Gasdynamics: Fundamentals and Applications
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Lecture 16
Normal Shock - II a

So in the previous class we had a introduction to shock waves and we learned that these waves, shock waves, are present only in supersonic flow. This is something that you have to bear in mind throughout this course that shock waves exist only in supersonic flows. We saw through a simple kind of examples or analogous example that presence of non-linearities actually sort of allows compression waves which are not really very strong.

These compression waves can coalesce with each other due to a process known as wave steepening this because of the non-linearities in the flow. Where flow variables are affecting each other, and the solution is dependent on flow variables and because of that they coalesce to each other and form a shock. Shocks are very very thin so when we look at gas dynamic analysis, we do not actually consider these shock waves, but they are considered as discontinuities.

So, they are not considered in solving the flow field and things like that but as discontinuity. So, a jump condition across the shock is what is applied when shocks are found in the flow field. So now we have to analyse these jump conditions across the shock. So, across a shock pressure, temperature and density increases and velocity decreases, Mach number decreases. So that is the essential picture.

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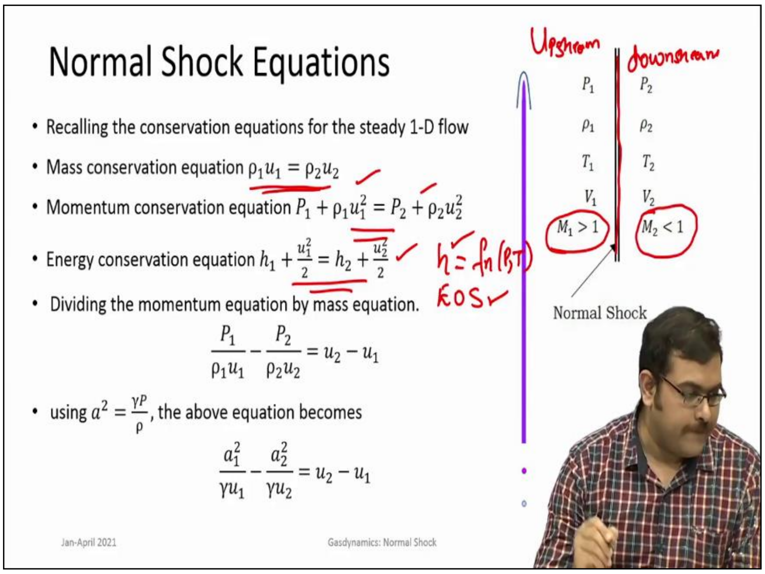

So now we will apply the conservation equations across the shock and here shock is treated as a discontinuity is extremely thin. You do not calculate structure of the shock wave. So, this is the schematic representation that is given here that you have a very thin shock. And before the shock these are conditions this is upstream, and this is downstream. So, you consider 2 states upstream and downstream of the shock. For sure the condition upstream of the shock has to be supersonic i.e., Mach number should be greater than 1.

Now this is a one-dimensional flow and there is no heat transfer or work done. So, it is an adiabatic flow. Please make a note it is an adiabatic flow. As we do the equations, we will find out that shocks generate entropy. So, it is not an isentropic flow, or it is not an isentropic process. So, the basic equations that you have to apply here the one-dimensional conservation equations. This we have already discussed. And here one-dimensional conservation equation is mass conservation, $\rho_1 u_1 = \rho_2 u_2$,is given over here.

Then momentum conservation in one dimension

$P + \rho u^2$ is constant. so $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$

Then energy conservation $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$

So, these are the 3 equations along with an equation of state and equations relating thermodynamic relations relating enthalpy as a function of pressure and temperature. This is the most general form and an equation of state. So, if you have these equations you can solve for these 3 equations. Now if you can see how many generally the upstream conditions are known so pressure density temperature and velocity are known.

So, we solve for pressure density temperature and velocity there are four unknowns here. There are 3 conservation equations and enthalpy is related to pressure and temperature in a general form by some form like this and there is a fourth equation of state. So, we have a consistent set of equations. So, this can be solved. This is the most general form. Here no assumptions are used, and it can be applied to any gas dynamic flow. So, including if there are changes in the Cp and it is not a calorically perfect gas, you can apply these equations in that case also.

