Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week 03
Lecture No: 17
Induced Strain Actuation – Static Analysis (Continued)
Part 05

Welcome to the fifth lecture on Static Analysis for Induced Strain Actuation.

In the last lecture, we talked about the block force method and saw a few cases. Today, we will see a few more cases and analyze them using the block force method. So, this case says that there is a beam, and this beam has one piezoelectric patch and it is at one side. So, this is a type of asymmetric actuation. Now, in this case, if the beam experiences an actuation force like this here, suppose it is F; the piezo patch experiences the opposite force, F. So, if you look at the beam section, at the mid of the beam section, here, maybe at the middle of it, we see that it is under an actuation force, F and a bending moment, M. So, if the difference from the mid part to the bottom part of the beam is t_b by two, given the fact that our beam thickness is t_b, then bending moment can be written as F multiplied by t_b by two.

$$M = F\frac{t_b}{2}$$

Now, here, there is a bending moment accompanied by axial force. So, in this section, if we draw the stress diagram, it will have a linearly varying stress, which can be represented as this. As well as, due to the force F, it will have a constant stress, which can be represented as this.

So, at the middle part of the beam, although the bending stress is zero, there will be an axial stress due to this force F. So, from the total stress, which is a combination of this and this, if I subtract this stress, then I can get my bending stress, which is shown here. So, this is the total stress, and this is the axial stress due to the axial force F. Now, this epsilon b zero, which is the axial stress due to this axial force F, is constant over the thickness. So, if I subtract this, I get my bending stress. Now, bending stress in terms of bending moment is minus M z by I_b, which we have seen before.

$$\sigma_b - \sigma_b^0 = -\frac{Mz}{I_b}$$

Now, from here, if we want to get the strain, we just divide this entire equation by the elastic modulus of the beam. So, this gives me the total strain at the neutral axis, and I subtract the strain due to the axial force F, and that gives me the bending strain part.

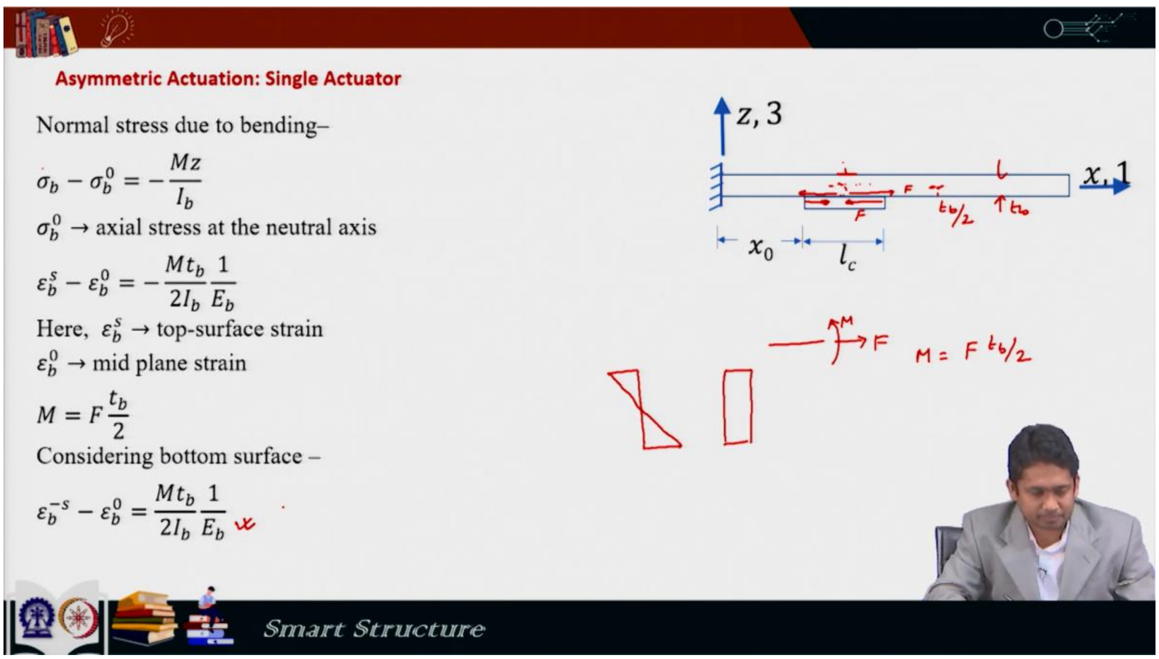
$$\varepsilon_b^s - \varepsilon_b^0 = -\frac{Mt_b}{2I_b}\frac{1}{E_b}$$

Now, if I want to find out the strain at the top surface of the beam, which is here in this zone, then I just replace z by t_b by two, and then we can get it. And similarly, if I want to get it at the bottom surface, then I replace z by minus t_b by two, and we can get the strain here. So, the bottom strain, as can be seen, is this.

$$\varepsilon_b^{-s} - \varepsilon_b^0 = \frac{Mt_b}{2I_b}\frac{1}{E_b}$$

So, after replacing z by minus t_b by two, we get this expression. And here, minus s denotes the bottom surface. So, this is my strain at the bottom surface.

(Refer Slide Time: 4:12)



And the midplane strain is – mid plane strain means this strain, and that is constant over our thickness, and that is the strain due to this axial force F, and we know that the stress due to the axial force F is this F divided by the cross-sectional area of the beam. And then, if I divide that further by the elastic modulus of the beam, that gives me the mid-plane strain, which is this.

$$\varepsilon_b^0 = \frac{F}{b_b t_b E_b} = \frac{F}{E_b A_b}$$

Now, in all these equations, we have these terms. So, EA_b is the extensional stiffness of the beam, which is E_b b_b by t_b, and extensional stiffness of the actuator, which is EA_{c1}, one means of actuator one; now here we have only one actuator. So, EA_{c1} and EA_c are the same, and that is E_c b_c multiplied by t_c.

$$EA_b = E_b b_b t_b$$

$$EA_{c1} = E_c b_c t_c$$

So, this b_c t_c, they carry the same meaning as they have been carrying: b_b means the width of the beam, t_b means the thickness of the beam and, b_c means the width of the actuator, and t_c means the thickness of the actuator. And similarly, we can find out the bending stiffness EI_b, which is this, and this expression E_b b_b t_b cube by twelve, it can be written in terms of E_b A_b as this also. And finally, we get EI_b in terms of EA_b as this. So, EI_b is equal to EA_b multiplied by t_b square by twelve.

