Smart Structures

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Week - 02
Lecture No - 09

3D Constitutive Modeling of Piezoelectric Materials -3

Welcome to the third lecture on Constitutive Modelling of Isoelectric Materials.

So, in the last lecture we talked about the first and second law of thermodynamics. So, today we will continue with that. Now, we will relate the first and second law and we will get some equation from where it becomes easier for us to derive the constitutive relations. So, combining first and second law of thermodynamics we get T s dot plus sigma i j epsilon i j plus E di minus u dot minus q i T multiplied by T greater than equal to 0. Now, U here is function of the strain electric vector electrical displacement and entropy s.

$$T\dot{S} + \sigma_{ij}\varepsilon_{ij} + E_i D_i - \dot{U} - \frac{q_i}{T}T_{,i} \geq 0$$

$$U = U(\varepsilon_{ij}, D_i, S)$$

$$\dot{U} = \frac{\partial U}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial U}{\partial D_i}\dot{D}_i + \frac{\partial U}{\partial S}\dot{S}$$

$$\left(\sigma_{ij} - \frac{\partial U}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(E_i - \frac{\partial U}{\partial D_i}\right)\dot{D}_i + \left(T - \frac{\partial U}{\partial S}\right)\dot{S} - \frac{q_i}{T}T_{,i} \geq 0$$

$$\sigma_{ij} - \frac{\partial U}{\partial \varepsilon_{ij}} = 0 \Rightarrow \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, E_i = \frac{\partial U}{\partial D_i}, T = \frac{\partial U}{\partial S}$$

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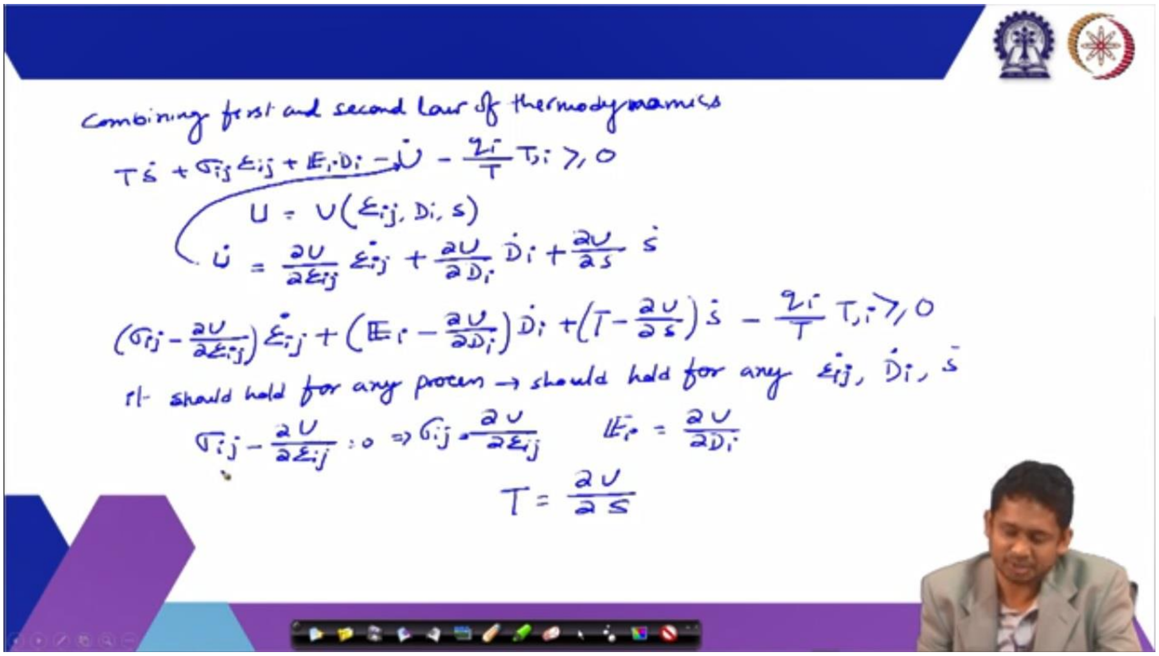

Now, so these are the state variables here. So, if that is the case then U dot should be is equal to del U by del epsilon i j divided by epsilon i j dot plus del U divided by del di multiplied by di dot plus del u by del s multiplied by s dot. So, if you put it here this becomes sigma i j minus del U by del epsilon i j multiplied by epsilon i j dot plus E i minus del U by del di into di dot plus T minus del U by del s multiplied by s dot minus q i by T multiplied by T comma i greater than equal to 0. Now, it should hold for any process.

