Introduction to CFD
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Module - 1
Lecture – 5
Classification of PDEs Continued

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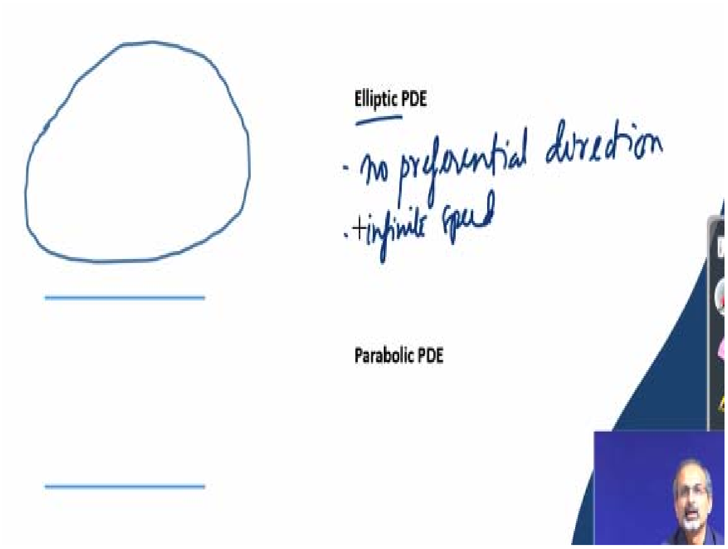

In the last lecture, we were talking about the behavior of elliptic partial differential equation and we tried to explain that in partial differential equations of elliptic nature, we do not have any preferential direction of disturbance propagation. So no preferential direction of disturbance propagation and disturbances propagate at infinite speed, i.e., there is no delay in propagation of the disturbance.

So, the system seems to come to equilibrium instantaneously. Let us discuss about how physically we can look at parabolic partial differential equations.

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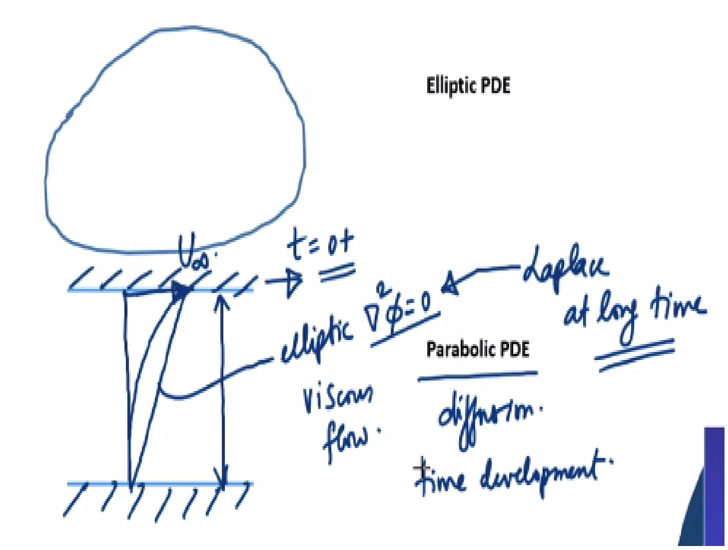

You remember we were talking about a problem earlier where we have a flow confined between two parallel infinite plates and at t = 0+ one of the plates start moving with a certain velocity. So, let us say that the upper plate starts moving with some velocity.

We were interested to know how the flow which is confined within this gap would respond to this instantaneous movement of the upper plate at t = 0+ when it starts impulsively and sustains its motion. We remember the parabolic partial differential equations model diffusion and they also model the time development of the flow. Remember that this is a viscous flow and therefore the flow immediately next to the moving plate would get dragged by the plate.

So, we can intuitively understand that after progress of time, the velocity will gradually become like this, which means that there is a time direction to the problem on one hand and then there is a diffusion which is going on within the thickness of this gap. So, parabolic partial differential equation is actually modeling both and if you give it enough time, then it reaches this profile which is a linear profile, which is nothing but solution of the elliptic equation like this.

We have solved this equation for one dimensional heat conduction problem earlier where you did see the linear velocity profile like behavior, but there it was a temperature profile we were talking about. So, this linear behavior comes from the Laplace equation which will be reached at long time. So, these are inherent characteristics of the partial differential equation of the parabolic kind.

