Gasdynamics: Fundamentals and Applications
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Lecture 22
Waves of Infinitesimal Amplitude

So this module we are looking into details of unsteady flow in one dimension and we are taking the shock tube problem. And we had discussed in the previous class the various flow features that are encountered in a shock tube. It starts with the initial condition that there are two sides to a shock tube. It is essentially a tube you have the high pressure side which is called the driver side and you have the low pressure side the driven side.

And at t = 0 the interface between do these two sides is suddenly ruptured or taken away you have very high pressure being interfaced to very low pressure gas, high pressure and low pressure. This high pressure differential essentially creates a shock and shock moves into the driven section. You have other waves and flow features also in the shock tube in expansion fans move towards the driver section and the region in between which is a interface between the driver gas and the driven gas shock process driven gas is the contact surface of contact discontinuity.

The shock as it goes through the driven section compresses the gas as well as carries gas along with it, it produces mass motion. So our idea here is to look at this complete problem and solve various regions and then put them together to look at the shock tube as a whole. So the first part of this is the moving shock that is already covered. So now let us start moving into the left hand side of the shock tube that we have been looking at which is the waves.

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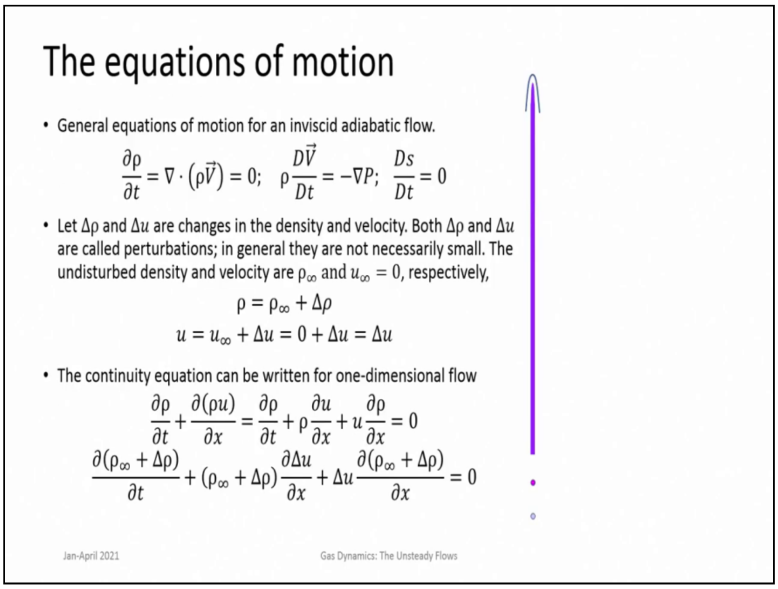

So, the expansion waves, so expansion waves before we go there we start with simple waves, waves of very, very small amplitude waves of infinitesimal small amplitude very small amplitude and look at the analyses of those waves. They are representative of sound waves and you can soon we will show their equations are contained within the system of flow equations that we are discussing.

So, there is a general equation of motion for an invisible adiabatic flow. So this is an invisible flow. So viscous effects are not considered it is a compressible flow. So we have the compressible equation, so $\frac{\partial\rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0$. This is continuity and $\rho\frac{D\vec{V}}{Dt} = -\nabla P$ .This is momentum and this movement of these waves are isentropic.

So, s is so entropy does not change, $\frac{Ds}{Dt} = 0$. So consider a quiescent medium where there is no initially there is no velocity and some small disturbance small changes in density or velocity or pressure is introduced. So $\nabla\rho$ and ∇u small very small changes in density and velocity are introduced. So they are small perturbations and so the undisturbed or the quiescent flow medium has ρ_∞ and u_∞ , P_∞ and T_∞ as properties.

And if you take a quiescent medium, $u_\infty = 0$. So, now what we are trying to do is to see how this these small changes propagate in the medium and we consider only the one dimension. So we are only considering one dimensional case here. So let us write the equations for these small perturbations. So ρ , the change in ρ is ρ actually is $\rho_\infty + \nabla\rho$, $u = u_\infty + \nabla u$ which is since u_∞ is 0 is basically ∇u , $u = \nabla u$.

Now the guiding principles here is that we consider changes to be or the terms in the equations to be significant if you have significant terms multiplications like $\frac{\partial \nabla u}{\partial x}$,these terms are significant. But if you have multiplications of small quantities ∇u , $\nabla\rho$ these are both are small quantities, two small quantities multiplying each other they become very, very small so they are taken to be 0.

