $\frac{dx}{dt} = u - a$ another set of characteristics which we called as $C^-$. If you take separately in these two characteristics and go back and look at the equations they transform into ODE's , $du + \frac{dP}{\rho a} = 0$ and $du - \frac{dP}{\rho a} = 0$. These are ordinary differential equations they can be integrated.

So now you see in this discussion on finite amplitude waves, what we see here is that your speed of propagation of the waves is not a infinity anymore. It is $u + a$ and $u - a$. These are the speeds with which the waves propagate and a is not a constant speed with which it propagates in space and time can vary. So that is the important difference between infinitesimal waves where you saw that the speed of propagation is the speed of sound which is $a_\infty$.

But here that is not so. So now how would we proceed ? We have to evaluate these integrals. So this is for $C^+$ characteristic the integral is $J^+$ these are called Riemann invariance because this has to be a constant if you integrate it $J^+ = u + \int \frac{dP}{\rho a} = constant$ so, along a $C^+$ characteristic. Secondly if you integrate the second equation that is this one and then you get $J^- = u - \int \frac{dP}{\rho a} = constant$ is constant along $C^-$ characteristic.

So this is along $u - a$ and this is along $u + a$ .This has to be borne in mind. It cannot be randomly done. So $J^+$ corresponds to $C^+$ , $J^-$ corresponds to $C^-$ . Now this is quite general. This is $\frac{dP}{\rho a}$ it is general.

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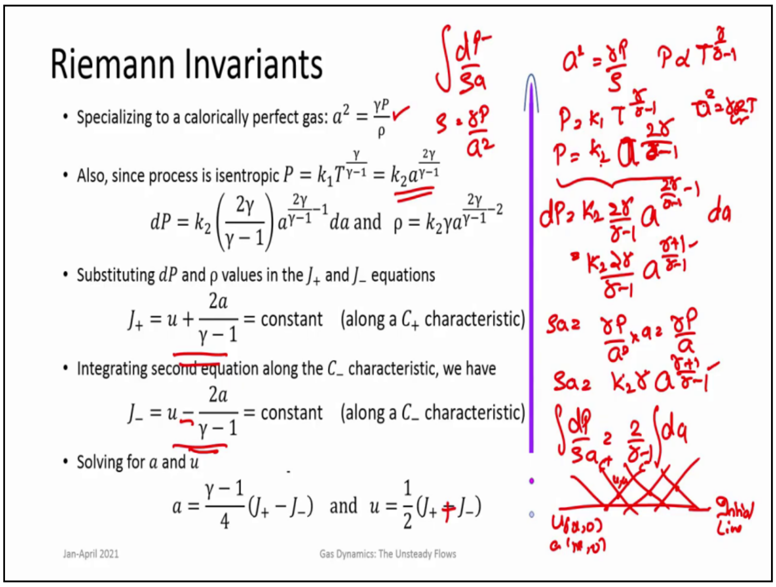


Now let us go into the specific case that we are discussing that is to a calorically perfect gas. So if you take a calorically perfect gas so you know a square i $a^2 = \frac{\gamma P}{\rho}$ s ,This is known. And these processes that we are discussing here is isentropic, we are looking at the expansion waves the travel. So they are isentropic waves so you know that isentropic waves or the isentropic process is related by, $P \alpha\, T^{\frac{\gamma}{\gamma-1}}$ .

So an arbitrary constant $k_1$ is introduced. $P = k_1\, T^{\frac{\gamma}{\gamma-1}}$ and of course T is actually $a^2$ . So $a^2 = \gamma RT$ . So you can use this fact here. so T can be replaced by $a^2$ with of course other additional constants getting multiplied so you get $P = k_2\, a^{\frac{2\gamma}{\gamma-1}}$ .

This is how we get this. Then once we have an expression for P, we can write dp and $dP = k_2(\frac{2\gamma}{\gamma-1})\, a^{\frac{2\gamma}{\gamma-1}-1} da$ . So you can do this, $dP = k_2(\frac{2\gamma}{\gamma-1})\, a^{\frac{\gamma+1}{\gamma-1}} da$ and we have to do integral $\int \frac{dP}{\rho a}$ . Now we have expressed P in terms of a.

Now can we express ρ in terms of a. We can use this equation, $a^2 = \frac{\gamma P}{\rho}$ or you can use $\rho = \frac{\gamma P}{a^2}$ . So you can use that equation and you have the term $\rho a$ . So $\rho a$ is a term that needs to be evaluated. This is nothing but $\rho a = \frac{\gamma P}{a^2} a = \frac{\gamma P}{a}$ , And $P = k_2 a^{\frac{2\gamma}{\gamma-1}}$ .. So $\rho a$ also turns out to be, $\rho a = k_2 \gamma a^{\frac{\gamma+1}{\gamma-1}}$ .

