Gasdynamics: Fundamentals and Applications
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Lecture 18
Normal Shock - III

So, until now we have been looking at the details of normal shocks. So, we have looked at how to analyse them and get relationship between all the flow parameters in terms of only the upstream Mach number and γ. Now let us continue with those discussions and see some more facts about these normal shocks.

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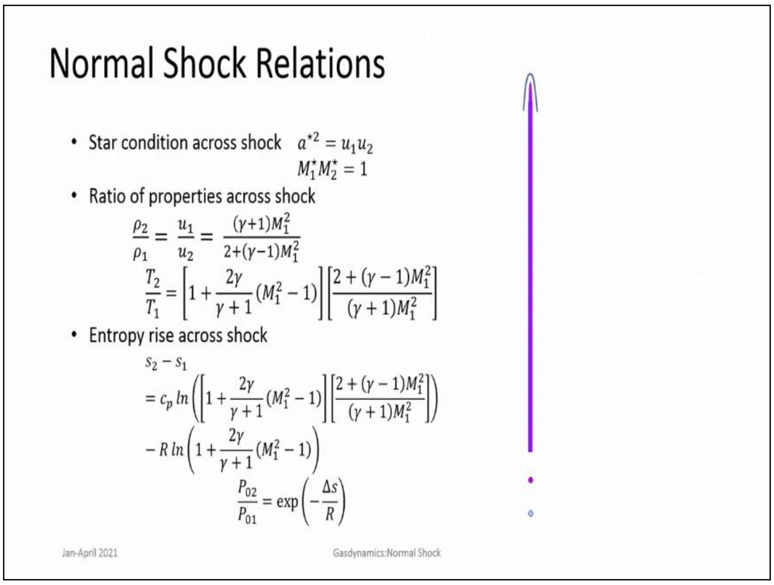

So just to give a quick recap, so the important relationship that we get is that of the Prandtl's relation for normal shocks which is $M_1^\star M_2^\star = 1$ and from this the main principle used here is that it is an adiabatic flow. And in adiabatic flow the star conditions remain constant, and you apply that condition along with the conservation equations and you get relationships for density ratio across the shock, temperature ratio, pressure ratio and entropy ratio as well as stagnation pressure ratio. Since shock waves generate entropy, the stagnation pressure decreases across the shock.

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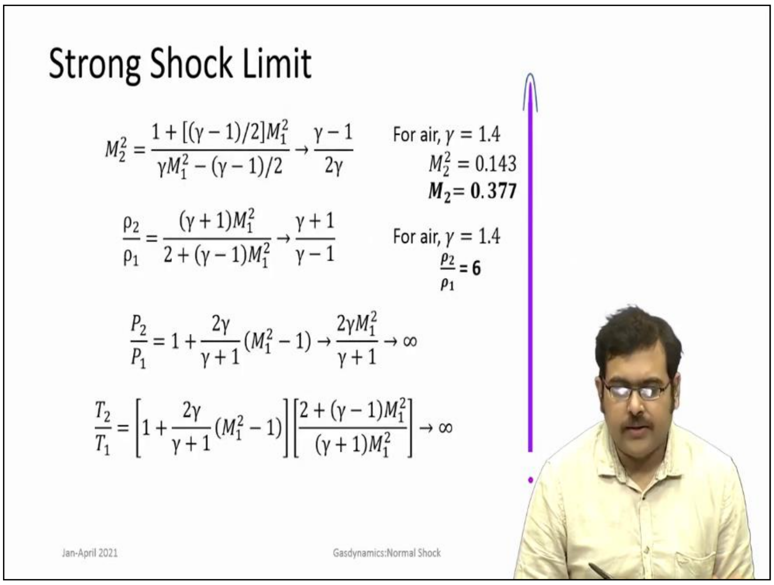

So, now with this information let us move ahead and look at what are the limiting conditions for the shock wave. So, one thing is clear that the characteristics of the shock depend only on the upstream condition that is Mach number and γ. So if Mach number, upstream Mach number is increased continuously so the strength of the shock depends on Mach number and it is generally referred to as the pressure ratio.

So as Mach number increases pressure ratio increases significantly. So now let us see what happens if Mach number tends to infinity. So it is the limit known as the strong shock limit because here the pressure ratio can become really high. So first let us look at the Mach number downstream of the shock. So the expression is over here, for Mach number downstream of the shock and Mach number tends to goes towards infinity.