So, this relation should hold for any process. So, it should hold for any so, which means that it should hold for any epsilon i j dot Di dot and s i dot. So, if that is the case then the only possibility is sigma i j minus del U by is 0 which means that sigma i j is equal to del u by del epsilon i j and then accordingly we get E i electric field i is equal to del U by del i and T is equal to del U by del s. Now, we saw that this sigma as a matrix can be simplified as sigma as a vector because we have already assumed that our stress matrix is symmetric by ignoring the moment due to electric field and polarization. So, if that is the case then sigma can be assumed to be a vector and accordingly epsilon i j can also be assumed to be a vector.

So, if we do that then we can write simply sigma i is equal to or maybe we can use a different index maybe sigma m is equal to del U by del epsilon m where m goes from 1 to 6 because it has because both stress and strain has 6 components. So, we wrote our stress matrix as a vector by assuming that the shear stresses are I mean complementary shear stresses are same and similarly ah the epsilon matrix the strain matrix can also assume to be a vector of 6 of 6 quantities assuming that the complementary shear strain strains are equal and that helps us write it in this way. Now, but again we will simplify the notations and with that simplification of notation we can write increment in sigma which is d sigma.

So, we are not carrying out with any suffix i j or m just for the simplification and we can write this as del sigma by del epsilon keeping D and s constant multiplied by d epsilon plus del sigma by del d keeping epsilon and s constant plus del sigma by del s keeping D and epsilon as constant. Similarly, we can take the sorry we have to multiply D and d s here also like what we did here.

$$\sigma_m = \frac{\partial U}{\partial \varepsilon_m}, m = 1, \ldots, 6$$

$$d\sigma = \left(\frac{\partial \sigma}{\partial \varepsilon}\right)^{D,S} d\varepsilon + \left(\frac{\partial \sigma}{\partial D}\right)^{\varepsilon,S} dD + \left(\frac{\partial \sigma}{\partial S}\right)^{D,\varepsilon} dS$$

$$dE = \left(\frac{\partial E}{\partial \varepsilon}\right)^{D,S} d\varepsilon + \left(\frac{\partial E}{\partial D}\right)^{\varepsilon,S} dD + \left(\frac{\partial E}{\partial S}\right)^{D,\varepsilon} dS$$

$$dT = \left(\frac{\partial T}{\partial \varepsilon}\right)^{D,S} d\varepsilon + \left(\frac{\partial T}{\partial D}\right)^{\varepsilon,S} dD + \left(\frac{\partial E}{\partial S}\right)^{D,\varepsilon} dS$$

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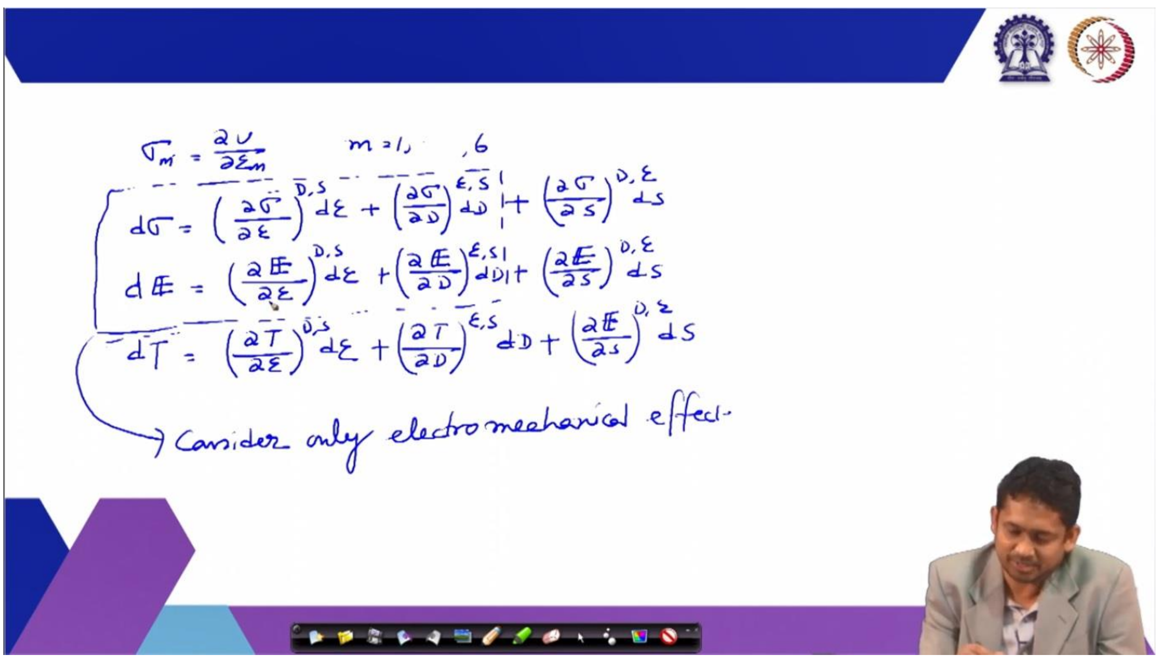