But then you may not get closed form solutions in that case. So, for this class we will seek closed form solution. So, we can understand something about the nature of these normal shocks how they behave with changes in Mach number and so on. So, we will take the assumption of a perfect gas and calorically perfect gas in particular. And let us go through the different steps to help us relate the various quantities. First thing that we would try to relate is the Mach number which is downstream of the shock to the upstream Mach number.

And the relation is for a normal shock and normal shock is normal to the direction of the flow and for that it is always that the downstream Mach number is subsonic and upstream Mach number is supersonic. So how do we go about this process?

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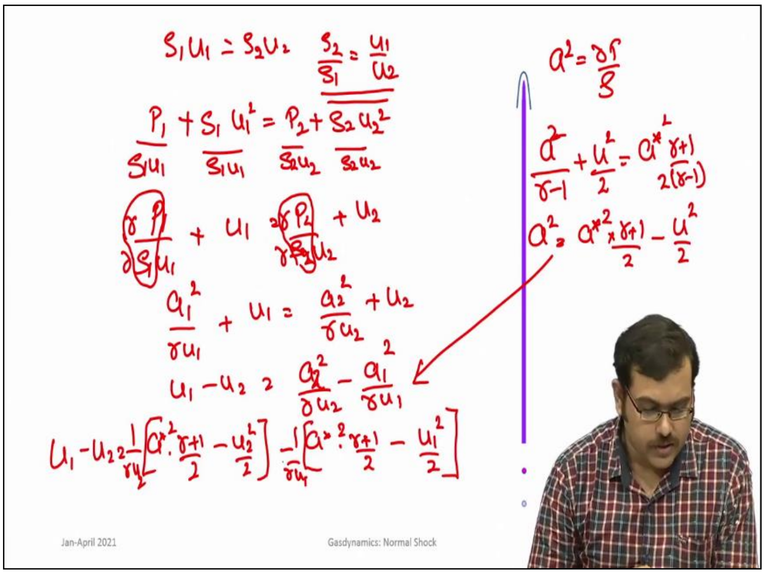

So first we consider that the continuity equation $\rho_1 u_1 = \rho_2 u_2$. And the direct relation that comes about from here is $\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$. Now we can keep this bear this in mind. Now second equation, so you if you look at the methodology that is followed over here, you see that this process is completely, I mean simultaneous, or you have to solve them together.

You cannot solve them separately you still do not know any variables as of now but we can through algebraic manipulations of these equations we can get to some relations which will help us to get to the final normal shock relations. So, if you consider the momentum equation $P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$ and divide this by the corresponding mass conservation equation.

So, I divide everything by the mass, but mass flow is constant. so I can divide this because $\rho_1 u_1 = \rho_2 u_2$, I can do this and on both sides of the equation. So, I get these terms P1 by rho1 u1. So this comes out $\frac{P_1}{\rho_1 u_1} + u_1 = \frac{P_2}{\rho_2 u_2} + u_2$. Now if you look at this and you identify in the equations here there is a combination $\frac{P}{\rho}$. Here there is a combination $\frac{P_2}{\rho_2}$ and we know that for a perfect gas $a^2 = \frac{\gamma P}{\rho}$ .So, we will try to see how we can bring this information into this equation and you can do that by multiplying and dividing by γ and $\frac{\gamma P}{\rho}$ is a factor here. This is one factor here a group and so you can write this as $\frac{a_1^2}{\gamma u_1} + u_1 = \frac{a_2^2}{\gamma u_2} + u_2$. Now at this you can even do a sort of change of you can write this only in terms of u_1 - u_2 is

$u_1 - u_2 = \frac{a_2^2}{\gamma u_2} - \frac{a_1^2}{\gamma u_1}$.

Now here we can introduce the using the alternate form of the energy equation which is

$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a^{*2}(\gamma+1)}{2(\gamma-1)}$.