So, you can use the appropriate limit that M_1^2 tends to infinity. So you can sort of take out M_1 from both of them and here you will get $\frac{1}{M_1^2}$. So as M_1^2goes to infinity $\frac{1}{M_1^2}$ goes to 0 and if you apply that particular limit you see that M_2^2 tends to a finite value which is $\frac{\gamma-1}{2\gamma}$for a perfect gas. Of course, it is for a calorically perfect gas.

So M_2 reaches finite values as a Mach number tends to infinity, we had an indication of this when we were looking at the graphs of the variables as Mach number (upstream Mach number) is increased in the previous class. So, we saw that the downstream Mach number saturates, and the reason is that as it tends Mach number tends to infinity M_2 reaches finite values and for air it is 0.377.

Similarly, if you look at the density ratio across the shock $\frac{\rho_2}{\rho_1}$, again this is the expression for $\frac{\rho_2}{\rho_1}$ follow the similar procedure that M_1^2 tends to infinity. And here also you observe that the density ratio across the shock reaches finite values. This is $\frac{\gamma+1}{\gamma-1}$ and for air where γ is 1.4, this is equal to 6. But for all other parameters like the pressure ratio $\frac{P_2}{P_1}$,this as M_1 increases, M_1^2 goes to higher and higher values. Of course, 1 becomes very small compared to that. Similarly this also is very small.

So, it approximates to $\frac{2\gamma M_1^2}{\gamma+1}$ which tends to infinity. So, pressure ratio goes to infinity and temperature ratio is nothing but multiplications of pressure ratio and density ratio. So, since pressure ratio increases to infinity even temperature ratio increases to infinity. So, this limit that shock waves are becoming very strong as Mach number increases is known as the strong shock limit.

And in strong short limit Mach number downstream of the shock and density ratio reach finite values but pressure ratio and temperature ratio can increase to infinite values.

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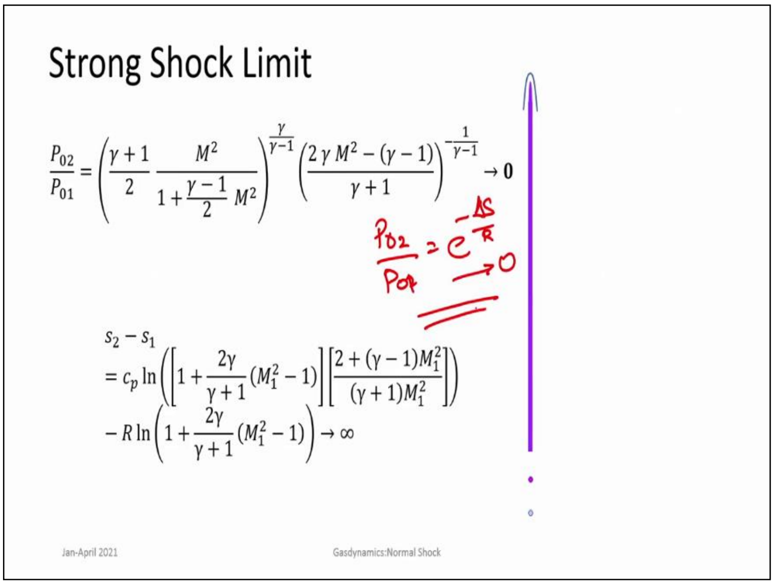

So, what about entropy? Entropy increases to infinity as a consequence the stagnation pressure ratio which is $\frac{P_{02}}{P_{01}}$ is nothing but $\mathbf{exp}\left(-\frac{\Delta s}{R}\right)$. So as ΔS increases to infinity $\frac{P_{02}}{P_{01}}$tends to 0. So stagnation pressure decreases to 0.