So, these are partial derivatives with respect to some state multiplied by d of that state and their summation. Accordingly we can write T of electric field E again electric field is a vector, but we have just simplified it and we are not carrying these suffixes. This keeping D and s constant multiplied by D e plus this divided by d keeping epsilon and s as constant by s keeping d and epsilon as constant. And accordingly we can write d T as delta T by delta epsilon keeping d and s as constant multiplied by d epsilon plus delta T by delta d epsilon and s as constant multiplied by again we have to multiply this d d and d s and d d and del s keeping d and epsilon as constant multiplied by d s. Now, this sigma has 6 components.

So, there can be 6 equations like that. So, this is this will give us a 6 by 6 matrix and there will be 6 of this d epsilons and there are 3 number of d's. So, this will give us a 6 by 3 system and and s is just 1 1 s. So, it will give us a 6 by 1 system. Likewise, we will get things here.

Now, what we do is we will simplify it further and we will assume a purely electromechanical process. So, assuming so, considering only the electromechanical effect if we consider only the electromechanical effect these 2 quantities can be taken out and similarly this entire equation can be taken out. So, we are considered only the electric and mechanical quantities. So, we do not incorporate temperature in our problem. So, we will make use of this equations after some time.

$$d\sigma = \frac{\partial^2 U}{\partial \varepsilon^2} d\varepsilon + \frac{\partial^2 U}{\partial \varepsilon\, \partial D} dD$$

$$dE = \frac{\partial^2 U}{\partial \varepsilon\, \partial D} d\varepsilon + \frac{\partial^2 U}{\partial D^2} dD$$

$$\begin{Bmatrix} \{\sigma\}_{6\times 1} \\ \{E\}_{3\times 1} \end{Bmatrix} = \begin{bmatrix} [C^D]_{6\times 6} & [n]^T_{6\times 3} \\ [n]_{3\times 6} & [\beta^\varepsilon]_{3\times 3} \end{bmatrix} \begin{Bmatrix} \{\varepsilon\}_{6\times 1} \\ \{D\}_{3\times 1} \end{Bmatrix}$$