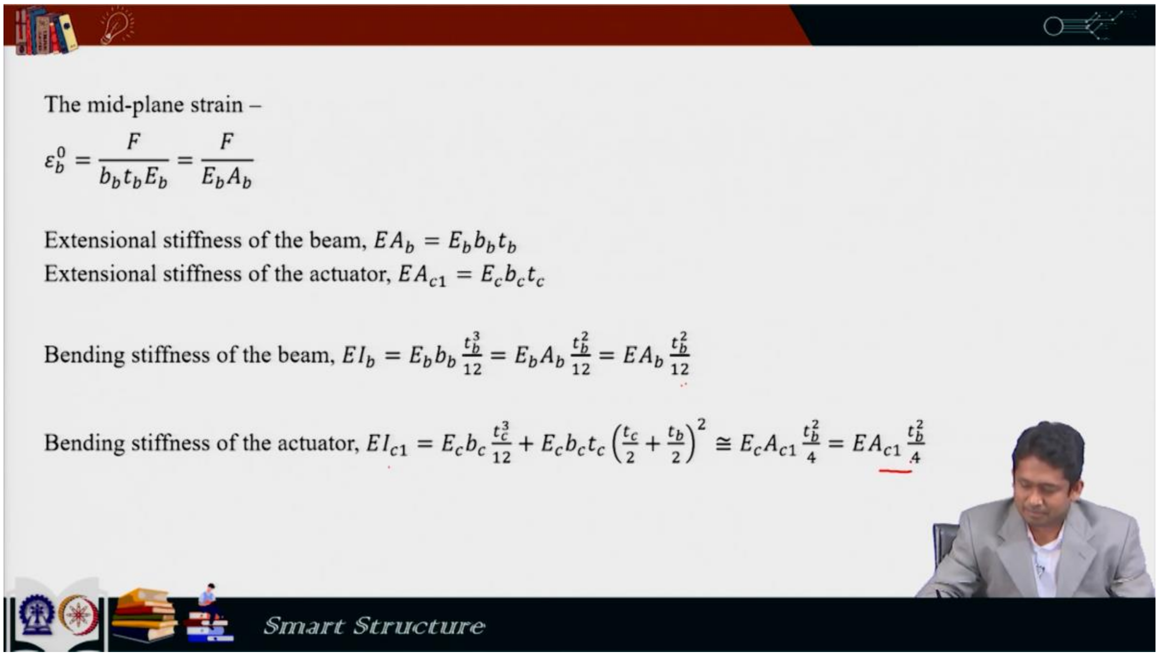
$$EI_b = E_b b_b \frac{t_b^3}{12} = E_b A_b \frac{t_b^2}{12}$$

Similarly bending stiffness of the actuator: it has two components; one is the bending stiffness of the actuator with respect to its own centroid and then another is the area multiplied by the distance of the centroid of the actuator from the centroid of the beam and the square of that. So, it is just shifted by the parallel axis theorem.

So, this is our EI_{c1}, and the same after making some approximations because the thickness of the actuator is small as compared to the thickness of the beam. So, ignoring the higher-order terms of t_c, we get this expression. And finally, EI_c is written in terms of EA_c as this.

$$EI_{c1} = E_c b_c \frac{t_c^3}{12} + E_c b_c t_c \left(\frac{t_c}{2} + \frac{t_b}{2}\right)^2 \approx E_c A_c \frac{t_c^2}{4} = EA_{c1} \frac{t_c^2}{4}$$

(Refer Slide Time: 6:36)



Now, at the bottom surface of the beam, we have already seen our expression for the strain. So, we know that this is our strain at the bottom surface of the beam minus epsilon b zero is this. Epsilon b zero, we have got it here. Then, finally, by combining these two, we can get strain at the bottom surface of the beam as this. We have just written the epsilon b zero in terms of the applied actuation force F. And again, the bending moment can be written in terms of the actuation force F as M is equal to F into t_b by two. And after doing all these substitutions, this entire expression can be written as four-F by EA_b.

$$\varepsilon_b^{-s} = \frac{Mt_b}{2I_b}\frac{1}{E_b} + \frac{F}{EA_b} = \frac{F\left(\frac{t_b}{2}\right)^2}{b_b\left(\frac{t_b^3}{12}\right)}\frac{1}{E_b} + \frac{F}{EA_b} = \frac{4F}{EA_b}$$

So, the strain at the bottom surface of the beam has been represented in terms of the actuation force F.

Now, from here, we can find out the extension of the bottom surface of the beam, which is in contact with the actuator, and that is nothing but this strain multiplied by l_c. l_c is the length of the actuator. So, if we go back, this is the length of the actuator, and we have found out the strain in this region. So, in this region, if we assume that the strain is constant, then if we multiply the strain with this actuator length, that gives me how much elongation our beam surface has experienced at this level.

So, which is this. So, we have this strain here multiplied by l_c, and that is the extension of the bottom surface of the beam.

$$\Delta l_b^{-s} = \frac{4F}{EA_b} l_c \qquad \Delta l_c = \left(\varepsilon_p - \frac{F}{EA_{c1}}\right) l_c$$

Now, for compatibility, that must be equal to the extension of the actuator. Now, the extension of the actuator is this: if the actuator is experiencing a force F. So, our actuator, as we drew before, is experiencing a force F here, and if we say that the epsilon p is the free strain of the actuator, then this becomes the expression for the change in length of the actuator, or the extension in the actuator. Now, for compatibility, this and this should be the same.

$$\Delta l_b^{-s} = \Delta l_c$$

So, once we equate these two, we get this expression. In this entire expression, the only unknown is F.

$$\frac{4F}{EA_b} = \frac{d_{31}V}{t_c} - \frac{F}{EA_{c1}}$$

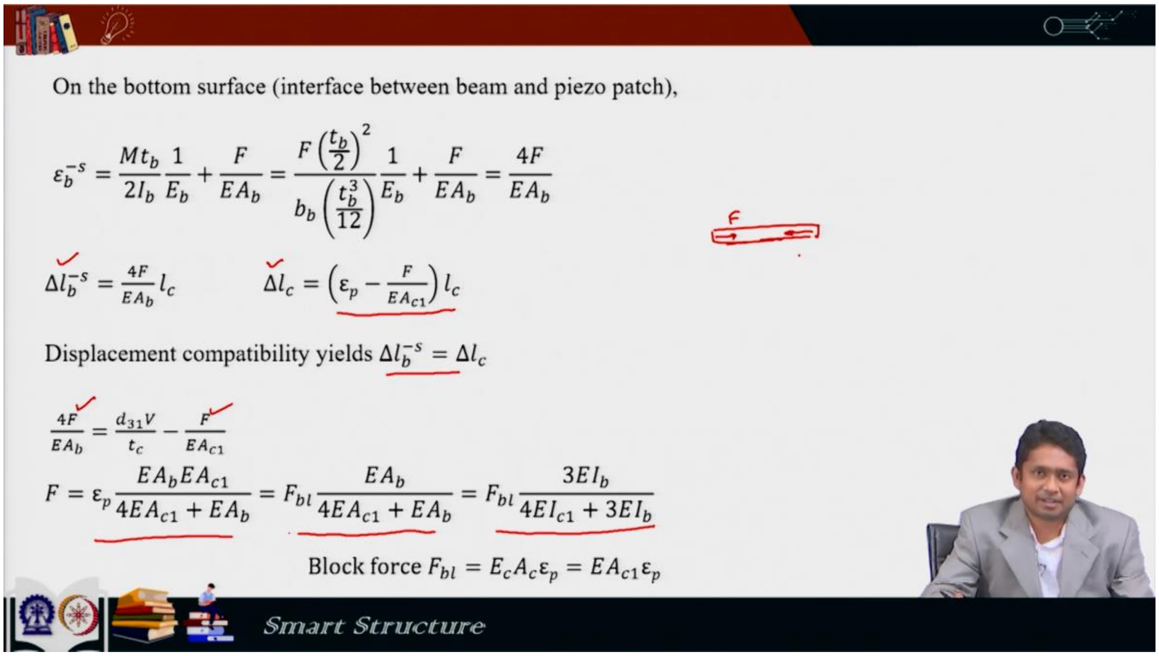
So, we can take it at one side and solve for F, and we get this expression.

