Smart Structures
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Week 09
Lecture No: 49
Constitutive Relations of Shape Memory Alloys - Continued
Part 05

So, today we will start from where we left in the last lecture. We were solving a problem of free recovery. So, there was a string made of safe memory alloy wire. So, that was fixed at one end and free at the other end and the other end had a weight attached to it. So, we saw what is happening when it is heating up. And we saw the strains as per both the Liang Rogers model and the Tanaka model. Now, we have to see what happens to it when it is cooled down.

Now, the temperature is heated up to 77 degrees centigrade, which means it crosses the austenite finish temperature and that means that, it becomes fully austenite and then the cooling process takes place. So, if I look at the diagram once again, this is our temperature and this is sigma. So, we were along this line. So, we are heating it up and we are somewhere here and this temperature we found it to be 75.5, whereas, we heated it up to 77 degrees centigrade that means, the material became fully austenite and then the cooling takes place. So, we have to find out when it is fully austenite what was the amount of strain here. So, to do that again we can write the constitutive equation, if I write it fully. So, sigma minus sigma 0 is equal to E multiplied by epsilon minus epsilon 0 plus omega multiplied by xi minus xi_0. This quantity is 0, because initial and final stress are same. Stress is maintained to be constant and in this case, we are finding it out. Finding it at A_f star which means at the point where austenite formation is just finishing.

$$\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0) + \Omega(\xi - \xi_0)$$

So, at that point we know that sigma should be close to 1. Sigma should be 1 ideally, sorry, 0, it is not 1, it is 0 because the austenite finish has taken place. And here, it started with a fully martensite. So, it is 1. Epsilon I have to calculate. Epsilon 0 is where we started from. And the epsilon 0 at that condition was 4.24 multiplied by 10 to the power minus 3. Now, because we have reached the austenite condition E is just E_A and accordingly omega is minus epsilon L multiplied by E_A. So, everything is known in this equation except for epsilon and this quantity as we know it is 0. So, if we solve for it, the epsilon comes to be minus 0.06276, which means this is the strain at the austenite condition, at the austenite phase.

$$\varepsilon = -0.06276$$

Now, when we cool it down, we are starting from the austenite phase which means, while cooling it, my initial strain should be 0.06276, while when the martensite formation takes place in this zone. If I want to calculate the strains at different temperatures, my initial strain should be taken as this. The austenite strain which is minus 0.06276. So, before we do that, we have to find out the xi's.

So, xi as per Tanaka's model and we are calculating it at T is equal to 48.5 degree centigrade. So, at that time, our xi is given by this equation. We know that xi T, when the martensite formation takes place is equal to 1 minus e to the power a_M, M_s minus T, plus b_M multiplied by sigma. Here sigma is equal to sigma 0, the same stress that is their constant throughout. Now in this case a_M as per Tanaka's model is natural log of 0.01 divided by M_s minus M_f, and that gives it as minus 1.6447 per degree centigrade and b_M is a_M by C_M and that is equal to minus of 1.6447 divided by 12 and we get the quantity as minus 0.137 per Mega Pascal. So, accordingly xi is calculated as 0.9717. In percentage, it becomes 97.17 percent.

$$\xi(T) = 1 - e^{a_M(M_s - T) + b_M\sigma}$$

$$a_M = \frac{\ln(0.01)}{M_s - M_f} = -1.6447\ /^0C$$

$$b_M = \frac{a_M}{C_M} = -\frac{1.6447}{12} = -0.137\ /\text{MPa}$$

$$\xi(48.5^0C) = 0.9717 = 97.17\%$$

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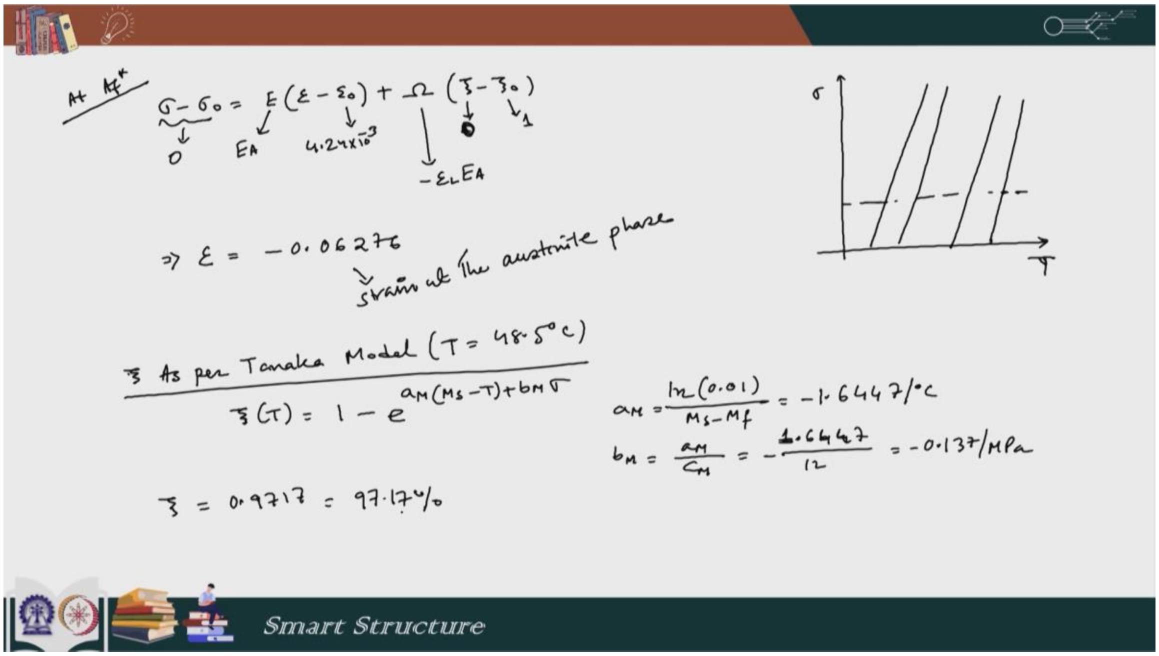