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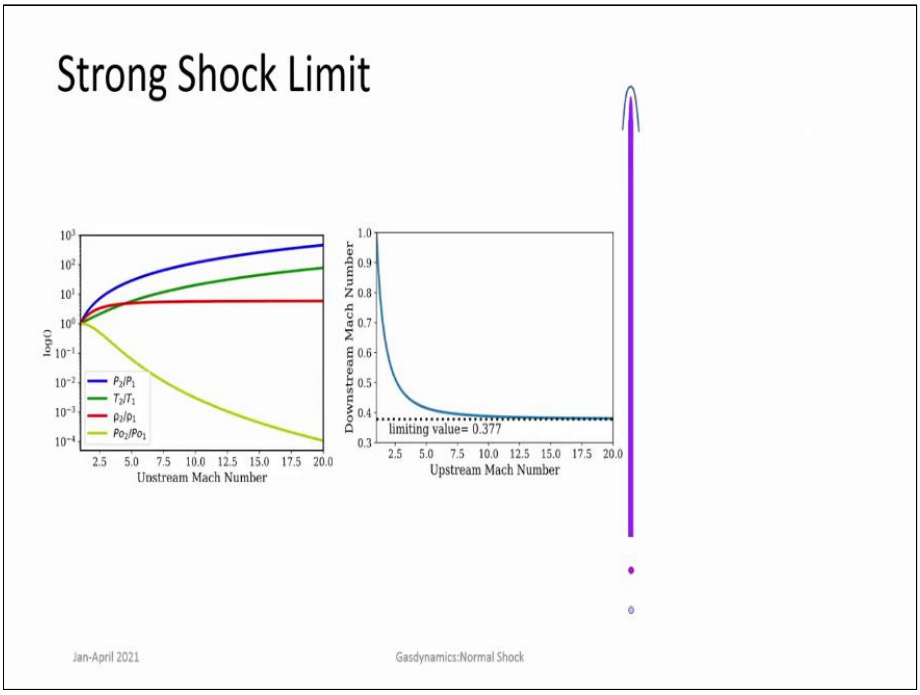

So again, let us review the graphs that we had plotted earlier and discussed it. So as for our discussions are strong shock limit you can see that density quickly saturates here and the value should be 6 and the Mach number (downstream Mach number) saturates to 0.377. While the pressure ratio and temperature ratio keep increasing and the stagnation pressure ratio continues to decrease and can decrease to 0.

So, this is a strong shock limit. Now the other side of the spectrum is when the Mach number is very very close to 1. So, you know that shock waves can be presented only in supersonic flows and that is condition is Mach number should be greater than 1. And the strength of the shock depends on the Mach number. So, the weakest possible shock is at Mach equal to one.

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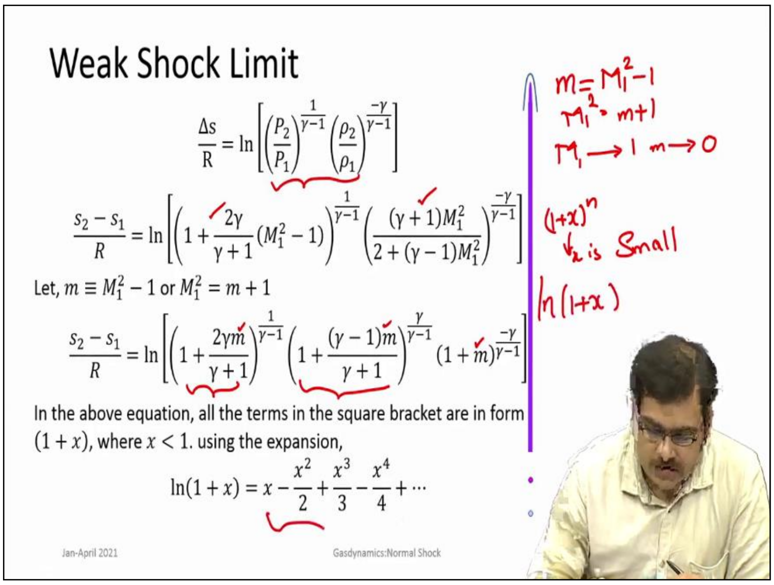

So, in the neighbourhood of Mach number equal to 1 we get the limit of very weak shocks. So, let us look at the limit of very weak shocks. So, to look at this limit, let us see what the entropy is change across the shock at very weak conditions. So, this is the same expression for entropy written in terms of pressure ratio and density ratios and pressure ratio and density ratio expressions for the normal shock is known so we substitute that in this expression over here and density ratio is substituted here.

Now this is in terms of parameter M^2-1 or $M_1^2 - 1$. This parameter is written as a coordinate or sort of variable transformation is done where $m \equiv M_1^2 - 1$ or $M_1^2 = m + 1$.

So, this is to make the analysis feasible. Now we are looking at the limit that M_1 tends to one. This is the weak shock limit so if $M_1 \to 1$, $m \to 0$. So, m tends to very small values.

So, if you now express the equations in terms of m as it is done over here then you find that these terms $1 + \frac{2\gamma m}{\gamma+1}$ or $1+ \frac{(\gamma-1)m}{\gamma+1}$ these terms are of the $(1+x)^n$ where x is small now here. So, this can be expanded using the series that is this is in terms of log. So, you can use the power series to expand this ln $(1+x)$. The expansion is given over here.