$$F = \varepsilon_p \frac{EA_b EA_{c1}}{4EA_{c1} + EA_b} = F_{bl} \frac{EA_b}{4EA_{c1} + EA_b} = F_{bl} \frac{3EI_b}{4EI_{c1} + 3EI_b}$$

$$F_{bl} = E_c A_c \varepsilon_p = EA_{c1} \varepsilon_p$$

So, here we have got the actuation force F in terms of the free strain and the sectional properties EA_b EA_c, I mean EA_{c1}. And then epsilon p can be written in terms of the block force. So, this expression can be written in terms of the block force as this. And again, we have seen that EA_b and EA_{c1} can also be related to the EI_b and EI_{c1}. So, the same thing can be written as this also.

(Refer Slide Time: 10:26)



So, that gives us the expression for our actuation force. Once we know the actuation force our problem is solved. Then, we can analyze the beam and find out its deformation and find out other quantities of our interest.

Now, if we know the actuation force, we know that bending moment is equal to F multiplied by t_b by two.

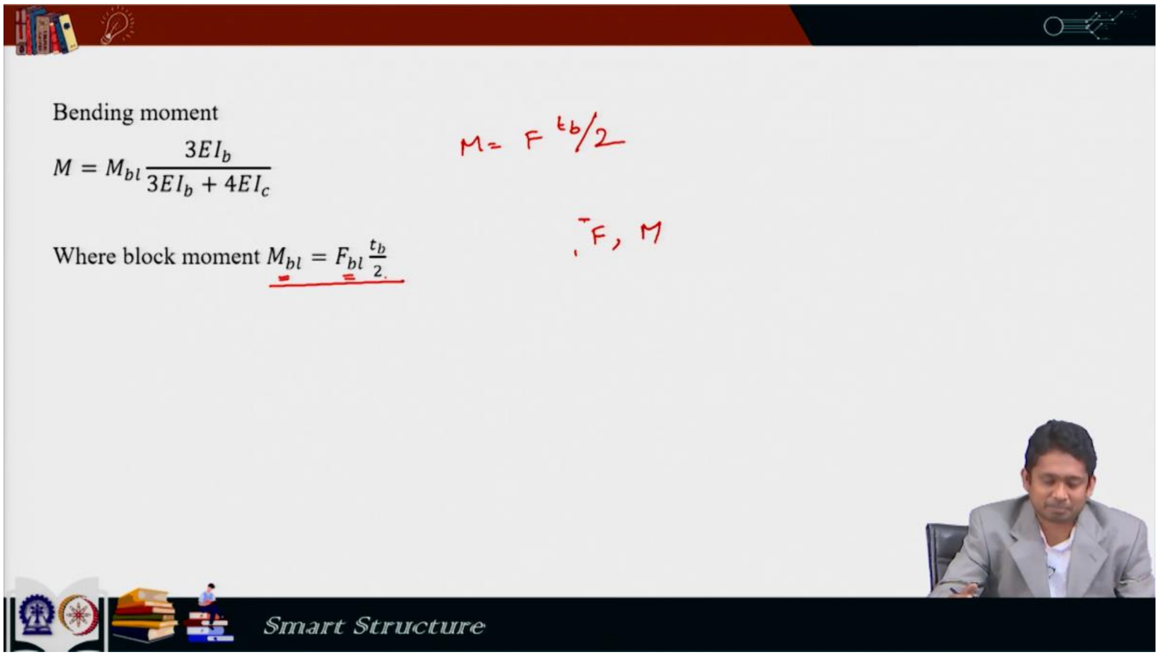
$$M = M_{bl}\frac{3EI_b}{3EI_b + 4EI_c}$$

And then, if you put it here and again, we have the expression for the block force which is M_{bl} is equal to F_{bl} into t_b by two.

$$M_{bl} = F_{bl}\frac{t_b}{2}$$

After all these substitutions, we can find out the bending moment also in terms of the block bending moment. So, actually we need both F and M. We have got those using both of these quantities, we can find out whatever we need in the beam: its deformation, its stress strain and so on.

(Refer Slide Time: 11:08)



Now, we look into another case where we have the symmetric configuration, but the actuation is not symmetric. So, the top and bottom piezo's are actuated with different voltages.

So, here the actuation voltage is V_{top} and here it is V_{bottom}. Now, as we did before, we will use a similar approach. So, we will decouple this problem into a bending problem and an extension problem. So, let us assume that the extensional part of the voltage distribution is V_1. So, it is V_1 and V_1 here, and the bending part of the voltage distribution is V_2 and minus V_2. So, we can say that V_1 plus V_2 is equal to V_{bottom} and V_1 minus V_2 is equal to V_{top}, which we can see here.

$$V_1 - V_2 = V_{top}$$

$$V_1 + V_2 = V_{bottom}$$

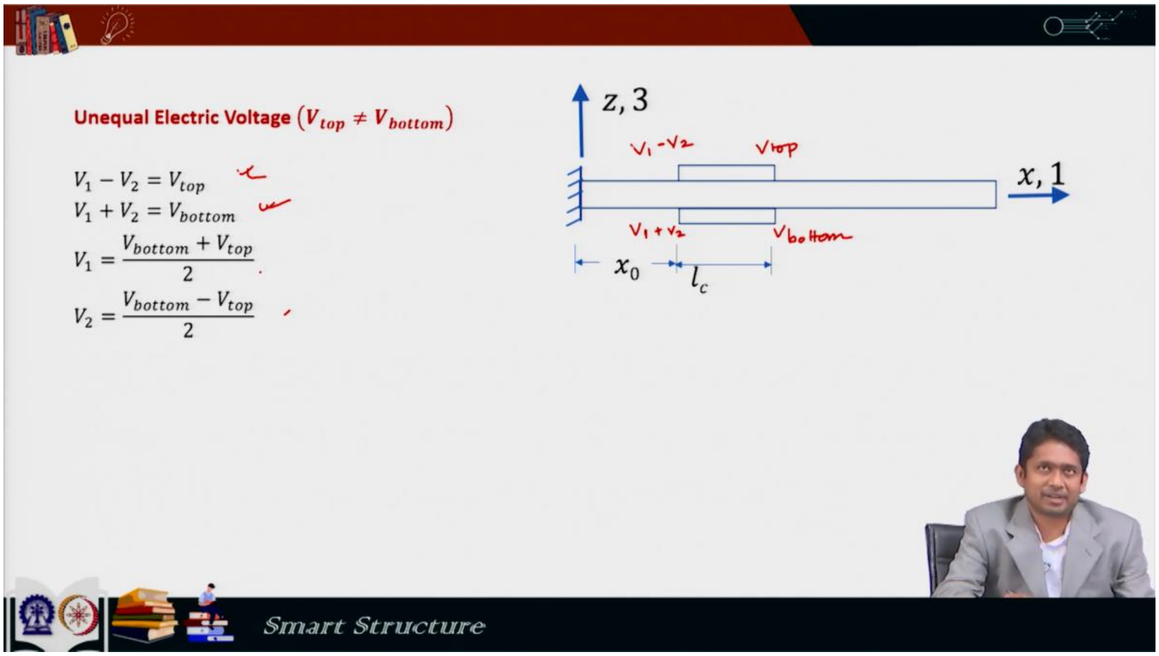
And then, once we solve these two equations, we get our expression for V_1, the extensional part of the actuation and V_2, the bending part of the actuation as this in terms of the actuation voltages.

