Introduction to CFD
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Module - 2
Lecture – 8
Methods for Approximate Solutions of PDEs Continued

In this lecture, we are going to start discussing about another method for approximation of partial differential equations, which we call as the finite volume method. In the previous two lectures, you have learned about the finite difference method and now we move on to the next method, the finite volume method.

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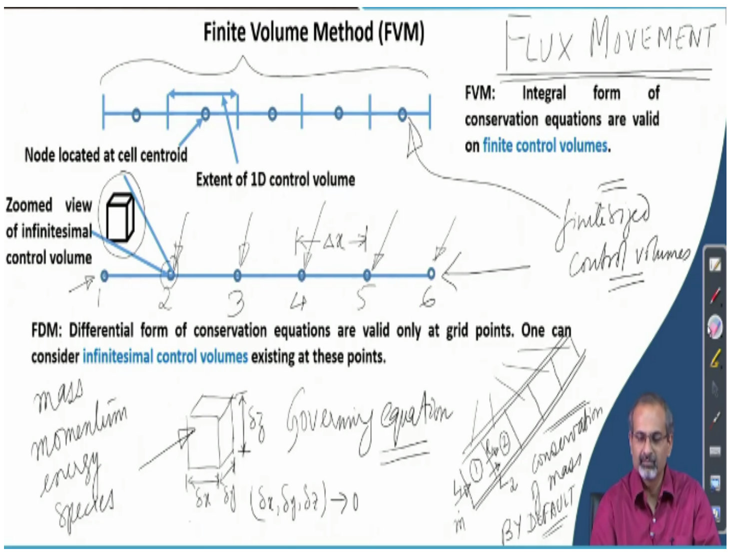

We will first look at the grid that we used for discussion in the context of the finite difference method. So, you can see the grid that we discussed about earlier. However, there is a small difference in comparison with the grid that we used for solving the problem in the last lecture. You can see that here you have 6 grid points rather than 5, which we had in the previous problem, but fundamentally there is no difference between the two grids.

We have constant spacing between adjacent grid points and we have boundaries at the two ends of the grid where we expect that some boundary conditions would be imposed. Then we would be told what kind of governing equation is to be solved on this grid, and then the next thing that we bother about is how to generate approximate expressions for the derivatives, which figure in the governing equation. So, we have broadly understood how in a finite difference method we go through these steps.

Just to recall, the whole process started with grid generation. We had a certain length over which a certain number of grid points were disposed in such a manner that we had constant spacing between adjacent grid points, and we understood that this way we can have a slightly simpler solution of the whole problem because the Taylor series would then be developed based on constant grid spacing, which is more convenient and more accurate in general.

Now, let us look at where from we actually obtained the governing equations, which we tried to approximate when we were discussing about the finite difference technique. So, if you look back at your basic courses in fluid mechanics and aerodynamics or hydrodynamics, you would recall that we looked at very very small control volumes which were infinitesimal in extent. So, we used to draw control volumes looking like this.

And then mark very small lengths along orthogonal directions and we used to say that each of these length scales should limit to 0. So, we were essentially talking about infinitesimal control volumes and the differential equations which form the governing equation was actually derived in such an infinitesimal control volume. Now, the moment we try to approximate the governing equation, we are introducing different errors in approximating them.

The first error that we commit when we have a finite number of grids to represent the domain is that we try exactly satisfying the governing equation only at a few discrete points and nowhere else in the domain. There are big gaps left in between where we have no clue how good or how bad the governing equation is being satisfied. Also at these discrete points, we are only approximately satisfying the governing equation, because we have left out many terms from the Taylor series through the dropping out of the truncation error terms. So there are different levels of approximation which are actually getting into the solution. Now, if we try to look at how that infinitesimal control volume fits in this perspective, you would have to zoom deep into a grid point and then if you zoom deeper and deeper, you will find that infinitesimal control volume sitting there around that point with very small dimensions.

So, essentially what we are doing is that we are trying to satisfy the governing equation as close to the exact governing equation as possible only in and around these grid points. Now, the question could be that why did we not think about making these control volumes bigger and giving them finite dimensions. So, that may bring in certain errors because the differential equation can actually be satisfied per se at distinct points.