So this is the guiding principle and once you apply these principles to the equations then we can see how they get transformed through the evolution equation for $\nabla\rho$ or ∇u that is the small perturbations. We take the continuity equation, $\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = \frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x}$

So now $\rho = \rho_\infty + \nabla\rho$ and $u = \nabla u$. So, we get these terms. Now this is directly by substituting these values into the equation and now we can use these principles. So you see that there is a $\nabla\rho$ here this term is significant ρ_∞ is the quiescent medium density this does not I mean this is a constant. So, $\frac{\partial\rho_\infty}{\partial t} = 0$.

So you can use those principles here. So now if you look at this a multiplication of $\nabla\rho\ \frac{\partial(\nabla u)}{\partial x}$ is going to be extremely small so that is not significant. And ρ_∞ $\frac{\partial(\nabla u)}{\partial x}$, partial derivative is significant.

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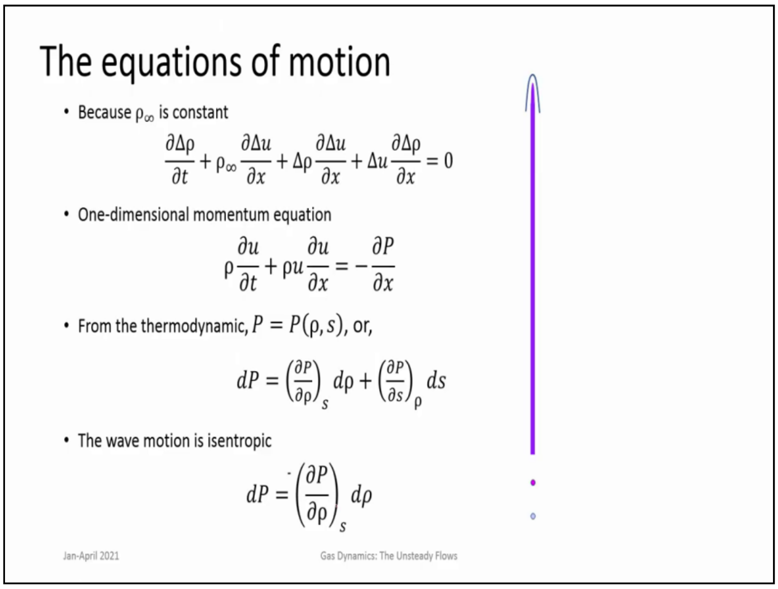

So using these guiding principles we can write down this equation in terms of $\frac{\partial(\nabla\rho)}{\partial t}+\rho_\infty\frac{\partial(\nabla u)}{\partial x}+\nabla\rho\frac{\partial(\nabla u)}{\partial x}+\nabla u\frac{\partial(\nabla\rho)}{\partial x}=0$. And you have the other term, $\nabla\rho\ \frac{\partial(\nabla u)}{\partial x}$, $\nabla u\ \frac{\partial(\nabla\rho)}{\partial x}$ this term is also extremely small. Now so this yields basically $\frac{\partial(\nabla\rho)}{\partial t}+\rho_\infty\frac{\partial(\nabla u)}{\partial x}=0$ This is what this equation yields.

Now if you take the one dimensional momentum equation $\rho\frac{\partial u}{\partial t}+\rho\, u\frac{\partial u}{\partial x}=-\frac{\partial P}{\partial x}$. Now this being an isentropic flow; so pressure can be related to density by the term. So P is a function of rho in general you can write P as a function of rho and s, P = P(ρ , s). If so

$$dp=\left(\frac{\partial P}{\partial\rho}\right)_s d\rho+\left(\frac{\partial P}{\partial s}\right)_\rho ds\ .$$

So here ds = 0 so its isentropic flow and this term dou P by $\left(\frac{\partial P}{\partial\rho}\right)_s$ at constant entropy, this is nothing but speed of sound a^2. So this term can be brought inside the equations and thereby $\frac{\partial P}{\partial x}$ can be converted to $a^2\ \frac{\partial\rho}{\partial x}$.