Now, we will do the same calculation as per Liang and Rogers model. So, xi as per Liang and Rogers model and our T is again 48.5 degree centigrade. As per Liang and Rogers model, the formula is xi T equal to 1 minus xi A and in this case because, we are starting from a fully austenite condition. So, xi A is 0 anyway. So, we can directly write that as 0. So, we can call it half, half multiplied by cosine of a_M, T minus M_f, plus b_M sigma plus half. Now, here a_M is pi by M_s minus M_f and that is 1.122 per degree centigrade and b_M is minus a_M divided by C_M and that is minus 0.0935 per Mega Pascal. So, now xi at 48.5 degree centigrade becomes 0.8804 which is 88.04 percent. So, that is my xi as per Liang and Rogers model.

$$\xi(T) = \frac{1}{2}\left\{\cos\left[a_M\left(T - M_f\right) + b_M\sigma\right]\right\} + \frac{1}{2}$$

$$a_M = \frac{\pi}{M_s - M_f} = 1.122\ /^0C$$

$$b_M = -\frac{a_M}{C_M} = -0.0935\ /\mathrm{MPa}$$

$$\xi(48.5^0C) = 0.8804 = 88.04\%$$

Now we have to find out the strains. We have already defined the initial conditions. So, now the equation is sigma minus sigma 0 equal to E xi multiplied by epsilon minus epsilon 0. We are neglecting the effect of phi as usual. So, it is omega xi multiplied by xi minus xi_0.

Then we have E xi then, we have epsilon minus epsilon 0. So, epsilon minus epsilon 0 was calculated as minus of 0.06276. So, it becomes plus 0.06276. And then we have minus epsilon L multiplied by E xi and then xi_0 is 0 because we started with a fully austenite condition. So, it is just xi.

$$\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0) + \Omega(\xi - \xi_0)$$

$$= E(\xi)(\varepsilon + 0.06276) - \varepsilon_L E(\xi)\xi$$

So, this equation we would use only thing is that xi have xi has to be calculated once using the Tanaka model and once using the Liang and Rogers model.

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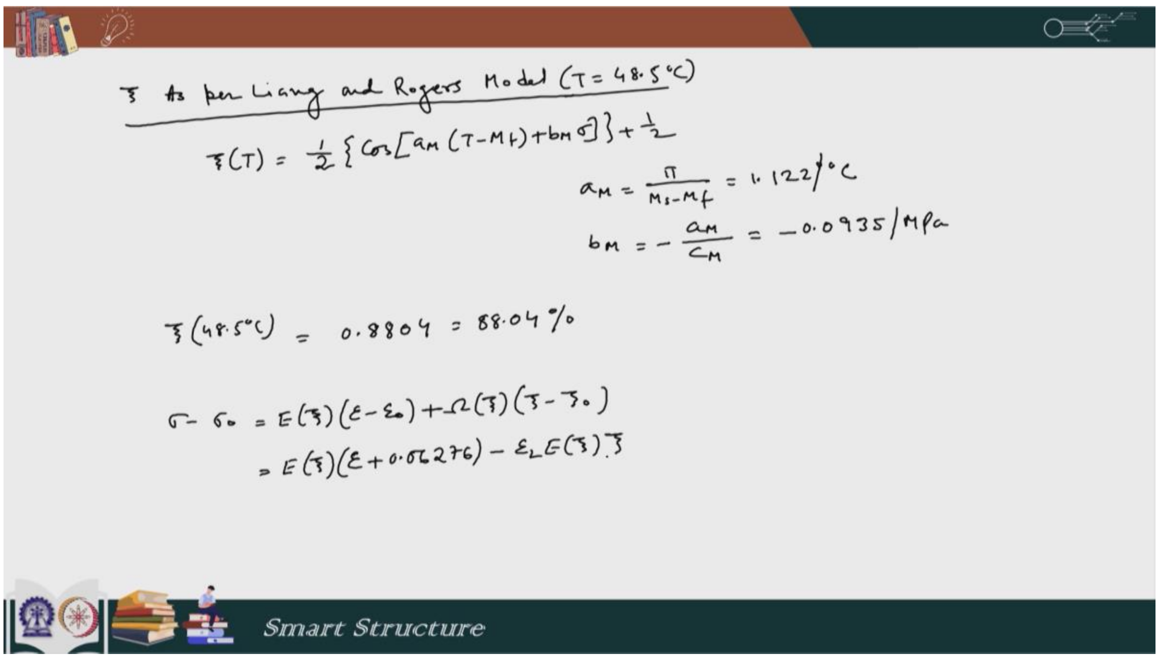