And the moment we try to make the control volumes bigger, then obviously there could be certain errors introduced in the process. But this could be a very interesting idea to pursue. Through this approach we will find a way which is better connected to the behavior of the physical world.

We are trying to scale up these infinitesimal control volumes and then give them finite spans and fill up a certain length scale using finite sized control volumes. We were just discussing about the fact that what makes them more appropriate and closer to the physical world. One big aspect that makes them more appropriate is that they are just juxtaposed next to each other and therefore many of the conservation laws which we deal with through these governing equations can be exactly satisfied.

When you juxtapose finite size control volumes and set them next to each other, so that one face of one control volume lets in a certain amount of mass for example, and sends the same mass out through another face which will be received by the next control volume. This principle is very close to the physical world, where we talk about conservation of mass for example when we are transferring it through different channels and pipes.

For example, if you had a pipe delivering some fluid, you could divide the length of the pipe into different control volumes and then what happens essentially is that if some mass is entering the first control volume through its left side, then it moves out through the right side of the same control volume. So, what goes out of the right side of the first control volume is essentially going to enter the second control volume without fail.

So L1 boundary, on the left side of the first control volume, lets in a certain amount of mass at a certain time rate. What enters through L1 is what leaves through R1 and then what leaves through R1 enters the second control volume through its left boundary L2 (R1 and L2 boundaries are identical). So what are we talking about? We are talking about conservation of mass by default if we are having such a system of side by side control volumes like the ones we have drawn here. So, the control volumes are essentially next to each other sharing boundaries.

And what these boundaries are doing is that they are transferring flux, it could be mass, it could be momentum, it could be energy, but by and large, we are talking about flux movement and this is something that is very closely connected with the physical world and it is quite intuitive as well. However, in the finite difference calculations, we never get the sense of the flux.

We just talked about locally satisfying the differential equation as best as we can with higher order of accuracy as far as possible and then we have no feeling of movement of flux, and this is what makes finite volume method a very strong method in the sense that it is very closely and inherently connected to the conservation equations. Therefore, the physical world is much more reliably and better modeled through finite volume technique because of this property that it inherently takes care of conservation of fluxes.

Again remember that it could be different types of fluxes that we talk about. It could be mass, it could be momentum, it could be Energy, it could be species and so on. Every time that we talk about a flux, we are certainly going to see conservation of the flux by default if we are properly implementing the finite volume method. So, we have to keep in mind that this is perhaps the strongest point of the finite volume method.

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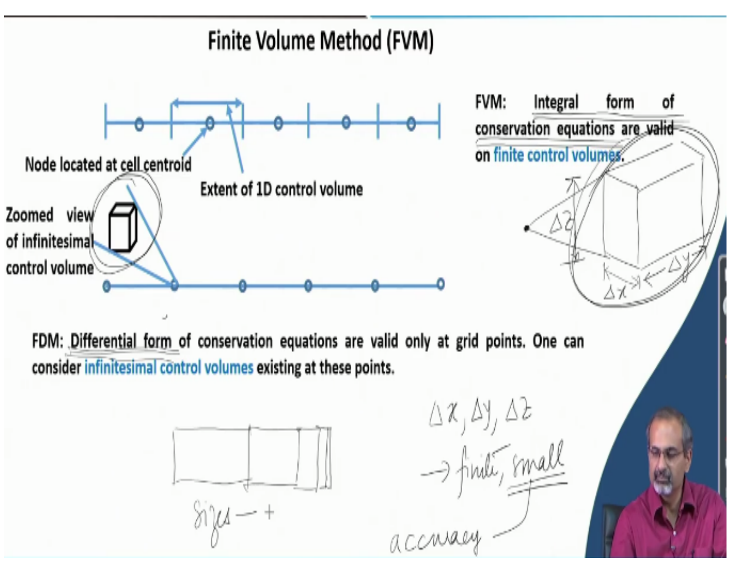

We were talking about governing equations where we see derivatives. In finite volume method however, we deal with the integral form of the conservation equations. So, we do not use the differential form, but rather we use the integral form. Keep in mind that finite difference method was dealing with the differential form of conservation equations, while here we are going to deal with integral forms.