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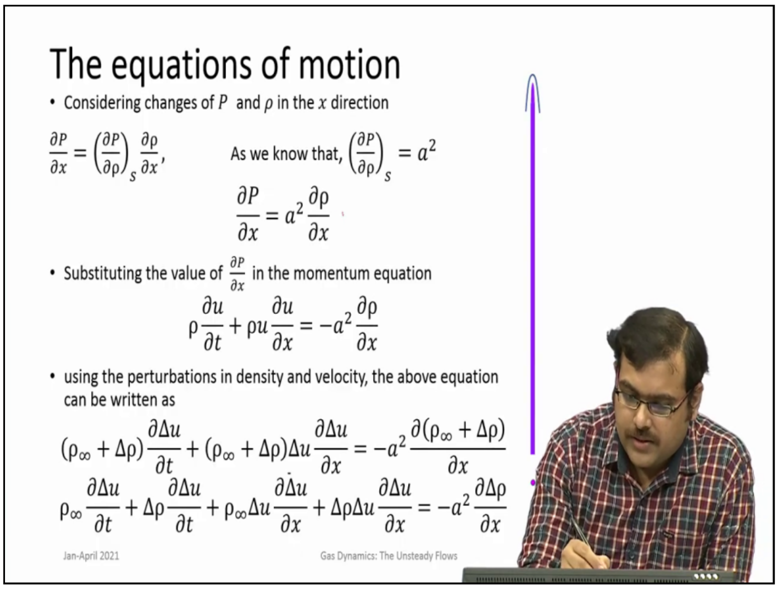

So that is what is done here. So it is been converted to the changes in density and now if a^2 in general can be a variable. So but for the quiescent medium there is a speed of sound which is a_∞ corresponding to the temperature T_∞ you know a_∞ is $\sqrt{\gamma R \ T_\infty}$. Now if there are small changes to pressure, temperature, density correspondingly ,we would expect small changes to a_∞. And therefore we need to consider them into the equations also.

So now you can write considering so now $\rho = \rho_\infty + \nabla\rho$, , $u = \nabla u$ and you can substitute them. Again follow the same guiding principles that such $\nabla\rho$, ∇u these terms are very small.

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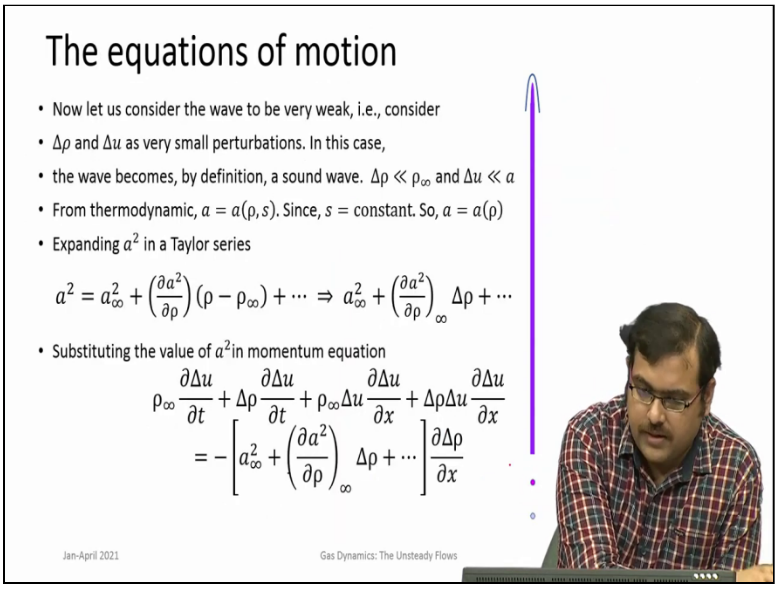

So there are two significant terms over there. Now this is very weak waves, infinitesimally small amplitude, very, very small perturbations. Now a that is a is again that speed of sound is a thermodynamic quantity. It can also be written as a function of density and entropy,

a =a(ρ, s). Now entropy is constant. So it becomes a function of density and it can be expanded by a Taylor series kind of an expansion. a square is a infinity square plus dou a square by dou rho rho minus rho infinity.

$$a = a_\infty{}^2 + \frac{\partial a^2}{\partial\rho}(\rho - \rho_\infty) + \ldots$$

So is basically the $\nabla\rho$ and higher order terms which are very small. Now this term here can be substituted for a^2 in the momentum equation. But again we find once you carry on the multiplication you have small quantity $\nabla\rho\ \frac{\partial\nabla\rho}{\partial x}$. So they are again very small. So finally the most significant terms of the momentum equation is (- $a_\infty{}^2\ \frac{\partial\nabla\rho}{\partial x}$) .