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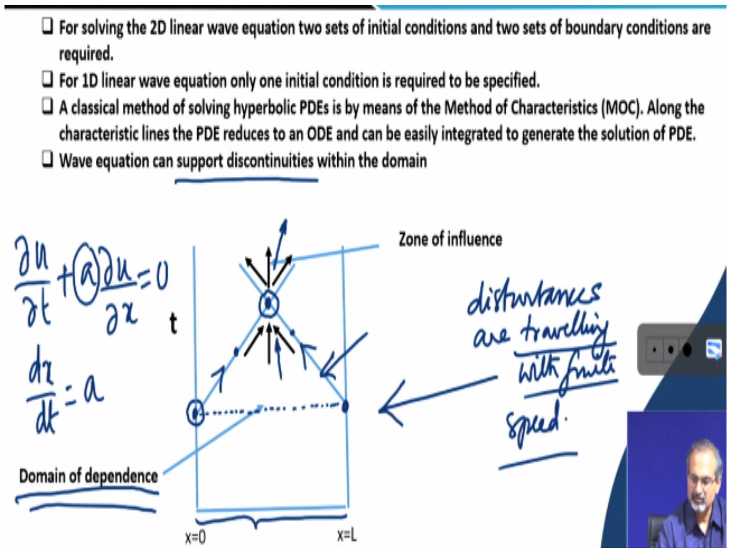

When we come to the hyperbolic partial differential equations, then we represent information often in the x-t plane like what we have drawn here. What we are showing over here is that this is the extent of the domain. At x = 0, you have a certain boundary condition, at x = L there is some other boundary condition. There seems to be movement of disturbance along these two lines into the domain.

Now, the fact that they are not inclined parallel to the x axis essentially means that the disturbances are actually traversing with finite speed. If the physical system was such that if you had a disturbance over here and that was propagating at infinite speed through the domain, then in no time it would have crossed this domain and reached the other end. This means that there would be no elapse of time and the disturbance would show up its effect all through the domain. But that does not happen.

Same is the case with this disturbance which is coming from the other boundary. So the moment the characteristics get slanted with respect to the x axis, it means that there is a lapse of time before the disturbances can make their way into the domain, which means that the disturbances are traveling with finite speed. This is one of the characteristics of hyperbolic partial differential equations.

You remember that when we were dealing with the linear wave equation, we said that the disturbances are propagating with speed ‘a’ through the domain and the characteristic was carrying this information. So, we have to deal with hyperbolic equations in this manner and therefore we have regions which would influence a certain point located in the x-t plane.

These regions which are ahead of that point which influence the happenings at the point would comprise the domain of dependence.

That means, the properties at this point depends on this domain which we call as the domain of dependence and then the properties at this point would end up influencing again some part of the domain, which we call as the zone of influence. So, the disturbances propagate in to influence the point from upstream, while disturbances from the point influence some other region further downstream in terms of time and space.

This is how the solution of the hyperbolic problem emerges. Another very important issue that we have to keep in mind is that hyperbolic equations can support discontinuities. So, some of these fronts could be carrying information about jump conditions like we see in shockwaves. So, there could be discontinuities existing inside the domain itself, which we did not see earlier in the case of elliptic or parabolic partial differential equations.

So, when you model hyperbolic equations using numerical schemes, you have to be particularly careful about handling these discontinuities.

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Let us try expressing the 2nd order linear wave equation as a system of first order PDEs
We propose two first order PDEs as trial equations given as follows:

$$u_1 = \frac{\partial u}{\partial t} = u_t \text{(i)}$$

$$u_2 = c\frac{\partial u}{\partial x} = cu_x \text{(ii)}$$

$u_1, u_2 \quad (x,t)$

Differentiating (i) and (ii) w.r.t. t we get

$$\frac{\partial u_1}{\partial t} = u_{tt} \text{(iii)}$$

$$\frac{\partial u_2}{\partial t} = cu_{xt} \text{(iv)}$$

Differentiating (i) and (ii) w.r.t. x we get

$$\frac{\partial u_1}{\partial x} = u_{xt} \text{(v)}$$

$$\frac{\partial u_2}{\partial x} = cu_{xx} \text{(vi)}$$

Now, we want to discuss a bit about handling a system of first order partial differential equations. We have looked at individual second order partial differential equations, which could have two independent variables to begin with and they could have N number of independent variables and we learned how to classify them and we also dealt with the physical behavior of such partial differential equations to some detail.

Now comes the point where we are looking at a system of first order partial differential equations which could very often occur in fluid dynamics because we are handling system of conservation equations. Let us say you have a system of partial differential equations which comprise of conservation of mass, momentum, energy or species and therefore you have a system to be handled.

So, you cannot just say that one of those equations will be classified independently and that way you will be able to classify the system's behavior. We have to deal with the system as a simultaneous set of partial differential equations. In that case, how do we go about classifying them? The best way to do that will be explained here with a simple example problem. We are talking about a trial solution to the second order linear wave equation.