Additionally, finite difference method is closely connected with the infinitesimal control volumes and therefore we try applying finite difference method to distinct grid points; while in finite volume method, because it has an integral approach instead of a grid point based approach, we scale up an infinitesimal control volume to a control volume which has finite dimensions. Note that the control volume no longer has infinitesimal dimensions.

This is the difference between a control volume of infinitesimal size limiting to a point in finite difference technique and a control volume with finite dimensions used in a finite volume technique. So, the control volume can have dimension delta x, delta y, delta z and these dimensions are not infinitesimal. They are finite, but of course they are small as well. We tend to keep them small for the sake of better accuracy. Again another very commonly used feature in finite volume method is that we tend to make these volumes look bigger or smaller by choice wherever we need them that way.

And this choice of sizes is done more conveniently in a finite volume method, usually more conveniently than the finite difference method.

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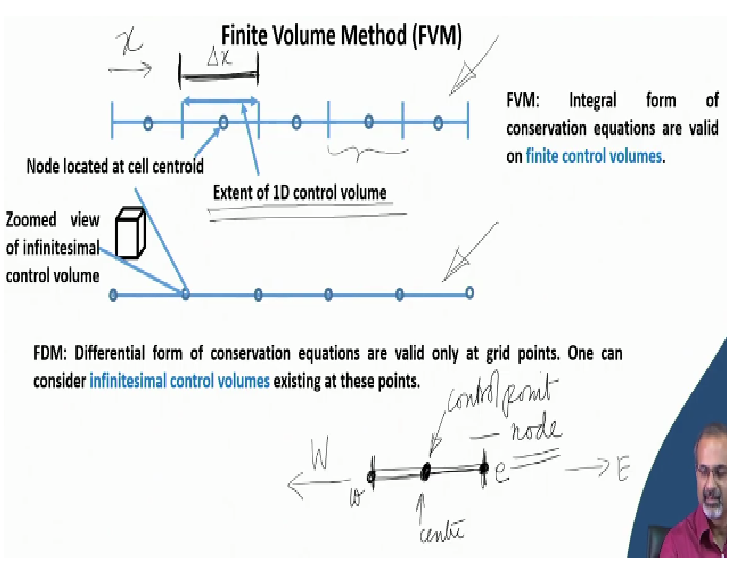

The fact that we have now expanded the control volume to make it occupy a finite dimension, we have the extent of a one-dimensional control volume here. We continue to use a one-dimensional domain like we did for the finite difference calculations and discussion. So, even here for the discussion of the finite volume method we are using the same one-dimensional problem or one dimensional domain.

But here we are noting that we have a certain extent which we are defining as a control volume, so it is nothing but a small element spanning a certain length along this direction. Of course, it also has dimensions in the two other orthogonal directions. We can always go about defining that. So, if this is direction x, then it does not mean that the control volume essentially needs to have infinitesimal dimensions along y and z.

As I discussed earlier, it can have finite dimensions along all 3 orthogonal directions. However, the nature of the problem is such that it only has x dependence and therefore essentially, we can represent the control volumes by just drawing a straight line and dividing it into number of segments and each segment would be representing a particular control volume.

Now, the control volume has a certain extent, say given by this interval Δx. We need to remember that this Δx is no longer the distance between two neighboring grid points, but rather it is the length of one control volume and then what we would end up doing is that if this is our control volume, we set a control point at the center of the control volume.

So, we choose to have a control point which is often called as a node and it is located at the center of the control volume. Now, since this is a one-dimensional control volume, we can call this direction as east, this direction as west and we can name this face of the control volume as small e, this face of the control volume as small w for example. So if you want to show more control volumes in the neighborhood, then you have to have a more elaborate diagram in place.