$$V_1 = \frac{V_{bottom} + V_{top}}{2}$$

$$V_2 = \frac{V_{bottom} - V_{top}}{2}$$

So, our problem has now been decoupled into an extension problem and a bending problem.

(Refer Slide Time: 12:26)



So, from whatever we did on the symmetric and anti-symmetric actuation, I mean the axial and bending actuations, we can say that the actuation force due to V_1, the extensional or axial force is F_{bl_1} multiplied by EA_b by EA_c plus EA_b.

$$F^e = F_{bl_1} \frac{EA_b}{EA_c + EA_b}$$

So, it is an axial actuation, where F_{bl_1} is the block force due to the voltage V and that is this.

$$F_{bl_1} = EA_c \frac{\varepsilon_{p1}}{2} = \frac{d_{31}V_1}{2t_c} EA_c$$

And similarly, for voltage V_2 which is a bending actuation the force is F^b is equal to this, which is equal to this in terms of the block force F_{bl_2} and the corresponding bending moment is M is equal to M_{bl_2} into this.

$$F^b = F_{bl_2} \frac{EI_b}{EI_b + EI_c} = F_{bl_2} \frac{EA_b}{EA_b + 3EA_c}$$

$$M = M_{bl_2} \frac{EI_b}{EI_b + EI_c} = \frac{2d_{31}V_2}{t_b t_c} \frac{EI_b EI_c}{EI_b + EI_c}$$

So, where the block force corresponding to V is this. And similarly, M_{bl_2}, the block bending moment would be F_{bl_2} multiplied by t_b by two.

$$M_{bl_2} = F_{bl_2}\frac{t_b}{2}$$

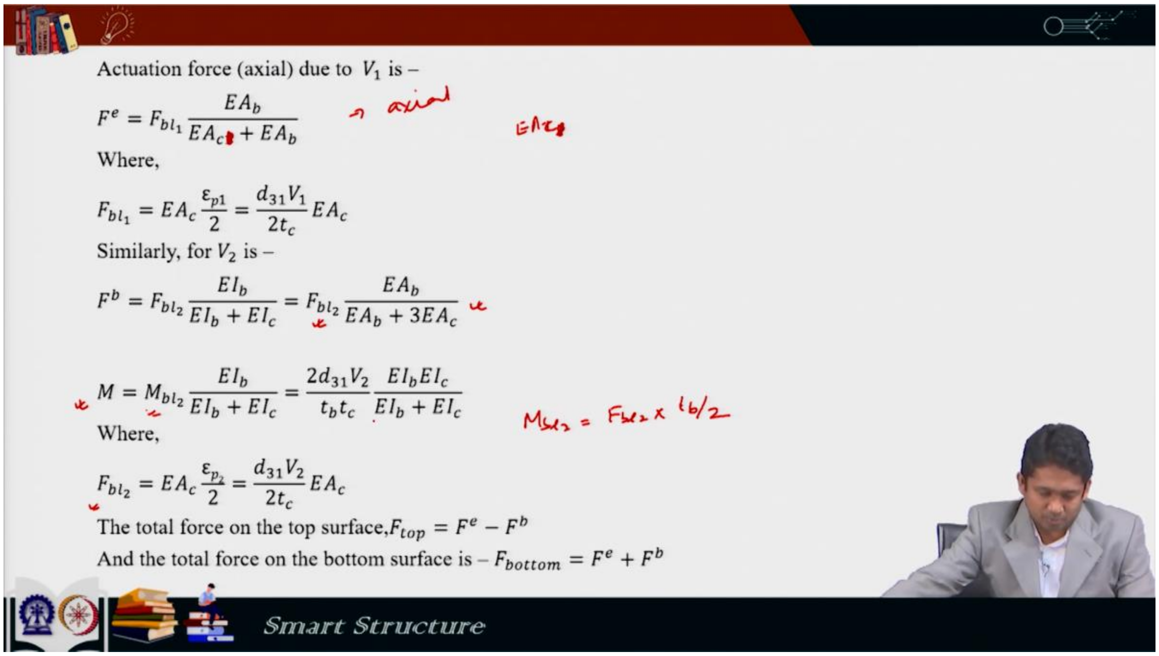
$$F_{bl_2} = EA_c\frac{\varepsilon_{p2}}{2} = \frac{d_{31}V_2}{2t_c}EA_c$$

Now, that we know F^e and F^b. We can say that at the top surface the actuation force is F^e minus F^b, and at the bottom surface the actuation force is F^e plus F^b.

$$F_{top} = F^e - F^b$$

$$F_{bottom} = F^e + F^b$$

(Refer Slide Time: 13:53)



Now, we will look into one more case, I mean two more cases. Again, these are asymmetric actuations and here the asymmetric is in the piezoelectric properties.

In the first case the asymmetry is in the geometric property of the piezoelectric patches. So, here the thickness at the top and the bottom piezo are different. So, we can call this as $t_{c_{top}}$ and we can call this thickness as $t_{c_{bottom}}$. Thickness of the beam remains same as we have been denoting by t_b. Now, here we are actuating them using the same voltage V. Now, as we have seen even though the voltage is same, but because the thickness is different, so, the actuation force can be different. So, if the actuation force at the bottom is say F_{bottom} and at the top is say F_{top}. Now, again we will decouple them into an axial actuation and a bending actuation. So, let us say that we have an axial part of the actuation, which we can call as F^e, and that force should be the same F^e and F^e. And similarly, we have a bending part of the actuation, and that is going to be different. So, again, we have F^b here and opposite force F^b here. So, we can say that F^b plus F^e is equal to the F_{bottom}, and F^e minus F^b is equal to the F_{top}, which is these two equations.

$$F^e - F^b = F_{top}$$

$$F^e + F^b = F_{bottom}$$

So, after solving, we can write the extensional actuation force F^e as this and F^b, the bending actuation force corresponding to the bending part, is this.