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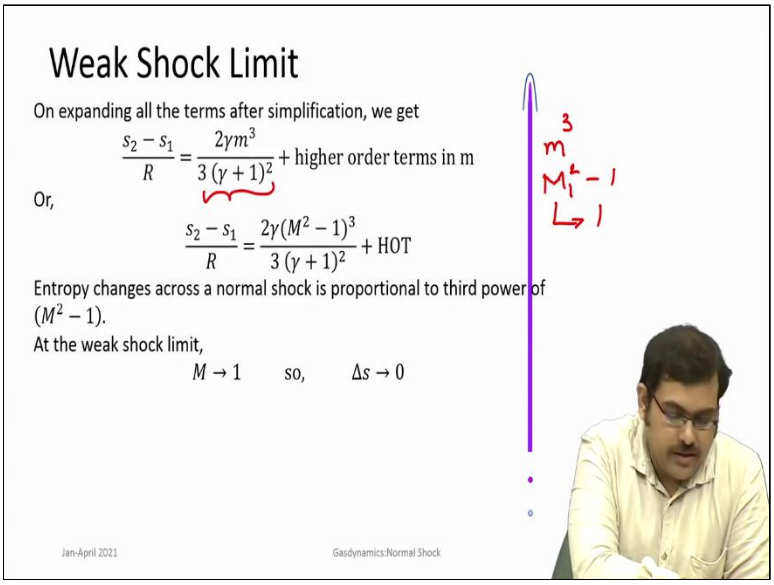

And you expand all the terms three terms and collect the terms together there will be some algebraic manipulations and cancellations and ultimately you end up with the term

$\frac{s_2 - s_1}{R} = \frac{2\gamma m^3}{3\,(\gamma+1)^2}$ + higher order terms in m.

The higher order terms are neglected. Even the first order term if you look at it is having a term m^3 that means it varies with the cube of m. Now m is already a small number because $(M^2 - 1)$ and M_1 is going towards one.

So, you see that as $M \rightarrow 1$, M_1 tends to one very quickly, rapidly $\Delta s \rightarrow 0$ that is entropy change goes to 0. That means in the weak shock limit essentially you are reaching isentropic conditions. So, the weak shock limit is almost like that of an acoustic wave or isentropic conditions.

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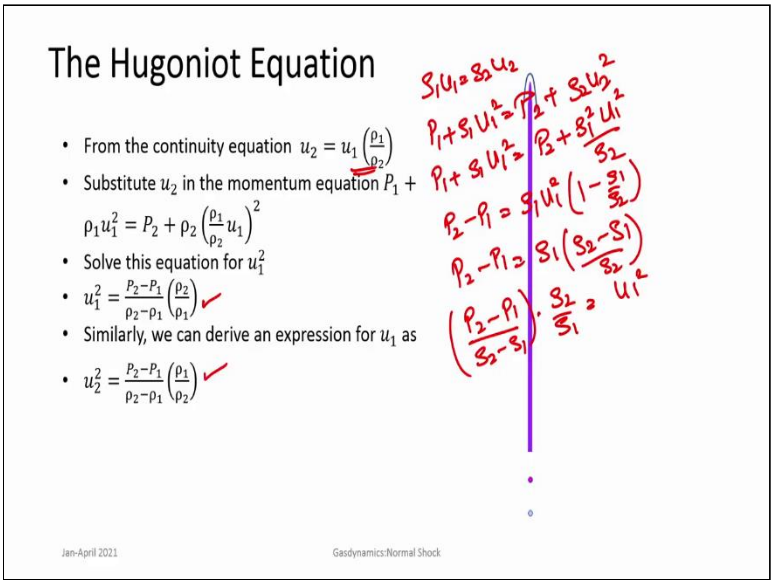

Having discussed the strong shock and weak shock limits, let us now look at another fact of these normal shocks that is the Hugoniot equation. The Hugoniot equation expresses the properties across the normal shock only in terms of thermodynamic variables and the velocity does not figure in the expression. So, one can look at all possible shock states and we will see how to do that.

Until now all the expressions for normal shocks had the upstream Mach number in place. So here we will look at how to write the equations for thermodynamic variables across the shock. So we have to appropriately combine the three equations and we begin from the continuity equation $u_1\,\rho_1 \;=\; u_2\rho_2$ and here directly you will get $u_2 = u_1\left(\frac{\rho_1}{\rho_2}\right)$.