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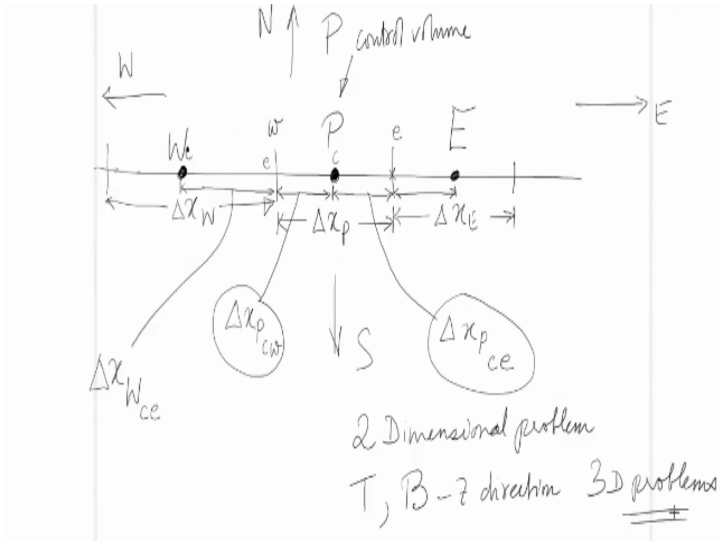

Let us try to draw a more elaborate diagram and show the lengths which are of interest to us. Let us say that this is a control volume, which is of direct interest to us. So, it has a certain length, a total length which we will mark as a Δx_P. We name this control volume as the P control volume and then this could be the east face of P, this could be the west face of P and then we can move on to neighboring control volumes.

So, maintaining the same direction, east, west, if you wish south and north. You could have a control volume here on the east, you could have another control volume here on the west, and note that this control volume definitely is wider than the control volume P. While this seems to be of the same order as P, it could be a little smaller or a little bigger. So, we have flexibility in defining the lengths of the control volumes.

Again, it is always better to have a clearer definition of these distances from the nodes to the faces. You would often need these distances when you do finite volume calculations. So let us say if you are looking at the distance between the node of the control volume P to its west face. So, you may like to put some nomenclature of your own saying that this is delta xp and if this is the center, then it is cw let us say, $\Delta x_{P_{cw}}$.

Again this distance may be marked as $\Delta x_{P_{ce}}$. You could have other ways of putting the nomenclature as well, but it is essential that you take note these lengths. Also as you can understand that if you go to a neighboring cell, let us say the west cell and you try to find out the Δx_W and then if you call this point again for the w cell as e it would be $\Delta x_{W_{ce}}$ and so on. Similar things may be done along the y direction.

So, then you would have neighboring cells along north and south. That will typically be the situation in a two-dimensional problem and you could have nodes along the direction coming out of the plane of the paper and into the plane of the paper where you can have nodes named as a top and bottom node, let us say along the z direction, and that would be required for 3D problems.

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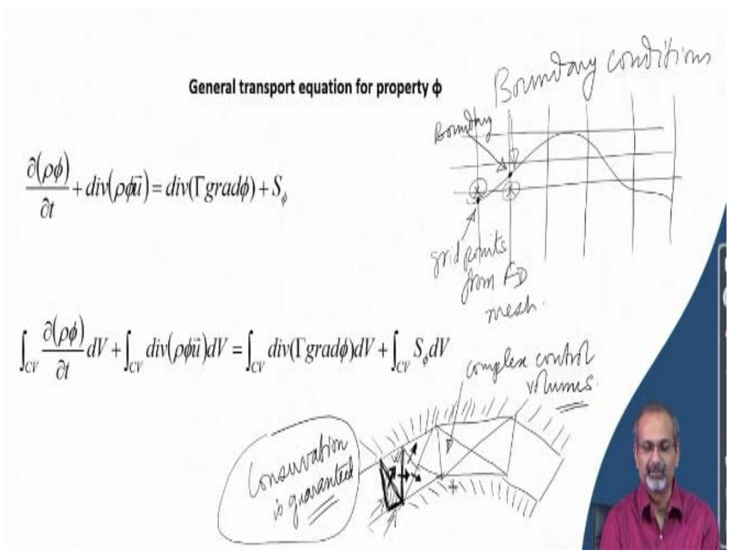

So, we briefly looked at the basic framework of discretization, the way we do it for the finite volume technique. In a way it is similar to finite difference because we are dividing a certain length scale into sub intervals, but then we are not distributing grid points, we are creating control volumes and the control volumes are set side by side facing common faces or rather sharing common faces.