This is the alternate form of the energy equation that we had discussed in terms of the sonic quantities. So, from here you can write $a^2 = \frac{a^{*2}(\gamma+1)}{2} - \frac{u^2(\gamma-1)}{2}$.

So now the process is just to use this in this particular equation. Import it here so you can get

$$u_1 - u_2 = \frac{1}{\gamma u_2}\left[\frac{\gamma+1}{2}\frac{a^{*2}}{1} - \frac{\gamma-1}{2}u_2^2\right] - \frac{1}{\gamma u_1}\left[\frac{\gamma+1}{2}\frac{a^{*2}}{1} - \frac{\gamma-1}{2}u_1^2\right]$$

So you get this equation over here now we will just consider just the right hand side for simplification and try to simplify that part.

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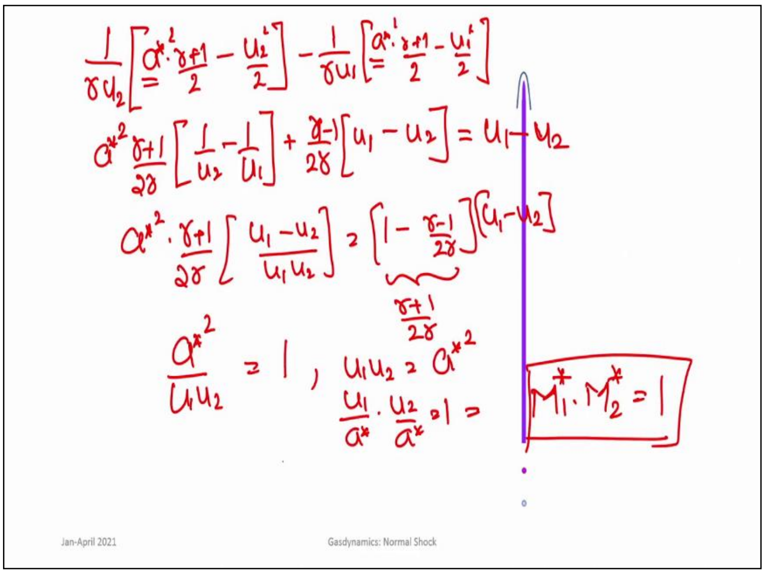

So let us look at this.

Now the principle being used here is this is an adiabatic flow. For an adiabatic flow the sonic state remains the same. So, a stars remains the same for both upstream and downstream flows.

So this, $a^\star$ is constant across the shock. So, we have now written it in terms of $a^\star$ and we can get quantities here common taking these common terms $\frac{\gamma+1}{2\gamma}\, a^{\star 2}$,

$$(u_1 - u_2) = \frac{\gamma+1}{2\gamma u_1 u_2}(u_1 - u_2)a^{\star 2} + \frac{\gamma-1}{2\gamma}(u_1 - u_2)$$

So this term turns out to be $\frac{\gamma+1}{2\gamma}$. So you find the common terms are $\frac{\gamma+1}{2\gamma}(u_1 - u_2)$ and that sort of cancels out. You get $\frac{a^{\star 2}}{u_1 u_2} = 1$ or $a^{\star 2} = u_1 u_2$.

So if you divide this by $a^{\star 2}$, you will get $\frac{u_1}{a^\star}\frac{u_2}{a^\star} = 1$. With this $\frac{u_1}{a^\star}$ is something that we already know from our previous discussions it is the star Mach number $M^\star$.

So this gives rise to the relation $M_1^\star M_2^\star = 1$. So, this relation is known as Prandtl's relation for the normal shock. And it gives us so you can observe these manipulations, algebraic manipulations, where all the equations were used simultaneously to arrive at these conditions.