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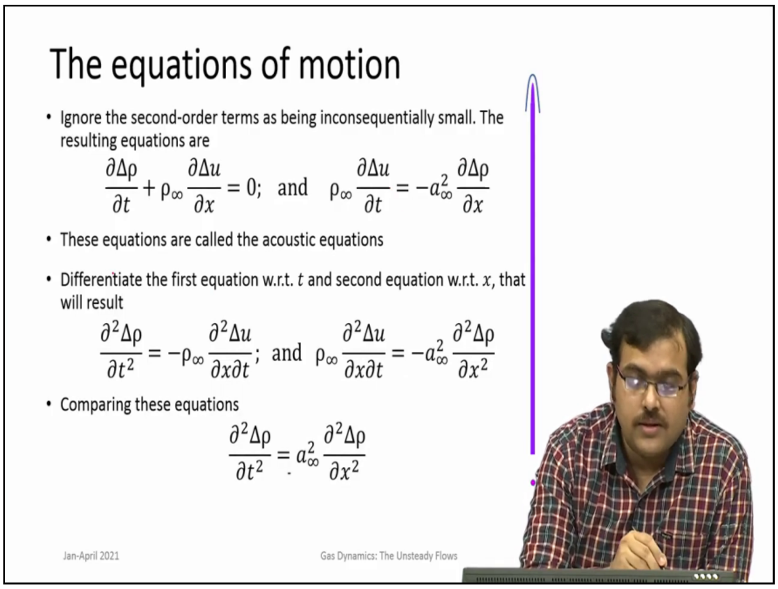

So you have two equations the first one comes from continuity and second one comes from momentum and these equations are now the equations for evolution of an extremely small perturbation in quiescent medium and that is typical of a sound wave. And can we get the equations of sound wave from here ? You can do that. You can just so if you take a partial derivative with respect to time here.

For this equation and take partial derivative with respect to x here. so you get,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = -\rho_\infty \frac{\partial^2 \Delta u}{\partial x\, \partial t}$$

and $\rho_\infty \frac{\partial^2 \Delta u}{\partial x\, \partial t} = -a_\infty^2 \frac{\partial^2 \Delta\rho}{\partial x^2} =$. And comparing these two equations you get,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = a_\infty^2 \frac{\partial^2 \Delta\rho}{\partial x^2}$$

which is what is written here is nothing but a wave equation.

Wave equation where the speed of the wave is a_∞^2 which is the speed of sound waves in that particular medium.

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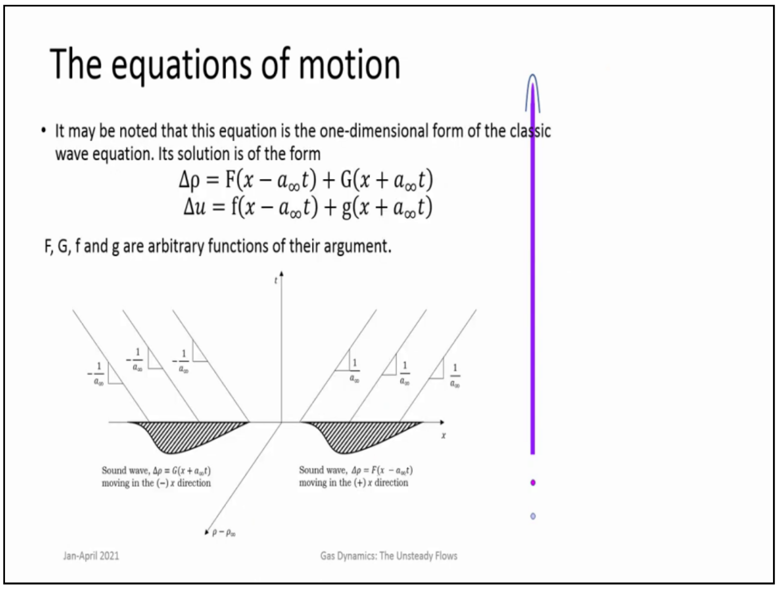

So the solutions to these equations in general are of the form,

$\Delta\rho = F(x - a_\infty t) + G(x + a_\infty t)$,that is $\Delta\rho$ that small change in density. You can get the same equation, the same wave equation for the change in velocity Δu and these are given ,

$$\Delta u = f(x - a_\infty t) + g(x + a_\infty t)$$

This is the solution to the wave equation. So where $(x - a_\infty t)$ *and* $(x + a_\infty t)$, they represent the directions of propagation of an initial disturbance so that is given over here you can see.