$$F^e = \frac{F_{bottom} + F_{top}}{2}$$

$$F^b = \frac{F_{bottom} - F_{top}}{2}$$

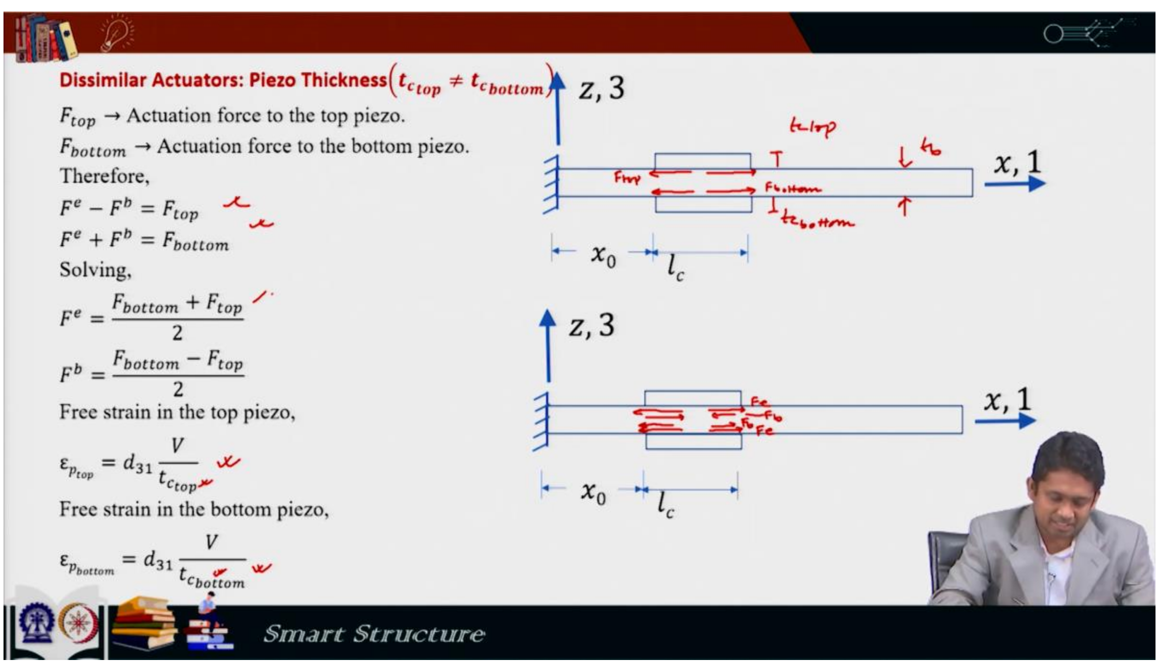
Now, the free strain in the top piezoelectric part is this, and the free strain at the bottom piezoelectric part is this.

$$\varepsilon_{p_{top}} = d_{31} \frac{V}{t_{c_{top}}}$$

$$\varepsilon_{p_{bottom}} = d_{31} \frac{V}{t_{c_{bottom}}}$$

So, they just differ by this term $t_{c_{top}}$ and $t_{c_{bottom}}$.

(Refer Slide Time: 16:19)



Now, all we need to do is to find out the displacements, axial displacements at this junction: the top surface of the beam and the piezoelectric patch where they are meeting, and at the bottom junction, the bottom part of the beam and the bottom piezoelectric patch where they are meeting. So, at the top surface of the beam, the extension is delta $l_{b_{top}}$, which is this.

$$\Delta l_{b_{top}} = \left(\frac{2F^e}{EA_b} - \frac{6F^b}{EA_b}\right) l_c$$

So, this is extension due to the axial force F, and this is contraction due to the bending actuation F^b. Because we know that M divided by I, multiplied by minus z, is our stress here due to bending moment b, and if I divide that by E^b, that gives me the strain here. And now this M can be related to F^b. So, F^b becomes M^b by t^b by two, or M^b is F^b multiplied by t_b by two, where M^b is the actuation bending moment. So, this M^b and M are basically the same. So, if we write this in terms of F^b, and then put z is equal to t by two, and after doing some mathematical manipulations, this expression comes. The extension of the piezo can be written as this. We have been doing this.

$$\Delta l_{c_{top}} = \left(\varepsilon_{p_{top}} - \frac{F_{top}}{EA_{c_{top}}}\right) l_c$$

And if we equate these two expressions, we get this equation.

$$\Delta l_{c_{top}} = \Delta l_{b_{top}}$$

$$F_{top}\left(\frac{4}{EA_b} + \frac{1}{EA_{c_{top}}}\right) + F_{bottom}\left(-\frac{2}{EA_b}\right) = \varepsilon_{p_{top}}$$

After that, we can do the same thing at the bottom piezoelectric patch and ensure that the extension at the bottom surface of the beam and the bottom piezo patch are same. And if we do that, we get this expression.

$$F_{top}\left(-\frac{2}{EA_b}\right) + F_{bottom}\left(\frac{4}{EA_b} + \frac{1}{EA_{c_{bottom}}}\right) = \varepsilon_{p_{bottom}}$$

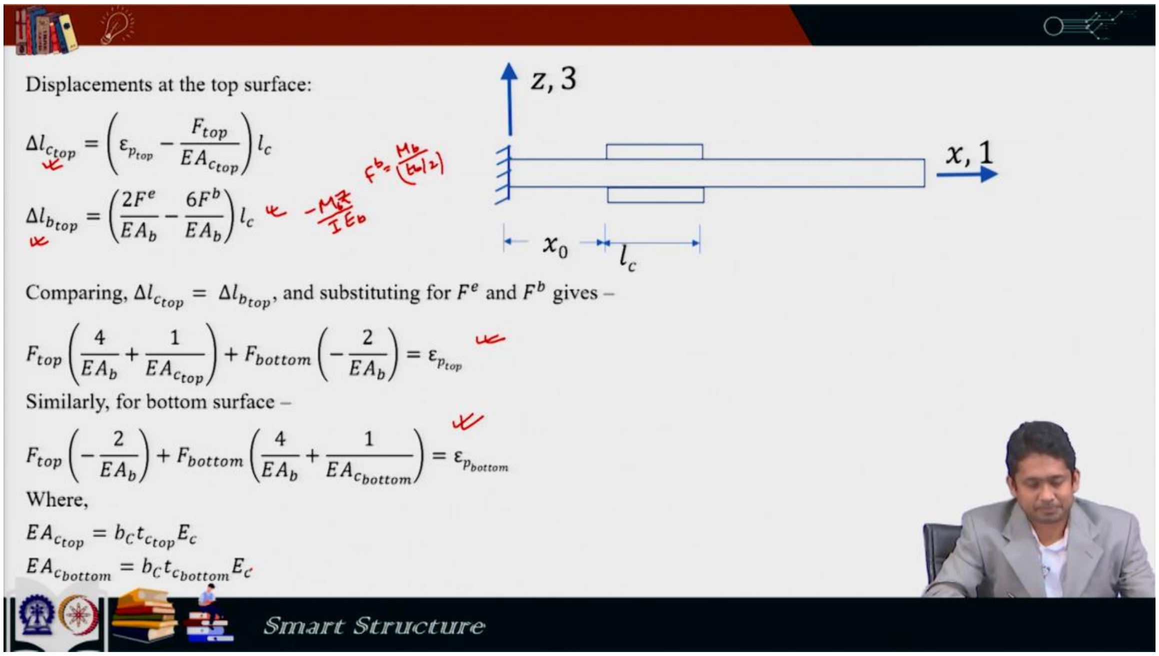
So, here we have used these terminologies $EA_{c_{top}}$ and $EA_{c_{bottom}}$.

$$EA_{c_{top}} = b_c t_{c_{top}} E_c$$

$$EA_{c_{bottom}} = b_c t_{c_{bottom}} E_c$$

They are just defined as this $EA_{c_{top}}$ is the axial stiffness of the top piezoelectric patch, which is $b_c t_{c_{top}}$ multiplied by E_c. $EA_{c_{bottom}}$ is equal to $b_c t_{c_{bottom}}$ multiplied by E_c.