Now $\frac{dP}{\rho a}$, now this term $\frac{dP}{\rho a}$ will now turn out to be if you put things together, $\int \frac{dP}{\rho a} = \frac{2}{\gamma-1} \int da$ So what you get is integral $\int da = a$.

So, $J^+ = u + \frac{2a}{\gamma-1} = constant$

so the Riemann invariant can be solved analytically. You can get close form solution, $u + \frac{2a}{\gamma-1} = constant$ along a $C^+$ characteristic.

Similarly for a $C^-$ characteristic is the same integration but you have this negative sign here. So, $J^- = u - \frac{2a}{\gamma-1} = constant$. So if you take any so the way to solve these is starting from an initial problem or initial value you draw the $C^+$ and $C^-$ characteristics. So they do go like this and you know the, u =u (x,0) ,,x the initial solution at x, 0 this is known.

So at intersections of the $C^+$ and $C^-$ characteristics you evaluate u and a. Because along a particular characteristic whether you take so if you take this is $C^-$ characteristics this is $C^+$ characteristic. So along the $C^+$ characteristic, $J^-$ is constant. Along $C^+$ characteristic $J^+$ is constant. So it can be evaluated starting from the initial value.

So this is initial line and at any point you can solve them by just an algebraic manipulation, solution of simultaneous equations.

$$a = \frac{\gamma-1}{4} (J^+ - J^-)$$

and $u = \frac{1}{4} (J^+ + J^-)$ ,So you have to look at so, it is $J^+$, $J^-$. So this is the equations. So for solving a and u separately. So the way the solutions of this kind are done is you have an initial line and then you can solve the field at any space and time.

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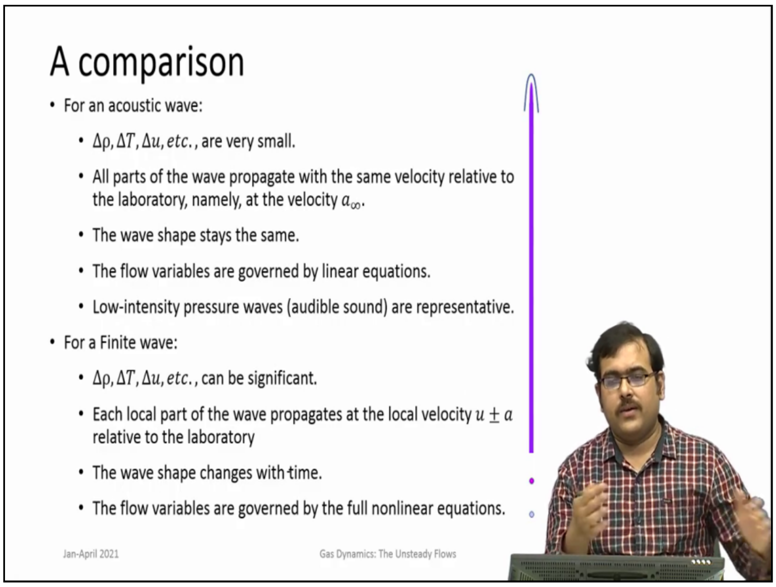


So now let us go and just compare whatever we had done in the previous class to this class. Previous class we had very small waves infinitesimal waves and this class we had finite waves. Now infinitesimal waves are very, very small and they propagate at constant velocity which is $a_\infty$. So when they propagate an initial disturbance propagates with the same form, it does not change form because it is just propagating all parts of the wave are propagating with the same speed a infinity does not change form.

So, the shape same just stays the same and the equations are linear hyperbolic equations. Typically these are sound waves representative of sound waves. But if you come to finite disturbances then the change is quite significant and so these waves they do not travel all the time with the velocity $a_\infty$. They travel with the propagation speeds $u + a$ or $u - a$ and it can vary from space and in space as well as at different times instant of times.

So because of this variation the initial shape of the wave it changes with time so it gets deformed. And these equations are fully non-linear.