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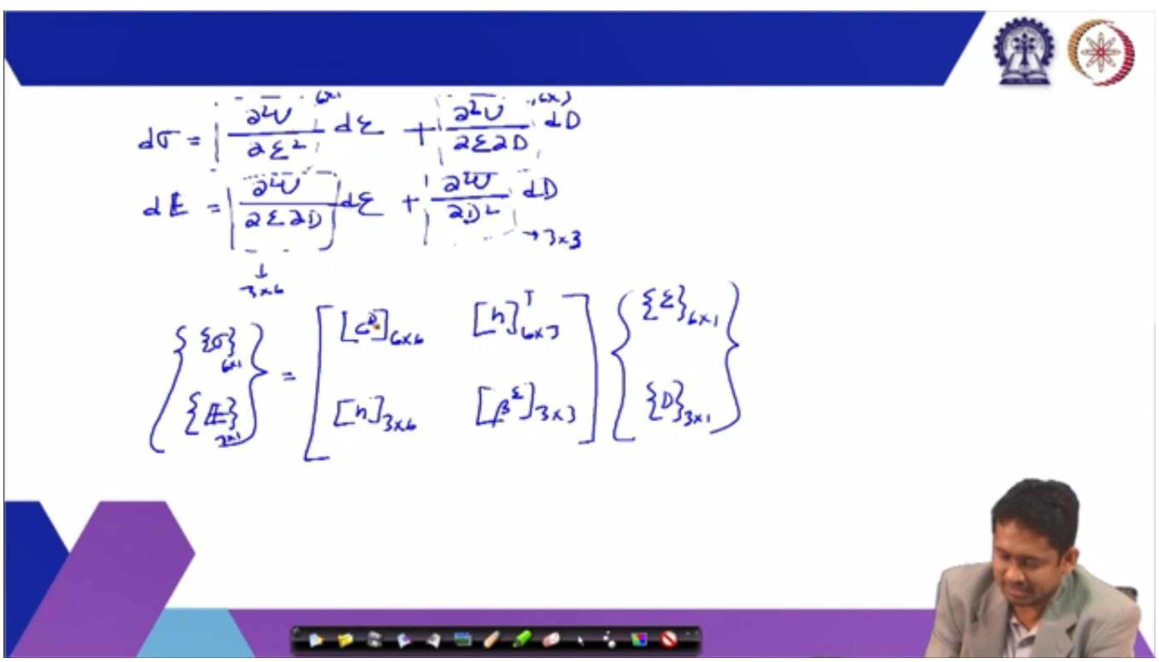

Now before that now let us write increment of sigma as del epsilon because we have seen that the first term here is del epsilon by del sigma multiplied by d e sorry del sigma by del epsilon multiplied by d e and sigma is already del u by del epsilon. So, we can write this plus del 2 u by del epsilon del d multiplied by d D. So, the second term in the expression for d sigma was del sigma by del d and again sigma is equal to del u by del epsilon. So, we can write this accordingly we can write d E as del 2 U by del epsilon del d into d epsilon plus del 2 u by del d 2 into d d. Now this is sigma is a 6 by 1 vector and epsilon is a 6 by 1 vector.

So, we will and d is a 3 by 1 vector. So, we are going to get 6 equations out of it and it is going to give me a 6 by 1 6 by 1 matrix and similarly this is going to give me 6 by 3 matrix this is going to give me 3 by 6 because e is 3 by 1 and epsilon is 6 by 1 and this is going to give me 3 by 3 matrix. So, with that we can write sigma epsilon is equal to C D this is epsilon 6 by 1 this is our D 3 by 1 and this is also 6 by 1 this is 3 by 1. So, this is our constitutive relation where epsilon and D strain and displacement are my epsilon and displacement are my state states. So, this is one form of the constitutive relations.

Now here I am writing C D. So, C relates strain C relates stress with the strain and we are putting this D superscript because the other state vector is D. Similarly we are writing beta here. So, beta is relating electric field with the di-elect displacement electrical displacement and we are putting a superscript epsilon because the other state vector is epsilon. Now we will try to write the constitutive relation in a different form.

So, the internal energy in the differential form can be written as d U is equal to sigma d epsilon plus E d D. Now we will define another ah we will define another potent thermodynamic potential enthalpy as H is equal to U minus E D. So, if I take dH is equal to d U then it becomes electric field multiplied by dD minus D multiplied by differential of the electric field. So, if I take this dU here finally, the expression becomes sigma D epsilon minus dD electric field E. Now we can write dH as also del of H divided by del of I mean del H by del epsilon keeping electric field as constant multiplied by D multiplied by D epsilon plus del H by del E keeping epsilon as constant multiplied by D E.

$$dU = \sigma d\varepsilon + EdD$$

$$H = U - ED \Rightarrow dH = dU - EdD - DdE = \sigma d\varepsilon - DdE$$

$$dH = \left(\frac{\partial H}{\partial \varepsilon}\right)^{E} d\varepsilon + \left(\frac{\partial H}{\partial E}\right)^{\varepsilon} dE$$

$$\frac{\partial H}{\partial \varepsilon} = \sigma, \frac{\partial H}{\partial E} = -D$$