(Refer Slide Time: 18:51)



The same expression can be rewritten in terms of this. So, we are introducing new variables alpha1, alpha2, and alpha3,

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix} \begin{Bmatrix} F_{top} \\ F_{bottom} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{p_{top}} \\ \varepsilon_{p_{bottom}} \end{Bmatrix}$$

where alpha1 alpha2 and alpha3 are this.

$$\alpha_1 = \frac{4}{EA_b} + \frac{1}{EA_{c_{top}}}$$

$$\alpha_2 = -\frac{2}{EA_b}$$

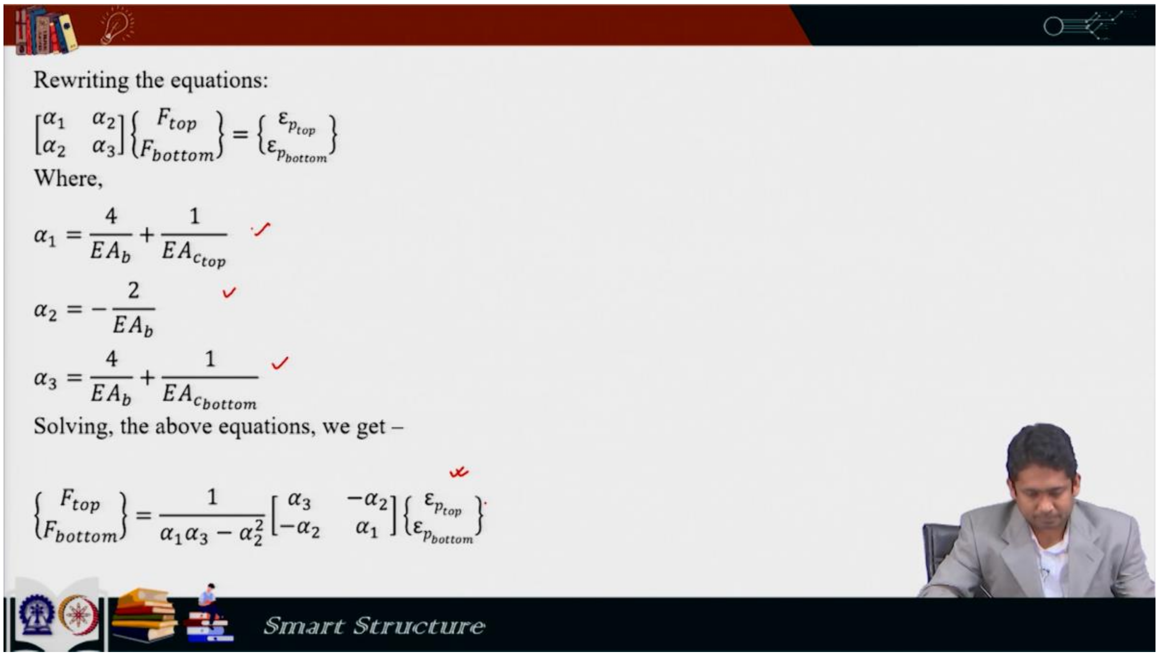
$$\alpha_3 = \frac{4}{EA_b} + \frac{1}{EA_{c_{bottom}}}$$

Now, this equation can be solved, and the expression for F_{top} and F_{bottom} can be found out.

$$\begin{Bmatrix} F_{top} \\ F_{bottom} \end{Bmatrix} = \frac{1}{\alpha_1 \alpha_3 - \alpha_2^2} \begin{bmatrix} \alpha_3 & -\alpha_2 \\ -\alpha_2 & \alpha_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{p_{top}} \\ \varepsilon_{p_{bottom}} \end{Bmatrix}$$

And this comes in terms of the all the material and the geometric properties and the free strain at the top and bottom piezo.

(Refer Slide Time: 19:27)



After solving, these are the expressions that we get.

$$F_{top}=\frac{1}{\alpha_1\alpha_3-\alpha_2^2}\left(\alpha_3\varepsilon_{p_{top}}-\alpha_2\varepsilon_{p_{bottom}}\right)$$

$$F_{bottom}=\frac{1}{\alpha_1\alpha_3-\alpha_2^2}\left(-\alpha_2\varepsilon_{p_{top}}+\alpha_1\varepsilon_{p_{bottom}}\right)$$

Now, once we know the F_{top} and F_{bottom}, and we know what is the extensional actuation force in terms of top and bottom. So, we can write this.

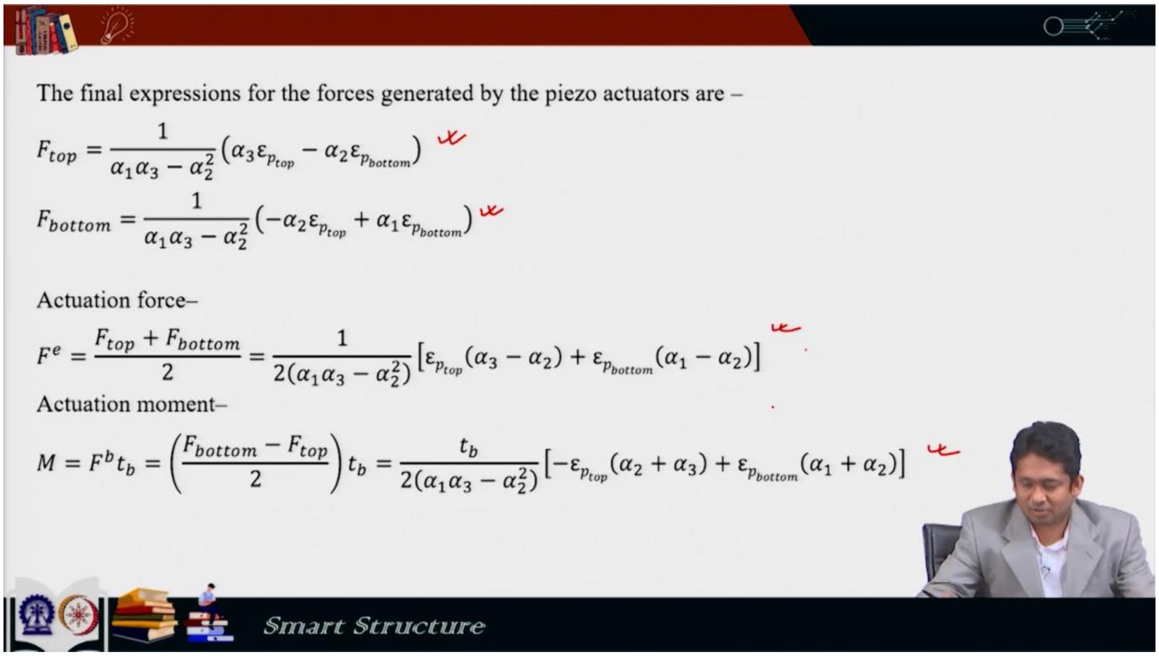
$$F^e=\frac{F_{bottom}+F_{top}}{2}=\frac{1}{2(\alpha_1\alpha_3-\alpha_2^2)}\left[\varepsilon_{p_{top}}(\alpha_3-\alpha_2)+\varepsilon_{p_{bottom}}(\alpha_1-\alpha_2)\right]$$