Now we take the momentum equation $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$ so we have this equation. Now here we substitute for u_2 so $P_1 + \rho_1 u_1^2 = P_2 + \rho_2\left(\frac{\rho_1}{\rho_2}u_1\right)^2$. Now $P_2 - P_1$ is, you can take this common, $\rho_1 u_1^2$ and you will get $P_2 - P_1 \;= \rho_1 u_1^2(1 - \frac{\rho_1}{\rho_2})$ and this can be written as

$P_2 - P_1 \;= \rho_1 u_1^2(\frac{\rho_2 - \rho_1}{\rho_2})$.

So $u_1^2 = \frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_2}{\rho_1}\right)$. I think that is the expression that we get over here. So, the same analysis can be carried out by expressing u_2, u_1 in terms of u_2 and then getting the equation for u_2 if you proceed along the same directions you get $u_2^2 = \frac{P_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_1}{\rho_2}\right)$. So, you get these two terms and now we have used the mass and momentum conservation.

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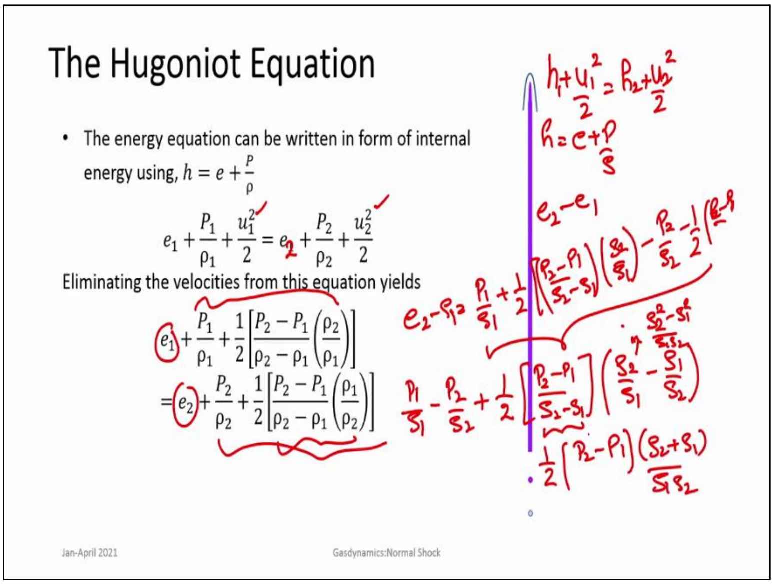

Now we move to energy conservation which is $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$

And here now h is nothing but $h = e + \frac{P}{\rho}$ and if you substitute that you get $e_1 + \frac{P_1}{\rho_1} + \frac{u_1^2}{2} = e_1 + \frac{P_2}{\rho_2} + \frac{u_2^2}{2}$. Now we have the expressions for u_1^2 and u_2^2. Just from the previous slide we substitute that here in these terms and we get $\left[\frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1}\right)\right]$.

Now this can be simplified because you have all the terms related to pressure and density over here and you have energy terms over here. So, now if you express this(e_2 - e_1). So, you get this term over there. $e_2 - e_1 = \frac{P_1}{\rho_1} + \frac{1}{2}\left[\frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1}\right)\right] - \frac{P_2}{\rho_2} + \frac{1}{2}\left[\frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_1}{\rho_2}\right)\right]$.

Now from here all we have to do is simplify the right-hand side of this equation and there you can observe that there is the group of terms like this ,

$$e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left[\frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1} - \frac{\rho_1}{\rho_2}\right)\right]$$

this is what you get.

So to do this term is nothing but $e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left[\frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2^2 - \rho_1^2}{\rho_1\,\rho_2}\right)\right]$. So this term will come out to be $e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left[\frac{P_2 - P_1}{1}\left(\frac{\rho_2 + \rho_1}{\rho_1\,\rho_2}\right)\right]$. So now this can be algebrically simplified.