So this is the initial propagation. So they move in this direction. So you can write down these solutions. This is normal, I mean this is the wave equation that is linear and solved in classes.

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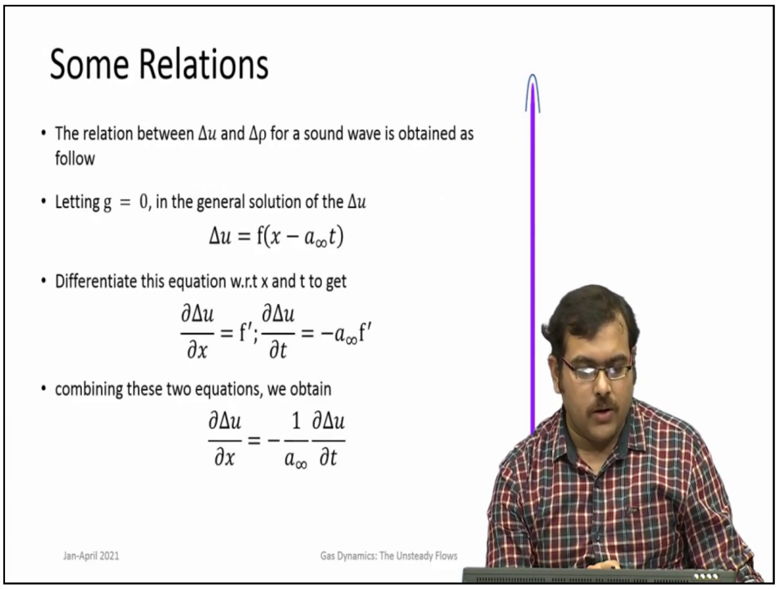

So now let us see how the various quantities that is density and velocity they are related to each other as well as changes in pressure. So, when you look at them, so if you say Δu consider for a general case let g be equal to 0. that is the small function, function with , g =0 then $\Delta u = f(x - a_\infty t)$,then $\frac{\partial \nabla u}{\partial x} = f'$, f' is the differential of f .

While,

$$\frac{\partial \nabla u}{\partial t} = -a_\infty f'$$

So from comparing these two we get,

$$\frac{\partial \nabla u}{\partial x} = \frac{-1}{a_\infty}\frac{\partial \nabla u}{\partial t}$$

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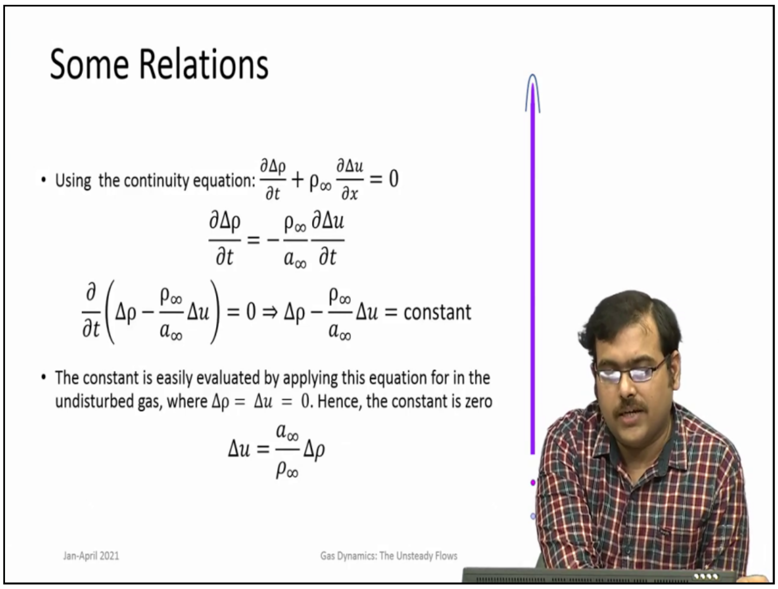

So now this can be combined along with the continuity equation. So continuity equation is,

$$\frac{\partial \nabla\rho}{\partial t} + \rho_\infty \frac{\partial \nabla u}{\partial x} = 0$$

So, from the previous equation you can write this as,

$$\frac{\partial \nabla\rho}{\partial t} = - \nabla\rho \frac{\partial \nabla u}{\partial t}$$