So, now let us do strain calculation using Tanaka model. So, to do the strain calculation using Tanaka model, all we have to use is – we have to just find out the xi that is obtained from the Tanaka model. So, if I put it in the equation, it is 0 is equal to E xi. Now E xi should be calculated first. So, E xi at this condition is E xi. So, when the xi that we obtained from the Tanaka model was xi is equal to, xi is equal to 0.9717. So, at that temperature at that xi E xi becomes 21 Gpa. And then accordingly, omega xi becomes minus epsilon L into E xi and that comes as minus 1.407 GPa, minus 1.407 GPa.

$$\xi = 0.9717$$

$$E(\xi) = 21\ GPa$$

$$\Omega(\xi) = -\varepsilon_L E(\xi) = -1.407\ GPa$$

So, if I now put everything here, if I now put everything here the equation becomes E in the same equation epsilon plus 0.06276. And then we have minus epsilon L E multiplied by xi. xi we know, we know E, we know epsilon E and everything. So, finally, after solving, the epsilon comes to be 0.00234 and if this is the epsilon then my delta L is L multiplied by epsilon and that gives a value of 0.0208. So, L final is L plus delta L and that is 0.321 meter which means the tip position of the wire at this temperature while cooling down is at a distance of 0.321 meter from the fixed end.

$$0 = E(\varepsilon + 0.06276) - \varepsilon_L E$$

$$\Rightarrow \varepsilon = 0.00234$$

$$\Delta l = L\varepsilon = 0.0208$$

$$L_{final} = L + \Delta l = 0.321 \text{ m}$$

Now, we will do the strain calculation using the Liang and Rogers model. Everything remains same except xi becomes different.

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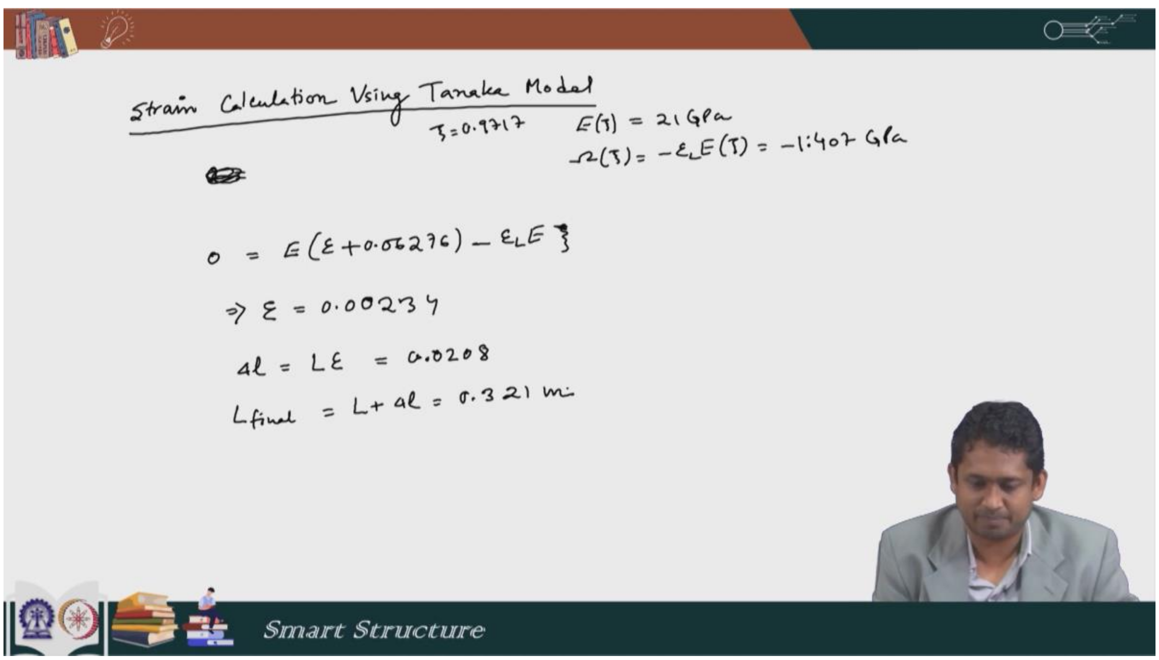