And therefore from a conceptual angle, there are distinct differences between the two, though often when you look at grids which are created for finite difference calculation and finite volume calculations, they may seem to be very similar to each other. Again remember that in finite difference calculations using non-uniform mesh is always more complicated, while that is rather easy to do in finite volume technique.

Again if you have very complex geometries to work with, finite volume is a much more flexible and easy to adapt method compared to finite difference method. For example, if you have very strange looking surfaces on which you have to implement certain boundary conditions in a finite difference framework, it could be quite difficult to do because in finite difference method we usually use grids of this kind.

And therefore in order to approximate the boundary, you will not be exactly able to match the points on the boundary and the grid points which are available. So, these are the points on the boundary while these are the grid points available from the finite difference mesh and they do not match. Thus, in order to enforce boundary conditions, it may be very cumbersome to use finite difference mesh.

However, it is quite easy to do it in finite volume technique because of the flux conservation approach which is inherent to finite volume method. So, if you have control volumes which are very arbitrarily oriented to each other and which are also quite arbitrary in shape, it creates no problems in ensuring flux balance. If you look at a cell like this, it may allow mass flux into itself from this direction.

If this is bounded by some solid surface, so if this is part of a solid surface at the bottom and this part on top and you have discretized the intermediate domain by using these cells, then whatever enters this cell from these two faces is bound to go out through this remaining face and then when it goes out, it goes out to the neighboring cell here through one face only and then the neighboring cell exactly transfers the same mass into this cell.

Once it gets into this cell, it has freedom in transferring mass through two faces to neighboring cells and so on. So mass will never get lost anywhere. Conservation is guaranteed and what you see over here is very complex control volumes, which is rather regularly used in finite volume technique without much difficulty. This amazing flexibility and the inherent ability of guaranteeing conservation of fluxes is the major strength, the cornerstone of finite volume technique.

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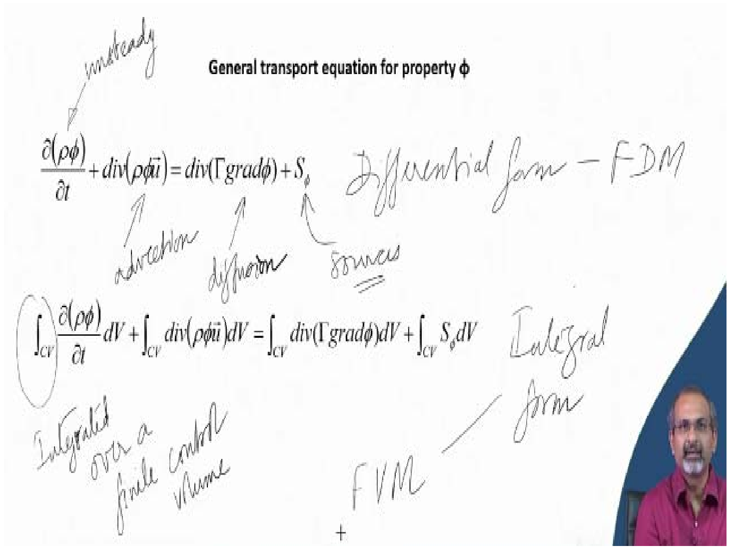

We now look at general transport equation for a property ϕ and we have written the transport equation in two forms. One is the differential form, the other is the integral form. We said that in the finite difference method, we use the differential form while in the finite volume method we use the integral form. So, we need to take note that both these equations are accounting for unsteady effects, they are accounting for advection.

They are accounting for diffusion and possible sources in the domain. In one case, the equation has a differential nature. In another case, the equation term by term is integrated over a finite control volume. So, obviously the second form is more appropriate for discussing the finite volume technique. We will discuss more of this in the next lecture. Thank you.