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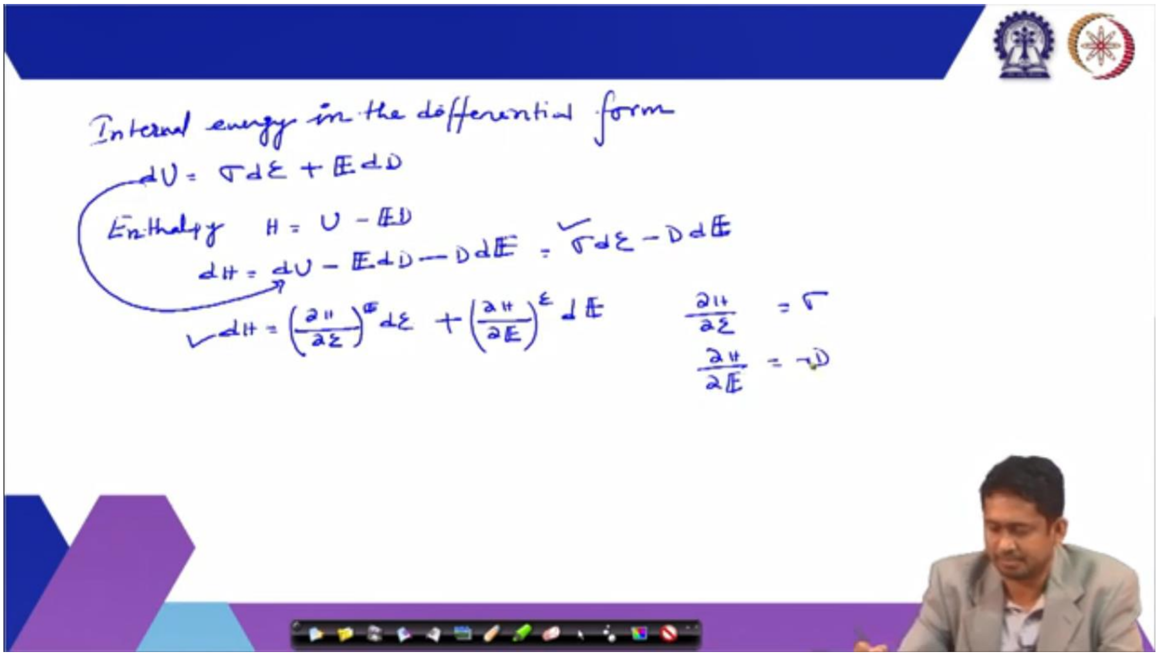

So, from here we can ah we can say that del H by del E is our sigma. So, if I relate this with this relation then we can say that del H by del epsilon is sigma and del H by del E is minus of D. Now what we do is we write D sigma increment in stress. So, we write D sigma as del sigma by del epsilon keeping E as constant multiplied by D E plus del sigma by del E keeping epsilon as constant multiplied by D electric field and that from that we can write this as del 2 H by del epsilon 2 into E plus del 2 H into del E and multiplied by del E. Accordingly we can write dD as dD by d epsilon keeping electric field as constant multiplied by D E plus D D by D E keeping epsilon as constant multiplied by dE and this we can again rewrite as if we put D in terms of H and then we can write as minus del 2 H plus D E.

$$d\sigma = \left(\frac{\partial\sigma}{\partial\varepsilon}\right)^{E} d\varepsilon + \left(\frac{\partial\sigma}{\partial E}\right)^{\varepsilon} dE = \frac{\partial^2 H}{\partial\varepsilon^2} d\varepsilon + \frac{\partial^2 H}{\partial\varepsilon\,\partial E} dE$$

$$dD = \left(\frac{\partial D}{\partial\varepsilon}\right)^{E} d\varepsilon + \left(\frac{\partial D}{\partial E}\right)^{\varepsilon} dE = -\left(\frac{\partial^2 H}{\partial\varepsilon\,\partial H}\right) d\varepsilon + \left(\frac{\partial^2 H}{\partial\varepsilon^2}\right) dE$$

$$\begin{Bmatrix} \{\sigma\}_{6\times1} \\ \{D\}_{3\times1} \end{Bmatrix} = \begin{bmatrix} [C^E]_{6\times6} & -[e]^T_{6\times3} \\ [e]_{3\times6} & [\varepsilon^{\varepsilon}]_{3\times3} \end{bmatrix} \begin{Bmatrix} \{\varepsilon\}_{6\times1} \\ \{E\}_{3\times1} \end{Bmatrix}$$