Liang and Rogers model and we are doing strain calculation, using Liang and Rogers model. So, in this case the xi that we got was 0.8804 and as per that our E is – if we apply the same formula, our E comes to be 25.89 Gpa. And accordingly, omega comes to be minus of 1.735 GPa. Then we apply the same formula we have 0 is equal to E multiplied by epsilon plus 0.06276 minus or we can say plus omega which is minus of epsilon L into E multiplied by xi.

$$\xi = 0.8804$$

$$E = 25.89 \text{ GPa}$$

$$\Omega = -1.735 \text{ GPa}$$

$$0 = E(\varepsilon + 0.06276) + \Omega\xi$$

So then, we can solve it and we can find it our epsilon. Once, we get our epsilon, we can again find out our tip position at that temperature. So, this completes the problem. So, again to summarize, because it is a free recovery problem, we could find out xi at this temperature during transformation. And xi gave us the elastic modulus and omega. And then, if we put it in the constitutive relation with proper initial conditions, we can find out unknown in the problem which is our epsilon.

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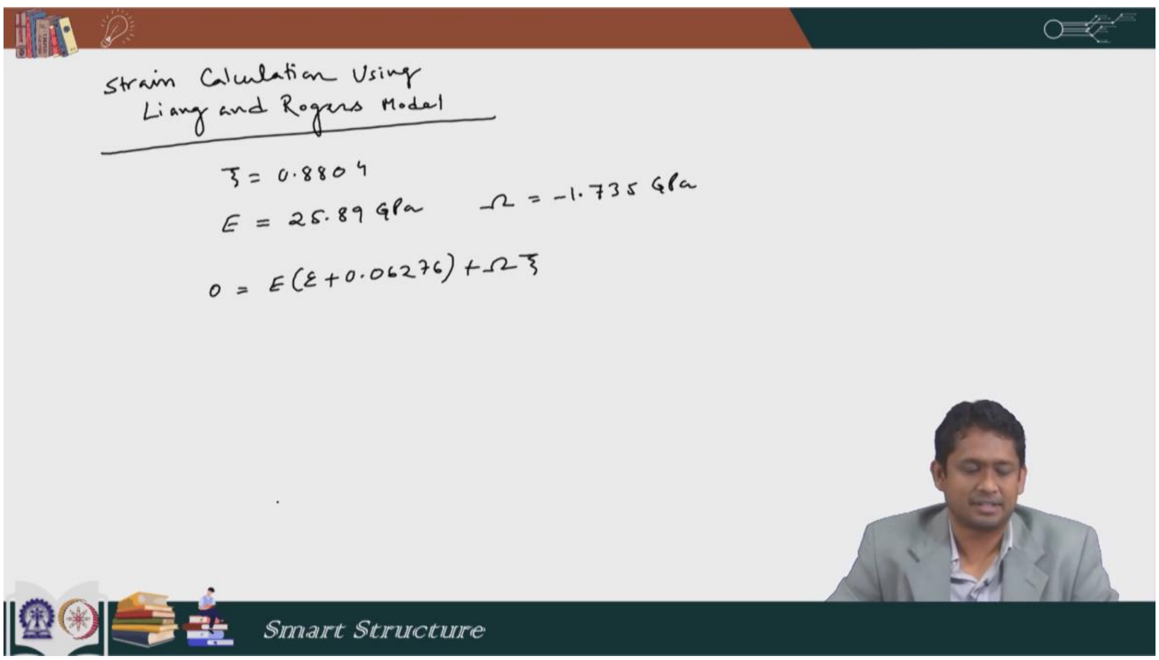

Now, we will look into another problem. That is a constraint recovery problem.

So, here is the problem. We have out shape memory wire of length 0.3 meter and circular cross section of with diameter of 0.5 millimeter is fixed at one end and connected to a linear spring of stiffness is equal to 2 thousand Newton per meter at the other end. The wire is a temperature of 15 degree centigrade in martensite phase. So, it is a martensite phase. And then, we have to find out the stress in the wire when the temperature is elevated to 60 degrees centigrade.

So here, the entire arrangement is looks like this. We have a fixed end from which a wire is coming out. And then, it is connected to a spring. The spring has certain stiffness. And the spring is held here. So, when this wire tries to expand or contract, it is resistant by the spring. So, it is not a - the wire is not free. So, when it is heated up, it tries to recover, but it is not a free recovery because of the resistance offered by the spring. So here, if I am again draw the phase diagram – in the previous case, the stress was constant and everything was happening at a constant stress.

Now, here initially, the stress is zero. So, when I am heating it up, till the austenite start temperature, the heating is zero. After it crosses the austenite start temperature, xi changes. And accordingly, it would try to shorten. And, when it tries to shorten, the spring would give some resistance, and that would generate some stress, tensile stress.

So, some stress comes into picture here. So, instead of following these zero stress lines, now, it would go towards some nonzero stress value. And again, because of the stress, xi changes. And as xi changes, the state would try to change. And, at the same time, temperature is also changing. So, strain would try to change. And that would again affect our stress. So, it becomes a nonlinear problem. So, instead of following this route, directly from A_s to A_f, it follows a route something like this. The transformation. So that, we would quantify it and see it numerically.