And similarly, we know the actuation moment. We know the bending actuation force F^b in terms of the F_{top} and F_{bottom}, and we can write this.

$$M=F^b t_b=\left(\frac{F_{bottom}-F_{top}}{2}\right)t_b=\frac{t_b}{2(\alpha_1\alpha_3-\alpha_2^2)}\left[-\varepsilon_{p_{top}}(\alpha_2+\alpha_3)+\varepsilon_{p_{bottom}}(\alpha_1+\alpha_2)\right]$$

So, here we have denoted this as M. We can call it M_b also because of this F^b. Now, this we get as the actuation moment.

(Refer Slide Time: 20:17)



So, once we know these two quantities, or I mean these quantities, then we can solve for the deformation, stress-strain, and all of our quantities of interest.

The next case is also an asymmetric actuation. Here, the asymmetry is again in the piezo properties, but here, the asymmetry is in piezoelectric material properties. So, d_{31} at the top and bottom are the same. Now, the thickness is the same. In the previous case, we had different thicknesses, but here, we can keep the thickness the same, or in fact, we can have both the properties also different, and we can solve it. So, again, a similar thing happens here; even though there is the same voltage applied at the top and bottom, the forces can be different.

$$\varepsilon_{p_{top}} = d_{31_{top}} \frac{V}{t_c}$$

$$\varepsilon_{p_{bottom}} = d_{31_{bottom}} \frac{V}{t_c}$$

So, we can write the same thing. Suppose the force here is F_{bottom}, and the force here is F_{top}. And then again, we decouple this into F^e, the extensional part, and a bending part, which is F^b. So, we can follow the same approach, and again, we can apply the compatibility condition here that the elongation is the same for the top surface of the beam and the top piezo, and also the bottom surface of the beam and the bottom piezo and that will give us two equations. But in those equations, now since I have the same material property for the piezoelectric patches, I mean, the same geometric property, we get alpha1 and alpha3 to be the same, and that is equal to four by EA_b plus one by EA_{c1}. So, here, EA_{c1} means EA_c of one of the piezo patches, and that is the same for both. And then alpha2 comes to be this.

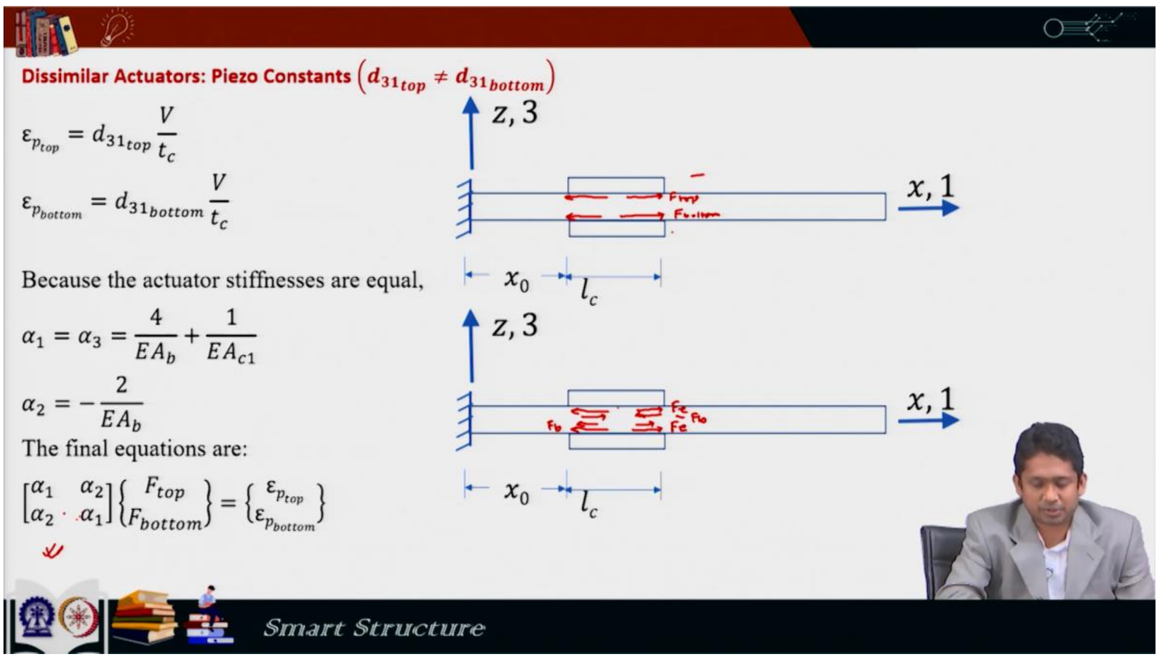
$$\alpha_1 = \alpha_3 = \frac{4}{EA_b} + \frac{1}{EA_{c1}}$$

$$\alpha_2 = -\frac{2}{EA_b}$$

So, finally, the equation looks like this. Alpha1 and alpha3 are the same, that is the only difference between these two equations.