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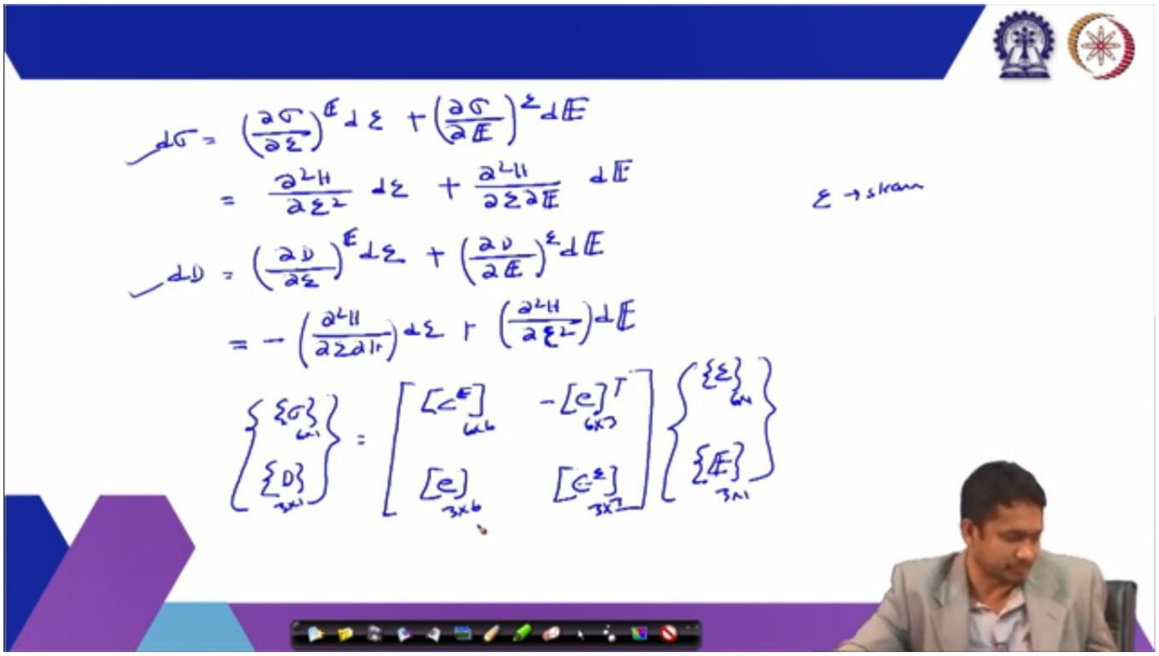

So, we have this and this and again we can combine these two relations we can write a constitutive relation in this form which is C E E. So, these are again some new constants that we are defining this. So, these are 6 by 1 vector, this is a 3 by 1 vector, this is 6 by 1 and this is 3 by 1. So, this is 6 by 6, this is 6 by 3, this is 3 by 6, this is 3 by 3. So, here we are using just if you look back, we are using two epsilons, one epsilon is for strain and then there is another form of epsilon that is here which is generally called a lunate epsilon and that is here to denote this constant.

So, two variations of epsilon we are using one for strain and one for this electrical constant and it is no a coupling constant it is just relating this electrical variable electric field to D. So, here this C relates stress and strain and this epsilon relates electrical displacement and electrical field and here E is a coupling constant which relates stress with electric field and electrical displacement with strain. So, that is how the electrical and mechanical terms are coupled here and it was it was similar in the previous constitutive equation that we derived 2. So, there the coupling constant was D. Now, sorry in that expression the coupling constant was H in the previous constitutive relation and we will get more coupling constants like D, G as we move along and as we keep changing the state variables.