So now, let us established the mathematical relation here first. The spring has a stiffness of k. So, if we say that, the force acting here at the junction between the spring and the wire material is F then, we can write that the strain in the material is – e is equal to minus delta L by L, change in the length by the initial length which should be is equal to minus of F by k. If I look at the spring, because k is a spring constant, so F by k is our delta L. And then, F is also nothing but sigma into A. Wire in the spring, sorry, stress in the wire multiplied by A. And that becomes, minus sigma A by kL. So, from this we can find out our epsilon as minus sigma A by kL.

$$\varepsilon = \frac{-\Delta L}{L} = \frac{-\mathrm{F}}{kL} = \frac{-\sigma\mathrm{A}}{kL}$$

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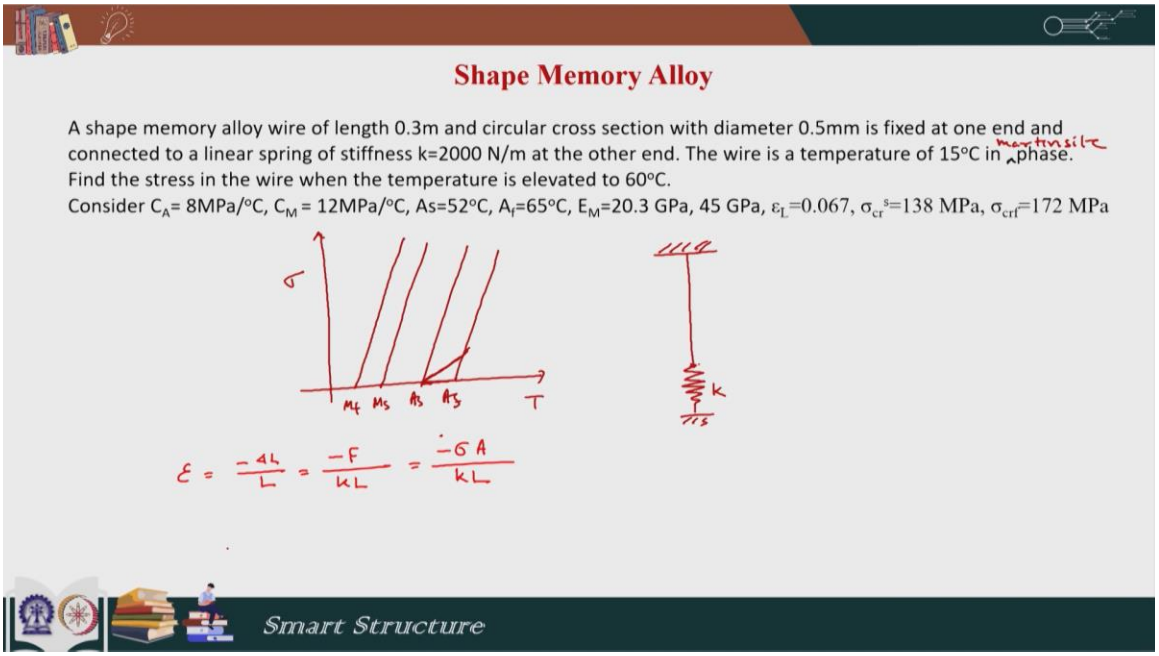

Now, if I look into the constitutive relation, it is sigma minus sigma 0 and that is equal to E as a function of xi multiplied by xi minus xi_0 plus omega xi multiplied by xi minus 1. Now, xi_0 is equal to 0 because there was no force applied. And till it reaches A_s there is no transformation. So, there is no change in length. So, xi 0 is 0. And accordingly, sigma 0 is also 0. Now, we can write sigma is equal to E of xi multiplied by epsilon and epsilon we already saw that it is minus of sigma A by kL. And then, we have epsilon L, E xi multiplied by xi minus 1.

$$\sigma - \sigma_0 = E(\xi)(\varepsilon - \varepsilon_0) + \Omega(\xi)(\xi - 1)$$

$$\Rightarrow \sigma = E(\xi)\left(\frac{-\sigma A}{kl}\right) - \varepsilon_L E(\xi)(\xi - 1)$$

So, this is a complicated equation because we have sigma here, we have sigma here, and also our xi is again a function of sigma. So, xi is e to the power a_A multiplied by A_s minus T, plus b_A sigma.

$$\xi = e^{a_A(A_s - T) + b_A \sigma}$$

So, that is as per Tanaka model. So, here we have xi as a function of sigma. So, if I put it here, this becomes again a function of sigma. So, this is a non-linear equation in terms of sigma which needs to be solved. Now, this kind of equation, there are many methods to solve it. So, here we will solve it using the Newton's method. So, to do that we define a function f of sigma which is equal to xi minus the entire right hand side which means a residue. So, sigma plus E of xi multiplied by sigma A divided by kL plus epsilon L multiplied by E of xi multiplied by xi minus 1.