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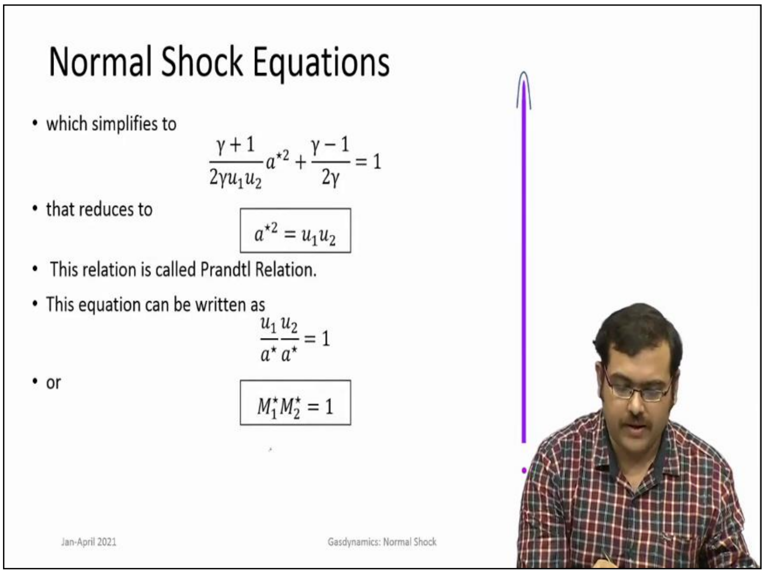

And so just to go over these so what we did was that we use the momentum equation and divided it by the mass equation as you have done over here and then written that equation in terms of the speed of sound over here. And then use the energy equation the modified form of the energy equation in terms of the star speed of sound and a star. And the idea is that a star remains constant in a normal shock across a normal shock.

So we use that information and import that into the equation here and then do the simplification using some algebraic manipulations. And finally arrive at the condition that $M_1^\star M_2^\star$ across the shock equal to 1. So now we know the upstream Mach number so this gives us a relation to find go ahead and find the downstream Mach number. Now this is in terms of star quantities.

So we can convert this into the Mach number directly into Mach number and that is possible so this is all the steps are detailed in the slides which just now we have discussed. So this is what we finally arrive at, $M_1^\star M_2^\star = 1$

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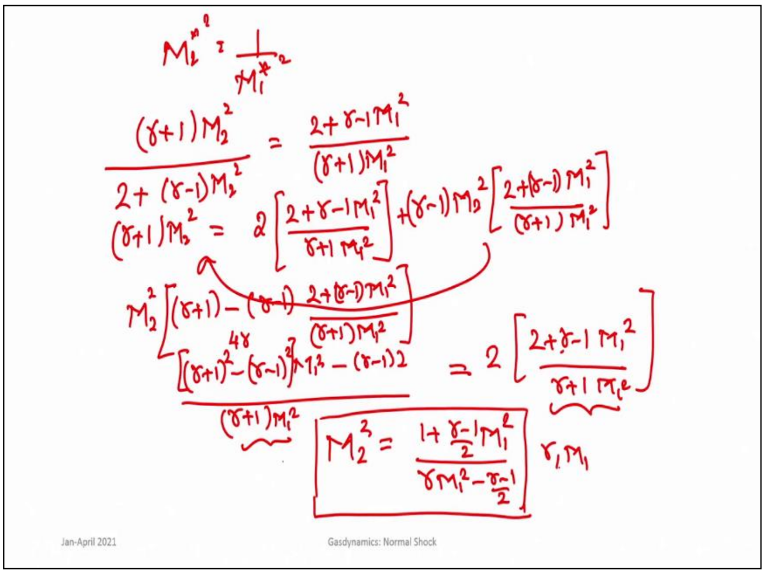

So now let us look at how to get a single relation between this downstream Mach number to upstream Mach number. We use the definition of so $M_1^\star = \frac{1}{M_2^\star}$ and so we can directly use the relation for $M_2^\star$ that is $\frac{(\gamma+1)M_2^2}{2+(\gamma-1)M_2^2} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}$ .So you get the inverse.

$$(\gamma+1)M_2^2 = 2\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right] + (\gamma-1)M_2^2\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$$

$$M_2^2\left((\gamma+1) - (\gamma-1)\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]\right) = 2\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$$

$$M_2^2\left(\frac{(\gamma+1)^2 M_1^2 - (\gamma-1)^2 M_1^2 - 2(\gamma-1)}{(\gamma+1)M_1^2}\right) = 2\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]$$

So, you have these terms are common they will get cancelled. So finally the equation is

$$M_2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}$$

So, you find that M_2 can be completely represented only in terms of γ and Mach number before the shock. So, now when we look at all other relations shock relations what we will observe is that all the properties across the shock can be written only in terms of the upstream Mach number and γ .