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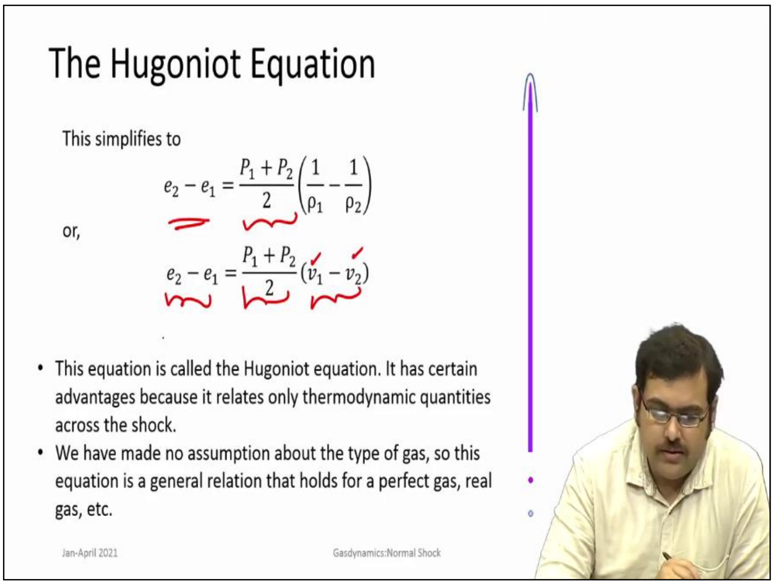

And after simplification this is the expression we get, $e_2 - e_1 = \frac{P_1+P_2}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$. So $e_2 - e_1$ is the energy difference, internal energy difference, across the shock wave while $\frac{P_1+P_2}{2}$is an average pressure written here and $\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$ written if expressed in terms of specific volume is $(v_1 - v_2)$ as change in specific volume. So, change in internal energy is equal to average pressure multiplied by change in specific volume, $e_2 - e_1 = \frac{P_1+P_2}{2}(v_1 - v_2)$.

So, this equation is known as the Hugoniot equation and one can readily see that there is no velocity terms over here. It purely expresses only in terms of thermodynamic variables. And also, here when getting to this equation we have never stated any assumptions of perfect gas, so this is valid for a general case. So Hugoniot equation is a general equation. It is valid for shocks in all kind of gas dynamic conditions.

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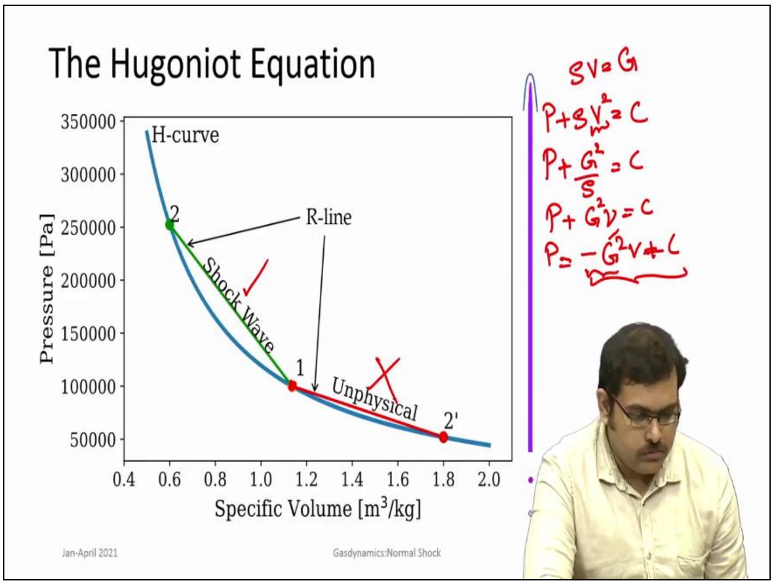

So, let us look at the plot. So, this is a Hugoniot curve. It plots all possible shock states starting from 1 where 1 represents the upstream conditions. So upstream once you know the upstream pressure and temperature consequently the specific volume. This point can be located and the Hugoniot curve can be drawn. And now how do we get to a particular shock if you know a particular velocity or you know the particular Mach number.

For that we draw what is known as the Rayleigh line. This is the Rayleigh line. Rayleigh line just comes about from the momentum equation, $p+\rho u^2 = constant$. We can use the energy equation to use to combine with this because ρu is constant across the shock. So if we can express u in terms of so this is $\rho u = G$ is another constant. So this becomes $p+\frac{G^2}{\rho} = constant$ or $p + G^2 v$ = constant. $P = -G^2 v + C$.

So this is the equation of a straight line so Rayleigh lines are straight lines in the PV diagram and you should also notice that it has a negative slope, G^2 is positive. So, it has its always having a negative slope. So now this Hugoniot equation as we all already know it just uses the conservation equations and the conservation equations do not specifically say whether a certain thermodynamic state is possible or not that comes from entropy conditions.