$$f(\sigma) = \sigma + E(\xi)\left(\frac{-\sigma A}{kl}\right) - \varepsilon_L E(\xi)(\xi - 1)$$

So, ideally when sigma is solved properly, then ideally this f sigma should be 0. So, this equation we will solve using the Newton Raphson's method. So, for that what we do is - start with initial guess, initial guess sigma 1 and then we evaluate f of sigma 1. And if f of sigma 1 is not satisfactorily close to 0, then what we do is we make a new guess sigma 2 equal to sigma 1 minus f of sigma 1 divided by del f by del sigma at sigma 1.

$$\sigma_2 = \sigma_1 - \frac{f(\sigma_1)}{\partial f / \partial \sigma \big|_{\sigma_1}}$$

Again, we check whether if I take sigma 2, then f sigma 2 is close to 0 or not. If not, then I make new guess and it continues until and unless I get f sigma to be very close to 0.

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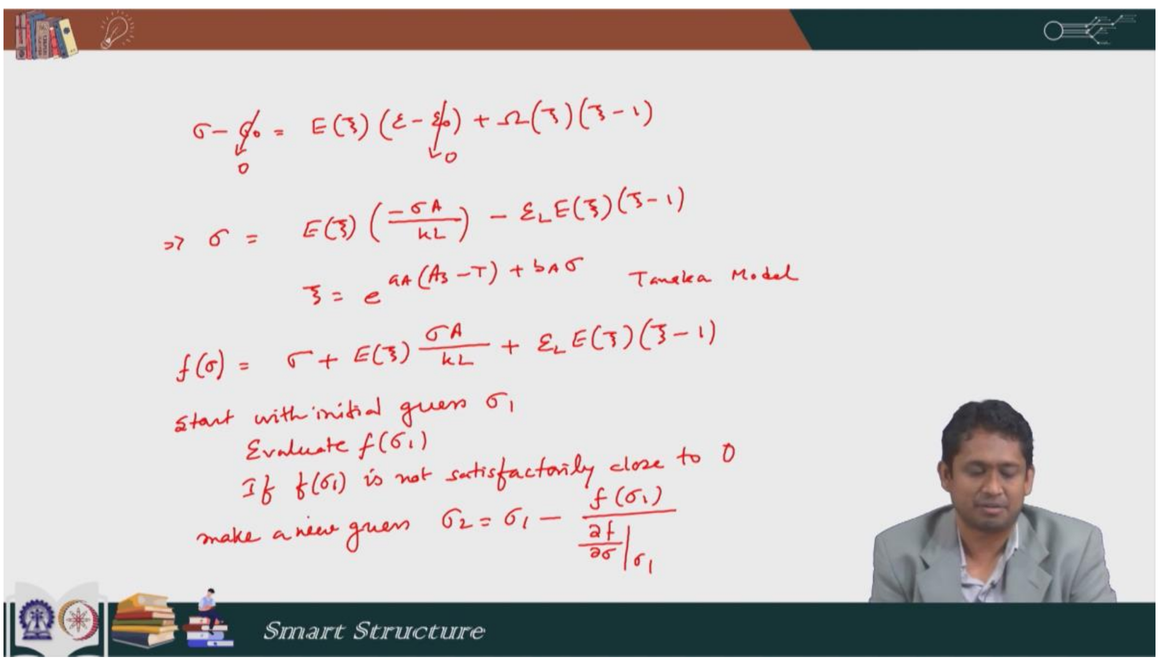

So, in this case f of sigma or sorry, f prime of sigma or del f by del sigma that we get is 1 plus E of xi multiplied by A by kL. So, if you put everything and then if you differentiate the entire expression by sigma, then the expression comes to be this, sigma A by kL multiplied by E_M minus E_A multiplied by b_A xi plus epsilon L multiplied by E xi multiplied by b_A xi plus epsilon L multiplied by xi minus 1 multiplied by E_M minus E_A multiplied by b_A xi.

$$f'(\sigma) = 1 + E(\xi)\frac{A}{kL} + \frac{\sigma A}{kL}(E_M - E_A)b_A\xi + \varepsilon_L E(\xi)b_A\xi + \varepsilon_L(\xi - 1)(E_M - E_A)b_A\xi$$

So, this is del f by del xi which we need to evaluate at each iteration for the current value of the sigma.