Further for the conditions that we are taking, which is calorically perfect gas, γ is a constant. If you know the gas then γ is known. So then the downstream conditions of the shock become only the functions of upstream Mach number. So now we can relate. So now since we know the relation between downstream Mach number and upstream Mach number, we can use this to relate all other flow variables.

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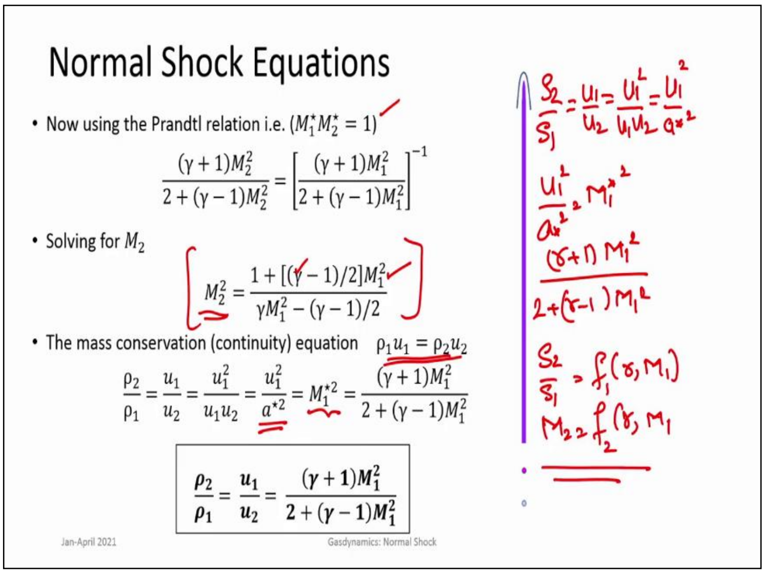

So, let us look at first, we look at the density ratio. So just to go over what we had just done, and we have expressed the Mach number that is Mach number downstream of the shock M2 in terms of only Mach number upstream of the shock and γ. So, we saw how to go about doing this using the Prandtl's relation. Now we will see how we can use the continuity equation to relate the density ratios across the shock.

What we need is $\frac{\rho_2}{\rho_1}$ and $\frac{\rho_2}{\rho_1}$ that is density increases across the shock and from mass conservation equation, $\rho_1 u_1 = \rho_2 u_2$, you directly get that $\frac{\rho_2}{\rho_1}$ is $\frac{u_1}{u_2}$. So now if you look and multiply and divide by u_1 then you get this as $\frac{u_1^2}{u_1 u_2}$. Now $u_1 u_2$ from our just recently discussed equations, $u_1 u_2 = M_1^{\star 2}$.

So, this is $\frac{u_1^2}{a^{\star 2}}$. Now that is what is written over here and $\frac{u_1^2}{a^{\star 2}}$ is nothing but the $M_1^{\star 2}$ for the upstream flow and star Mach number is written in terms of Mach number itself.

$M_1^{\star 2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$. So now we see that the density ratio is also $\frac{\rho_2}{\rho_1}$ is also only function of γ and upstream Mach number. M_2 is function of γ and upstream Mach number.

So, to just to distinguish I can put this as f_1 and f_2. So, we will now go ahead and do it for all other variables. So, we have pressure ratio, we have temperature ratio, we have to look at what happens to stagnation temperature, pressure, entropy generation across the shock. So, each of them we will go through in detail and that we will do in the next class and see how these variables vary across the shock wave.