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \begin{Bmatrix} F_{top} \\ F_{bottom} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{p_{top}} \\ \varepsilon_{p_{bottom}} \end{Bmatrix}$$

(Refer Slide Time: 22:36)



And after that, we solve the equations, and we get our expression for F_{top} and F_{bottom}.

$$F_{top} = \frac{1}{\alpha_1^2 - \alpha_2^2}\left(\alpha_1 \varepsilon_{p_{top}} - \alpha_2 \varepsilon_{p_{bottom}}\right)$$

$$F_{bottom} = \frac{1}{\alpha_1^2 - \alpha_2^2}\left(-\alpha_2 \varepsilon_{p_{top}} + \alpha_1 \varepsilon_{p_{bottom}}\right)$$

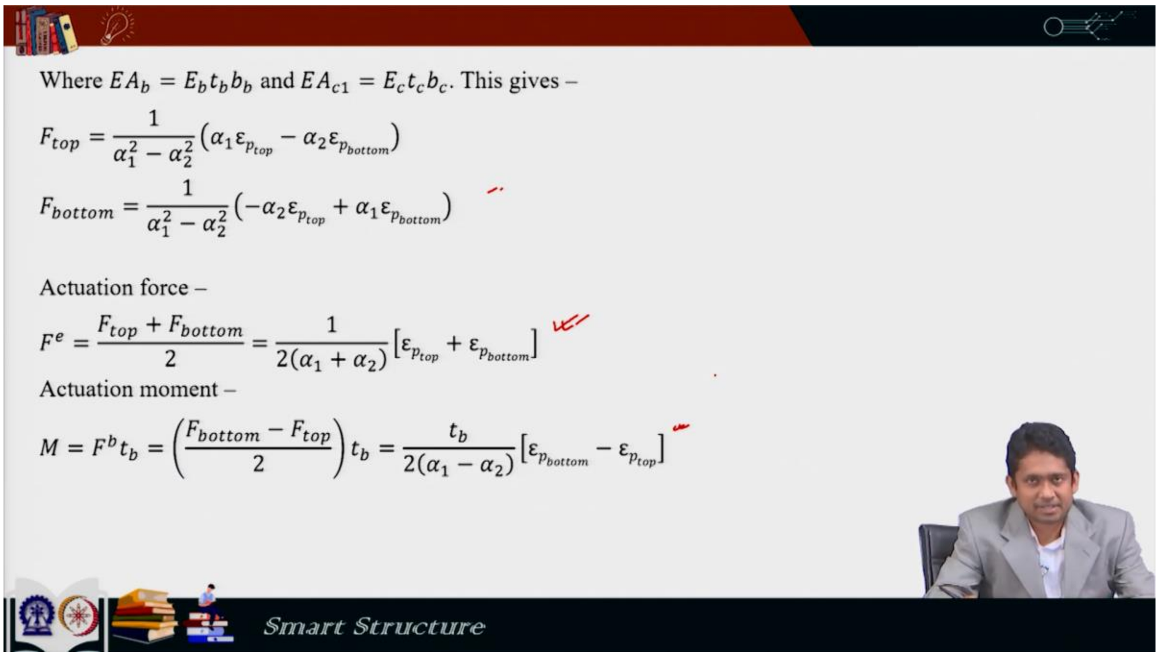
And from there, we can find out the actuation force, I mean, the extensional part of the actuation, that is, for that, we get the actuation force. And for the bending part of the actuation, we can get that force or the corresponding moment.

$$F^e = \frac{F_{bottom} + F_{top}}{2} = \frac{1}{2(\alpha_1 + \alpha_2)}\left[\varepsilon_{p_{top}} + \varepsilon_{p_{bottom}}\right]$$

$$M = F^b t_b = \left(\frac{F_{bottom} - F_{top}}{2}\right) t_b = \frac{t_b}{2(\alpha_1 - \alpha_2)}\left[\varepsilon_{p_{bottom}} - \varepsilon_{p_{top}}\right]$$

And then we can analyze the same beam.

(Refer Slide Time: 22:52)



Now, we look into the assumptions that we made in this: the major assumptions in this analysis.

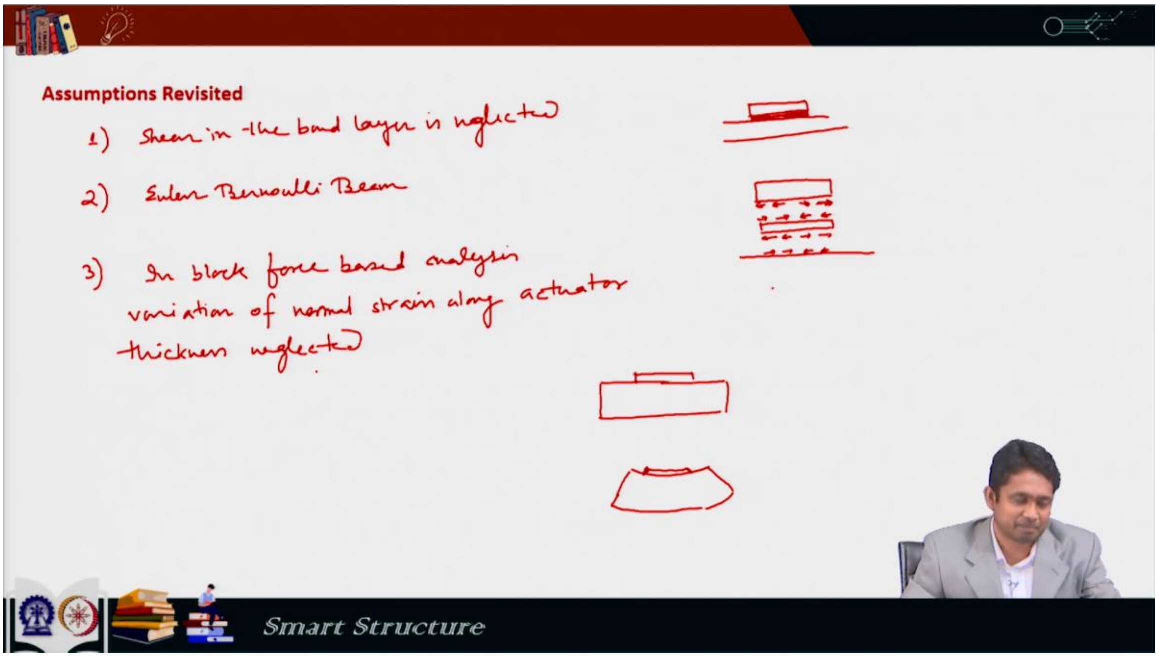
First of all, the shear in the bond is neglected. So, this piezoelectric patch it is connected to the beam with a bond. So, there is a bond layer between the piezo patch and adhesive that joins these two. So, if you look at it carefully, we will see that, I mean, if we separate them and draw the diagrams, the shear diagrams, we would see that, we can see the shear at the interfaces.

So, this is how the shear would look. Now, the effect of the shear has been on this bond layer has been neglected. In fact, in all these analyses, we did not consider the bond layer at all because it is quite thin as compared to the piezo or the beam. Now, even if we consider the bond, our analysis technique will let us consider it for only its extensional and bending stiffness but not for this shear effect.

Next is the Euler-Bernoulli assumption that we made. So, it says that there is no shear deformation in the beam that we neglected. And also, while doing the block force methodbased analysis, we neglected the variation of normal strain along the thickness of the piezo actuator.

So, in the variation of normal strain along actuator thickness was neglected. So, as we saw that, if I have a beam, and if it has a piezo patch, and if this bends, then in the bend shape, it would look like this. So, there will be a variation of strain along the thickness of the piezo also, but that is neglected in the block force-based analysis.

(Refer Slide Time: 25:45)



So, these are the major assumptions that we made while doing this analysis.

With that, I would like to conclude this lecture.

Thank you.