Now, let us start with sigma 1 equal to 200 Newton per meter square. So, at that sigma, sigma equal to sigma 1. We have f equal to minus of 1.6803 into 10 to the power 9. And del f by del sigma at sigma 1 equal to minus of 8.2990.

$$\sigma_1 = 200\, N/m^2$$

$$f(\sigma_1) = -1.6803 \text{ x } 10^9$$

$$\left.\frac{\partial f}{\partial \sigma}\right|_{\sigma_1} = -8.2990$$

And then, we have sigma 2 we apply the same formula and then we get sigma 2 from sigma 1 and f and del f by del sigma and sigma 2 comes to be 1.6747 multiplied by 10 to the power 7 Newton per meter square. And at that stress, we again evaluate our f sigma 2 and again we evaluate our del f by del sigma at sigma 2 and we continue it.

$$\sigma_2 = 1.6747 \text{ x } 10^7\, N/m^2$$

$$f(\sigma_2) = \cdots\cdots$$

$$\left.\frac{\partial f}{\partial \sigma}\right|_{\sigma_2} = \cdots\cdots$$

Then, we see that at sigma 6 we get our value as 1.9413 into 10 to the power 7 Newton per meter square. And at this value of sigma, the residue function f comes to be quite less which is minus of 1.148 into 10 to the power minus 7.

$$\sigma_6 = 1.9413 \text{ x } 10^7\, N/m^2$$

$$f(\sigma_6) = -1.148 \text{ x } 10^{-7}$$

So, we start with a residue of in the order of 10 to the power 9 and then we came at 10 to the power minus 7 in just 6 iterations. So, this is quite satisfactory. So, we can say that this is the value of the stress for this condition at that temperature.

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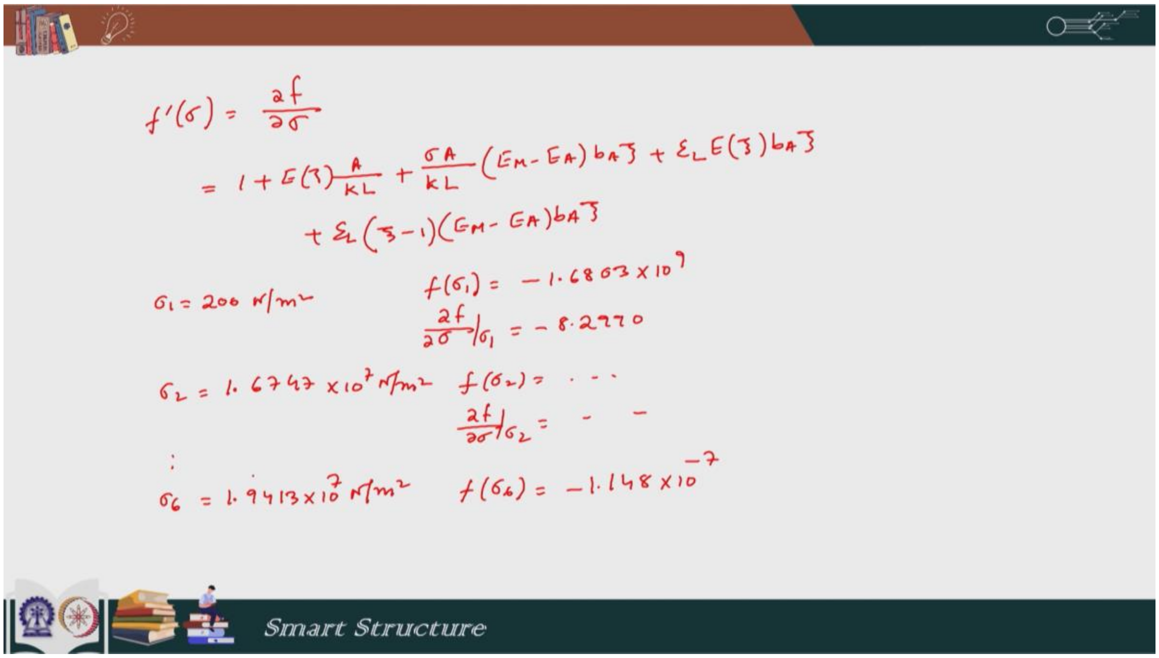

Now, that is how the entire transformation takes place. Here we can see the how the entire system evolves. So, at the temperature 52 degree centigrade which is our A_s, the transformation starts. So, stress stress is 0 still here, then gradually as the temperature increases, the stress also increases and it increases in this non-linear fashion.