So, from 1 we can draw Rayleigh lines going either way with a decrease in pressure and increase in specific volume or an increase in pressure and decrease in specific volume and this is the case of an expansion shock. It is unphysical. So this is not possible while only the shock wave which is a compression shock is possible. So with the help of Rayleigh line which has the information of velocity which will figure in the term G^2. One can locate a particular shock.

So Hugoniot can give you all possible shock states passing through an initial point one. So Hugoniot is a general equation. You can use it for any type of shocks in any gas dynamic medium.

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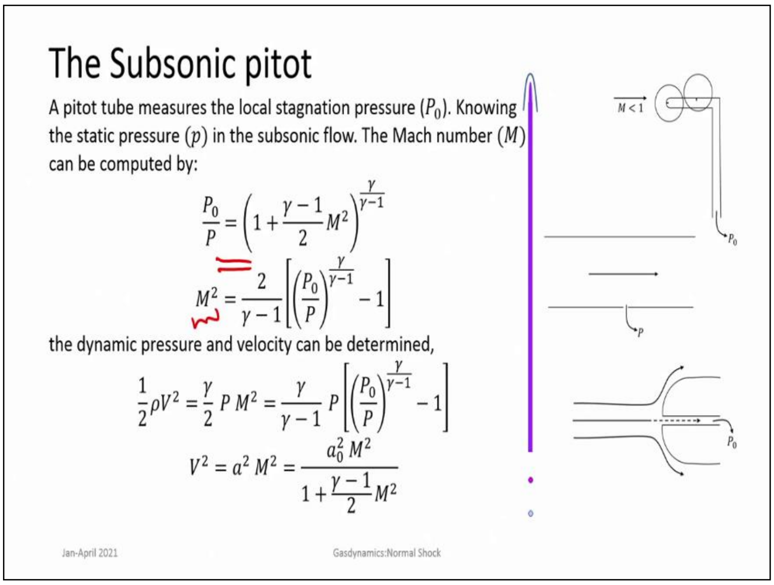

So now we come to a particular application of the shocks. Until a few classes ago we were discussing the applications of isentropic equations to the problem of flow measurement and the Pitot. And we discussed the subsonic Pitot where in a compressible medium or in compressible flow one cannot use the Bernoulli's equation but rather we have to use the condition that you achieve stagnation pressures within the Pitot. Pitot measures stagnation pressure.

And the relationship with stagnation pressure and static pressure is given by an isentropic process and from this one can get the Mach number. So this is for a subsonic Pitot.

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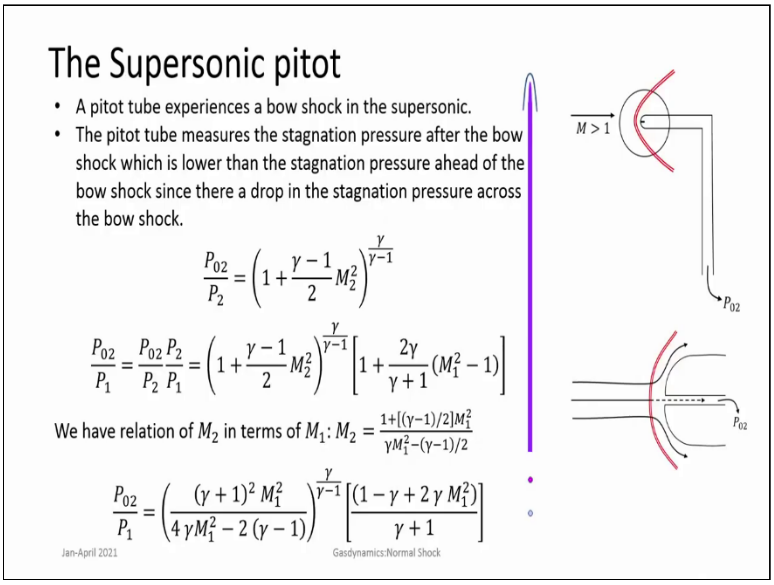

Now let us go to the condition that the same Pitot this device Pitot is used is to measure the flow speed or flow velocity what happens when we put this Pitot inside a supersonic flow. So it is now very clear that in supersonic flows if you place some bodies or you put some devices then from our earlier discussions you should understand that in order for the flow to turn over the body, the flow has to know that the body is present over here.