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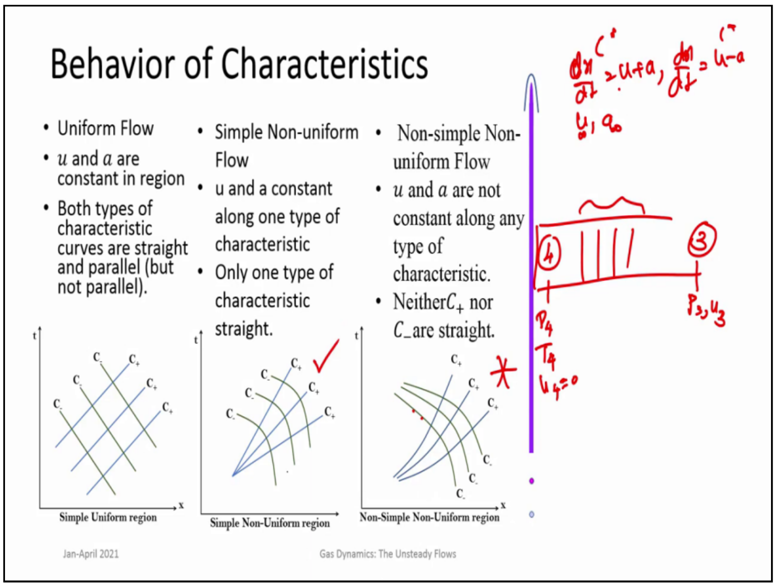


So how can we apply this to shock waves shock tube problem. To get the connection we need to understand a little bit about how these characteristics behave? So, if you have a region of uniform flow that is the velocity, pressure and temperature do not depend on space and time and there you draw the characteristics $C^+$ and $C^-$ . The $C^+$ and $C^-$ are nothing but $\frac{dx}{dt} = u + a$ and This is $C^+$ and $\frac{dx}{dt} = u - a$. This is $C^-$.

So if you draw them at all points u and a are the same. So you can take them as $u_\infty$, $a_\infty$. They will same everywhere in a uniform flow field. Therefore $\frac{dx}{dt}$ is the same along $u + a$ and $u - a$. So you have 2 sets of straight parallel lines. So they are always parallel they are straight in a uniform flow. Well if you take a non completely, non uniform flow that is represented here.

Represented here u and a are not constant anywhere so in general they are changing everywhere. So they at every point of course you will have $\frac{dx}{dt} = u + a$ but as you move away from the point you have another $u + a$ or $u - a$. As a consequence these are curves. They are not straight lines and so you have two sets of curves. So this is completely non uniform flow.

So an intermediate between these two is a simple non uniform flow. It occurs when you have a region of non uniform flow bounded by two uniform flows. And if you think of it this is typically in a in the expansion side of the shock tube where you have region 4 here and region

3 here. Region 3 is uniform, you have pressure is P3, speed is u3. And region 4 is having no velocity, it has pressure P4, T4, u4 is 0.

And the expansion waves move between them. So you have a non uniform flow which is the expansion, motion of the expansion waves between two regions of uniform flow. So that follows the simple non uniform flow problems. So that is how it comes into picture.

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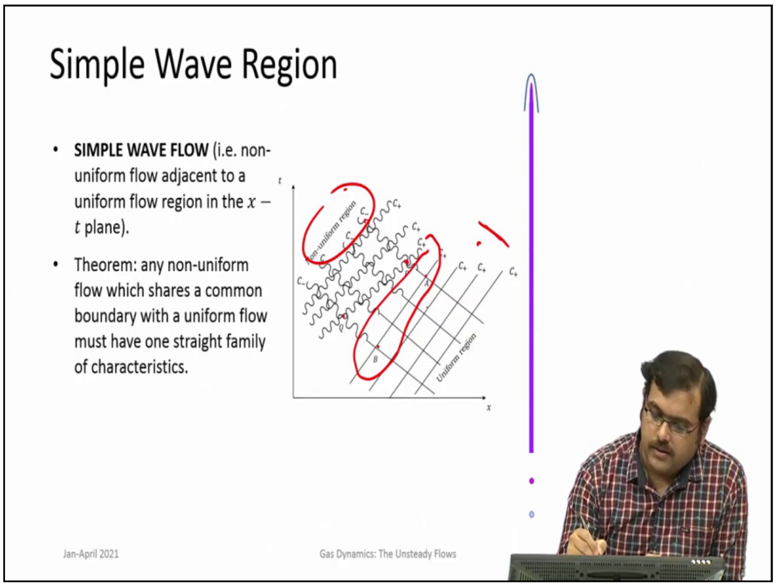


And what is important there is an understanding that when you have such a flow. So we already talked about uniform flows here it is always straight that is rather straight forward. But if you have a simple wave region a region where it is simple non uniformity not completely non uniforms then you have a property that one set of waves either one set either $C^+$ or $C^-$ it depends on the problem.

One set of the waves will be straight lines the other one will be curved and you can show that simply by considering the case that you have a non uniform region here and you have a uniform region here. And this forms the boundary of non uniform region and the uniform region. So consider so these $C^+$ characteristics here and and $C^-$ characteristics are coming in they are coming in here from the non uniform region.