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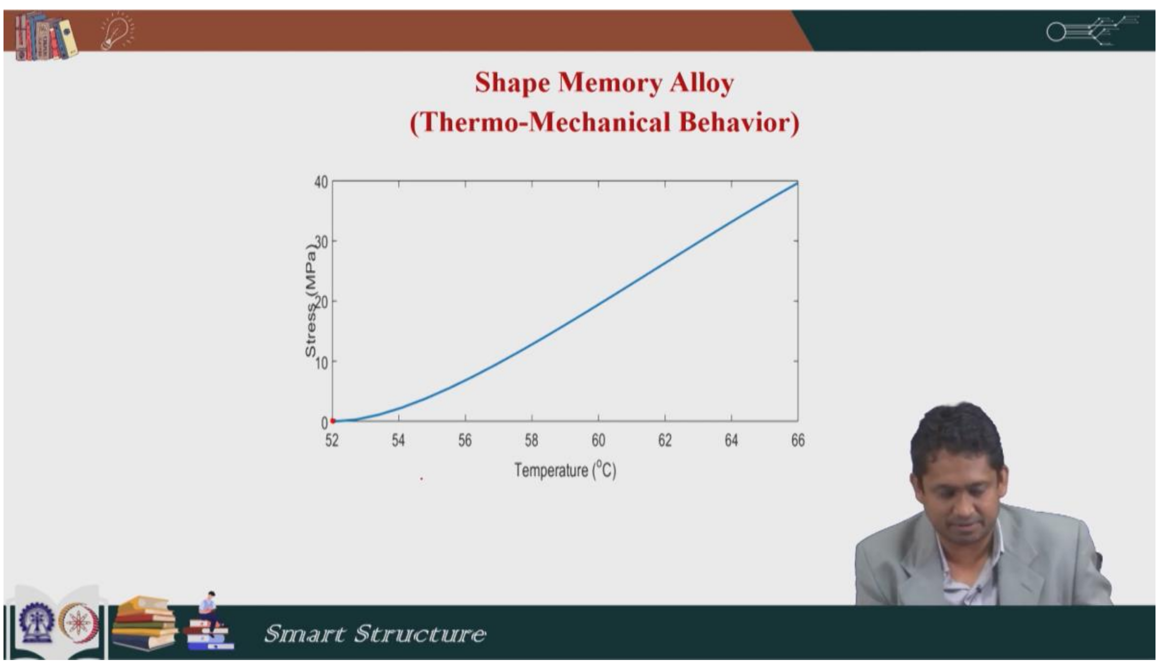

Now, what if I change my spring stiffness. When the when the spring stiffness is 0, which means in this case when the stiffness is 0 that means, the stiffness is not existing in this case, our shape memory alloy wire is free to move. So, in that case there is no stress generated. So, we can see that it is 0. Now as we keep increasing the stress, as we keep increasing the spring stiffness, the stress starts increasing at a certain temperature. So, this is for a fixed temperature. This graph is drawn. But the slope of this curve becoming less as we go on, which means that after a high value of stiffness, this almost becomes like a fully constrained end. So, after that there is no further significant change in stress that we observe if I keep increasing the spring stiffness to a very high value and that is what we can see here. So, when k is equal to 0, it is almost it is like a free end. So, the shape memory alloy is free to expand or contract. When k is equal to a very high value, it is almost like a constraint end. So, there is no strain possible and the stress is very high in between them if k varies, we can see a graph like this.

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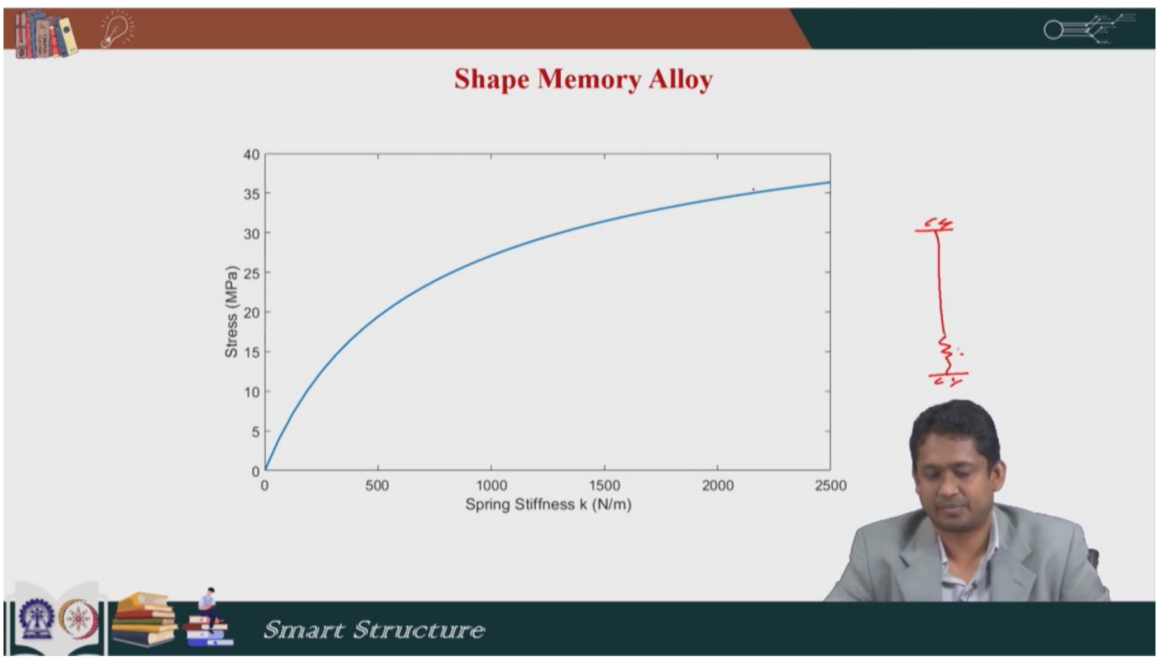

So, with these two examples, we would like to finish these lectures on the constitutive relations and their users here.

Now, we will gradually move to some applications.

Thank you.