This can be taken together, $\nabla\rho$, ρ_∞ and a_∞ are constants. For this problem,

$$\frac{\partial}{\partial t}(\nabla\rho - \frac{\rho_\infty}{a_\infty} \nabla u) = 0$$

So this implies that,

$$\nabla\rho - \frac{\rho_\infty}{a_\infty} \nabla u = \text{constant}$$

Now this constant can be evaluated by taking the equation for the undisturbed gas and there when you started off with $\nabla\rho$ and ∇u both were zeros hence at initial this is 0 therefore it is constant. So this constant is 0 and so it relates, $\nabla u = \frac{a_\infty}{\rho_\infty} \nabla\rho$.

So this is the equation. So you can evaluate how much change in velocity is produced by a small change in density.

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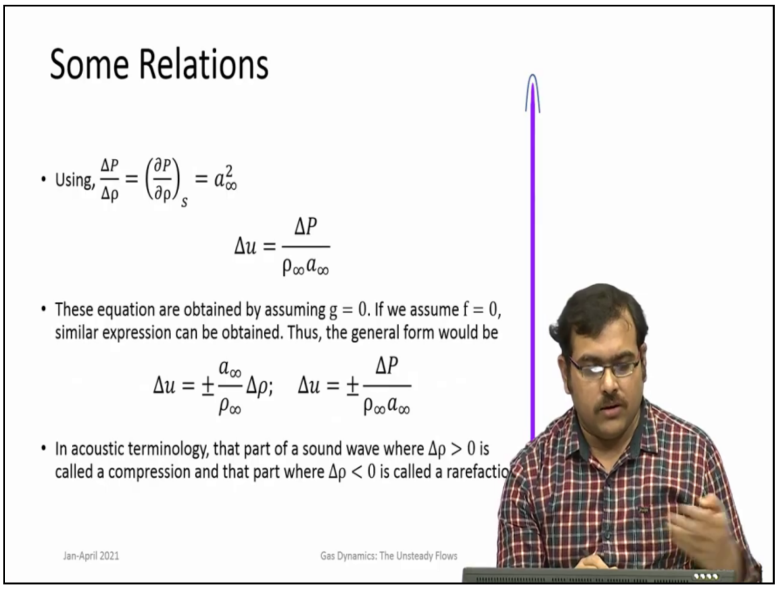

Now because this is motion of sound ,

$$\frac{\nabla P}{\nabla \rho} = a_\infty{}^2$$

Small pressure perturbations. You can use that and relate change in velocity to change in pressure. So in general so we started off with the general equation

$$\Delta u = f(x - a_\infty t) + g(x + a_\infty t)$$

and we set g =0. We can also set f = 0 and get other forms of this equation. So in general you will get,

$$\nabla u = \pm \frac{a_\infty}{\rho_\infty} \nabla \rho \quad ; \quad \nabla u = \pm \frac{\nabla P}{a_\infty \rho_\infty}$$

So these are very, very small changes what we are talking about in this. So this is infinitesimally small and they these are corresponding to sound waves which is very small. And when pressure increases, density increases in the sound wave that is compression. So that is a compression part of the sound wave and when the pressure decreases then that is called the rarefaction part of the sound wave.

So, this some of these things known to you but now we have seen that these waves are also part of the fluid dynamic equations that we have for the general fluid flows. And we saw how waves fit in there and these waves are extremely small in amplitude. Now we move on in the next class we will move to finite amplitude waves where waves now have significant strength in there pressure changes or density changes or velocity changes.

Now in this, what you noticed was that you get finally the wave equations. So

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = a_\infty{}^2 \frac{\partial^2 \Delta\rho}{\partial x^2}$$

This is a wave equation with a_∞ speed of sound that is a constant. So this is the case of infinitesimal waves. When we go to finite waves we see that this speed of sound at that is not a constant anymore it can change from point to point time to time.

So that is going to be the additional complexity that comes into picture in this if there was an initial waveform it just propagates along these lines. These are lines of propagation they are also known as characteristics. And this methodology that is the way they propagate along certain lines known as characteristics will be very useful when we look at finite amplitude waves. Here $(x - a_\infty t)$ and $(x + a_\infty t)$ represent two sets of characteristics along which the solution propagates.

Since a_∞ is constant here these characteristics are straight lines but when we go to finite waves we will see that that need not always be true. So in the next class we will go to finite amplitude waves.