So so you can consider these 2 points this is a uniform flow while PQ lie in the non uniform flow along the $C^+$ characteristics.

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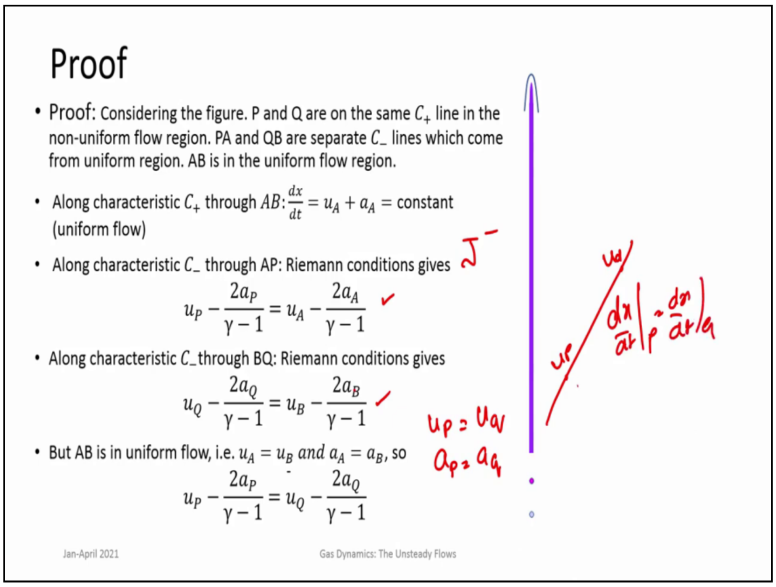


So if you do look at that problem then what you can show is so along $C^+$ $\frac{dx}{dt} = u + a$ but $C^-$ characteristics are the ones which are moving from the uniform region to the non uniform region. So along $C^-$ characteristics the $J^-$ that is Riemann constants or invariant. So you can write, $u_P - \frac{2\,a_P}{\gamma-1} = u_A - \frac{2\,a_A}{\gamma-1}$ .

Similarly you can write it for points Q and P so these two equations you can write. But since it is coming in from uniform region A and B , $u_A$ and $u_B$ they are the same. So as a consequence you can get that $u_P = u_Q$ and $a_P = a_Q$. So along the $C^+$ characteristic essentially $u_P = u_Q$ . They are here along the $C^+$ characteristic in the non uniform region but they have the same value. That means the $C^+$ characteristics have the same $\frac{dx}{dt}$ at P is equal to $\frac{dx}{dt}$ at Q and their values are also the same they are straight lines. $\left(\frac{dx}{dt}\right)_P = \left(\frac{dx}{dt}\right)_Q$

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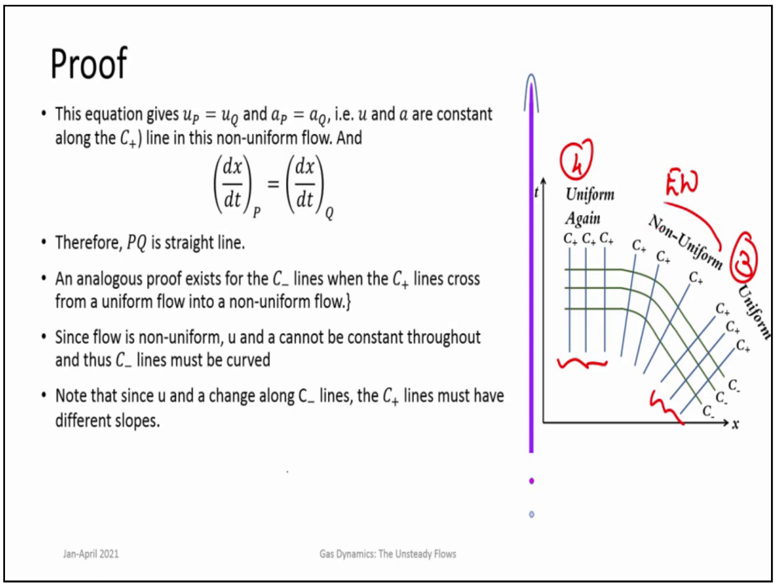


So this is the picture that comes out that you have a region of uniform flow another region of uniform flow in between if you have a non uniform flow then one set of characteristics are straight lines ,the others are curved. So keep this idea in mind as this is important when we now look at the shock tube in whole and region 3 and region 4 are uniform regions which bound. So this is region 4 which is region 3 and in between you have the expansion waves.

So the expansion fan lies in between those two and this information is useful when we look at the solution of the shock tube problem.