So, initially we had internal energy as our thermodynamic potential and from there we change to enthalpy as our new potential. So, the stress one of the state variable was D in the internal energy and then after we change the change it to a new potential enthalpy then instead of electrical displacement D our electric field E become the state vector. Now, that change of this potential through that transformation that we did is called Legendre transformation. So, we will do another Legendre transformation to get a new potential and this potential is Helmholtz free energy. So, let us define this as F as internal energy minus stress multiplied by strain.

$$F = U - \sigma\varepsilon$$

$$dF = dU - \sigma d\varepsilon - \varepsilon d\sigma = \sigma d\varepsilon + EdD - \sigma d\varepsilon - \varepsilon d\sigma = -\varepsilon d\sigma + EdD$$

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^{D} d\sigma + \left(\frac{\partial F}{\partial D}\right)^{\sigma} dD$$

$$\frac{\partial F}{\partial \sigma} = -\varepsilon, \frac{\partial F}{\partial D} = E$$

$$d\varepsilon = \left(\frac{\partial \varepsilon}{\partial \sigma}\right)^{D} d\sigma + \left(\frac{\partial \varepsilon}{\partial D}\right)^{\sigma} dD = \frac{-\partial^2 F}{\partial \sigma^2} d\sigma - \frac{\partial^2 F}{\partial \sigma\, \partial D} dD$$

$$dE = \left(\frac{\partial E}{\partial \sigma}\right)^{D} d\sigma + \left(\frac{\partial E}{\partial D}\right)^{\sigma} d\sigma = \frac{\partial^2 F}{\partial \sigma\, \partial D} + \frac{\partial^2 F}{\partial D^2}$$

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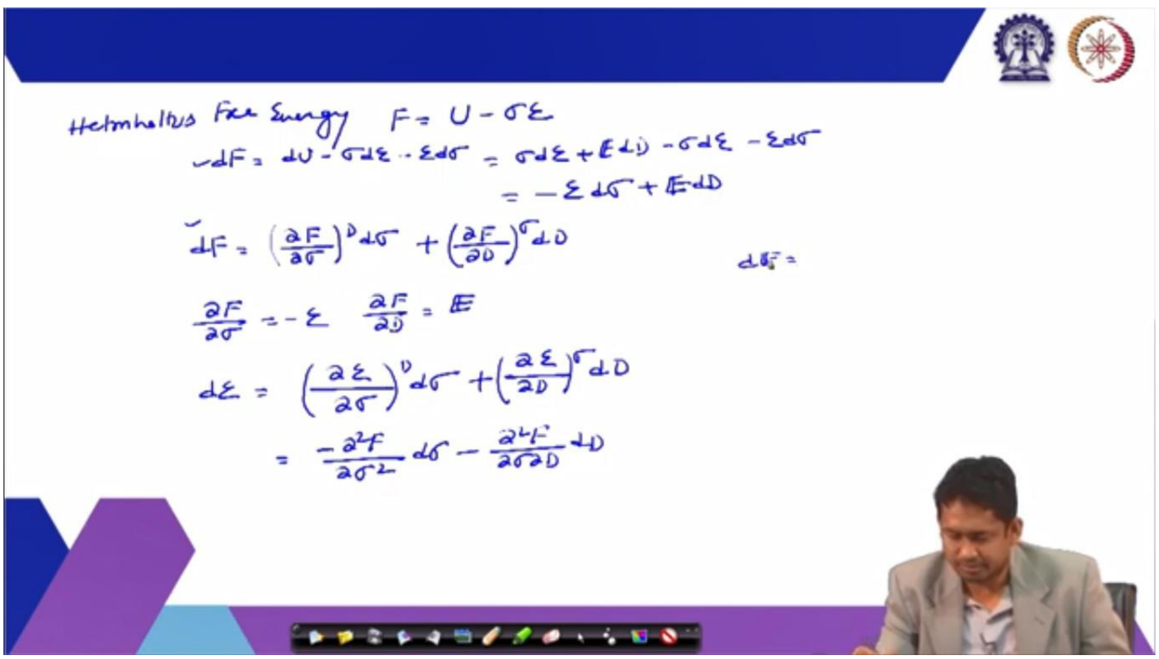

So, in the differential form this can be written as dU minus d sigma minus sorry epsilon d epsilon multiplied by sigma minus d sigma multiplied by epsilon and then becomes sigma d epsilon plus E dD minus sigma d epsilon minus epsilon d sigma and finally, the quantities minus epsilon d sigma plus E dD. Now, we can also write dF as dF del F by del sigma keeping D as constant multiplied by d sigma plus del F by del D keeping sigma as constant multiplied by D D. Now, if I compare this with this, we can write del F by del sigma as minus of epsilon and we can write del F by del D as electric field E. Now, we can define we can write the differential form of the strain del epsilon by del D keeping D as constant multiplied by D sigma plus del epsilon sorry it was it should be sigma. So, del epsilon by del sigma keeping D as constant multiplied by D sigma del epsilon by del D keeping sigma as constant multiplied by dD.