In a supersonic flow this cannot be accomplished because information does not propagate in all directions. In order to facilitate this ultimately get a shock wave a shock wave envelops the body this is what is coming over here. So a shock wave envelops the Pitot. So there is a Pitot, there is a shock wave in front of the Pitot. And what the Pitot measures actually is stagnation pressure downstream of the shock.

So if you take a look at the zoomed in picture this is the idea over here, that you have the stagnation stream line. It is going through right here and this is the Pitot and very close to the nose of the Pitot. There is a shock and this shock is normal to the free stream line at this point. So one can apply the normal shock conditions across the shock. And the Pitot actually measures the stagnation pressure downstream of the shock wave.

So if you put a Pitot in a supersonic stream then it does not measure the stagnation pressure of the stream rather it measures the stagnation pressure downstream of the shock. And by now it is very clear to us that stagnation pressure decreases across the shock. So P_{02} is less than P_{01} and it is not the same as P_{01}. So now we have to express so just going by the similar measurement.

So how measurements are done usually in some experiments and practically is that usually the Pitot is placed inside the duct or within the free stream and P_{02} is measured and you need always one more measurement that is the static pressure you need to measure. And for that the measurement is usually taken on the walls of the tunnel somewhere where static pressure of the stream can be easily measured.

So this is P^1 so usually the measurement is P_{02} and P_1. The stagnation pressure ratio is expressed for P_{02} by P_2 that is for the downstream conditions of the shock and it is dependent on M_2 square. But we know that all these values P_2, P_{02} ,M_2 all of them depend only on the upstream Mach number. So they can be expressed as functions of only upstream Mach number. So that is what is done over here P_{02} / P_1 is what we are usually measuring.

And $\frac{P_{02}}{P_1}$ is written as $\frac{P_{02}}{P_1} = \frac{P_{02}}{P_2}\frac{P_2}{P_1}$. $\frac{P_2}{P_1}$ can be expressed in terms of upstream Mach number. $\frac{P_{02}}{P_2}$ expressed in terms of M_2^2 . M_2^2can be written in terms of M_1^2 and so we get this expression for $\frac{P_{02}}{P_1} = \left(\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} \left[\frac{(1-\gamma+2\gamma M_1^2)}{\gamma+1}\right]$. This is known as Rayleigh's Pitot formula. And this formula is generally given in normal shock tables along with all other equations and the pressure ratios and temperature ratios.
Because this is a very common measurement that is made and we would like to know the Mach number. So once you know $\frac{P_{02}}{P_1}$ either by using the tables or by using calculators we can get back what is Mach number. So this is Rayleigh Pitot formula.

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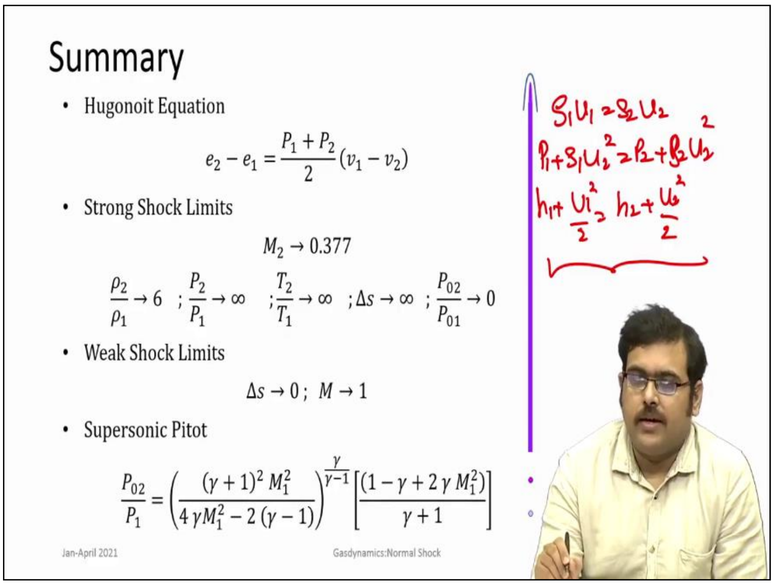

So with this we come to the end of discussions on stationary normal shocks in this particular discussion we used $\rho_1 u_1 = \rho_2 u_2$, $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$ and $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$. So in all these are conservation equations for a steady flow and this is what is considered is a normal shock is steady in a supersonic flow.

Now next what we would discuss is the moving shocks. The shocks can also move. Typical cases that you one can encounter is inside shock tubes or in cases of explosive events when there is a blast wave. The blast wave also moves rapidly at supersonic speeds. So we will see how we can analyse such shock waves in the next class.