Now, we know that epsilon is equal to minus of del F by del sigma. So, with that we can write this and this. Similarly, we can define the differential form of the electric field and that can be written as keeping D constant plus del D keeping sigma constant and again, we know that E is equal to del F by del D. So, this becomes del 2 F by del sigma del D plus del 2 F by del D 2. Now, if we combine these two, we can write one more constitutive relation and that can write we can be that we can write So, again it is 6 by 1 vector and this is a 3 by 1 vector this is 6 by 1 vector.

$$\begin{Bmatrix} \{\varepsilon\}_{6\times1} \\ \{E\}_{3\times1} \end{Bmatrix} = \begin{bmatrix} [s^D]_{6\times6} & [g]^T_{6\times3} \\ -[g]_{3\times6} & [\beta^\sigma]_{3\times3} \end{bmatrix} \begin{Bmatrix} \{\sigma\}_{6\times1} \\ \{D\}_{3\times1} \end{Bmatrix}$$

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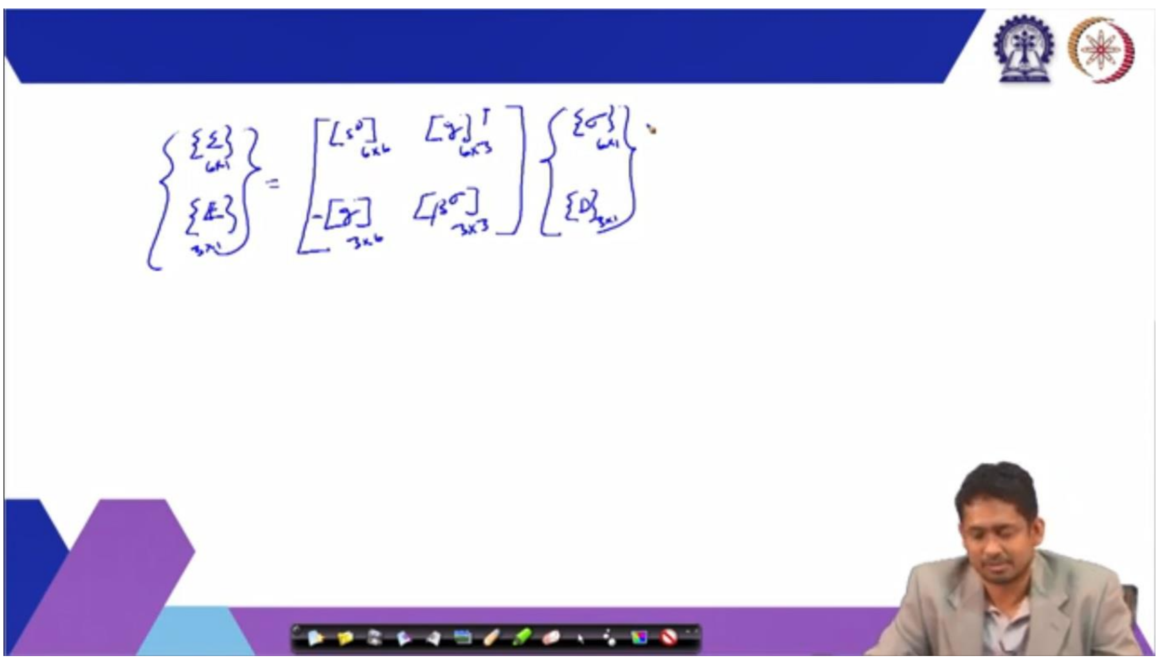

So, this is 6 by 6 this is 6 by 3 this is 3 by 6 and this is 3 by 3. So, we got another constitutive relation where my state vectors are stress and electrical displacement D. So, with that we would conclude this lecture. So, there is one more form of the constitutive relations that we will see in the next lecture and after that we will see the inter relation between the constants that we get from each of these constitutive relations.

So, thank you.