We want to split the second order linear wave equation into a system of first order partial differential equations. We are proposing a trial solution like this where we have introduced two variables u_1 and u_2 and we have defined them using these two equations. You notice that one of the variables has been linked with the time derivative of u, while the other has been linked with the space derivative of u as well as the wave speed.

Remember that we are handling the second order linear wave equation where c is the wave speed along positive x axis and -c is the wave speed along the negative x axis. Now, we go about doing partial differentiations of the two functions with respect to t and again with respect to x. Remember that u, u_1 and u_2 are functions of both x and t.

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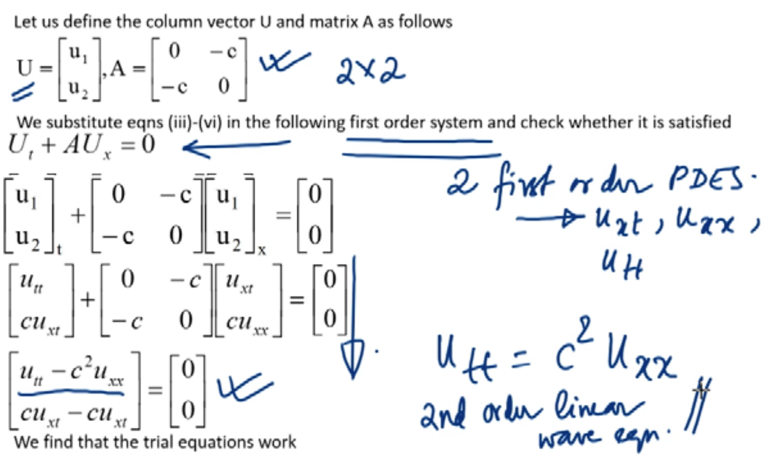

A column vector **U** comprises of the two variables u_1 and u_2. A matrix **A** is defined, which has the following entries 0, –c, –c, 0. So, it is a 2x2 matrix. We are substituting the equations that we got by taking the time and space derivatives of u_1 and u_2 in the first order system which we are mentioning here. Because **U** is a column vector and **A** is a matrix, **U**t + **AU**x = **0** gives a system of equations. They comprise of two first order PDEs which have to be modified slightly for getting second order terms like **U**xt, **U**xx, **U**tt satisfying the equation **U**t + **AU**x = **0**. Now, we are trying to figure out whether the trial solution works or not. So, whether the right hand sides of the modified system of equations will actually turn out to be 0 or not is something that we are going to test and see as we proceed.

You find that the first equation is nothing but the second order linear wave equation and the second equation is an identity. This proves that the trial solutions have worked. So, u_1 and u_2, the way we expressed in equations 1 and 2 have represented the second order linear wave equation in the form of a system of first order PDEs.

Why was this exercise important for us? It is because we now want to see whether this equivalent first order PDE system would end up giving the same information regarding the nature of the partial differential equation, which they represent. When we try to classify the second order partial differential equation that is the linear wave equation by itself, we know it will be of the hyperbolic kind. So whether the first order system will also give us the same answer or not, that is what we need to check.

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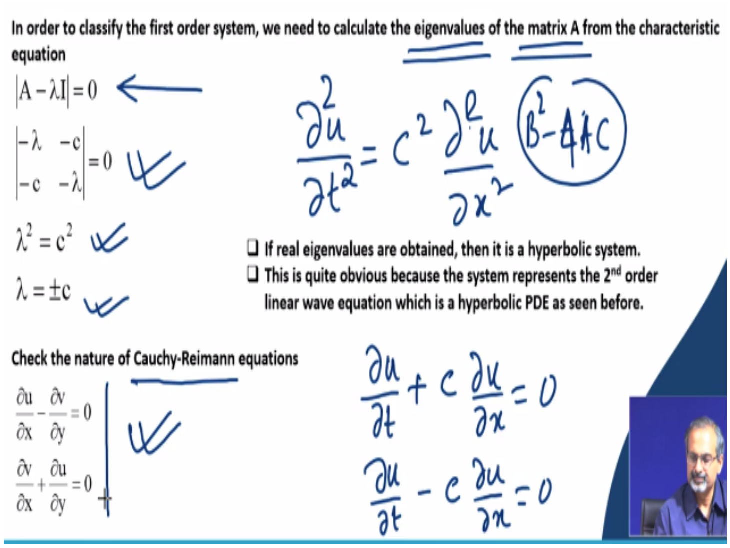

Here the idea is that in order to obtain the behavior of the first order system of partial differential equations, we follow a route which is very similar to what we saw earlier when we were dealing with a single second order partial differential equation with more than two independent variables. There we had to build a matrix comprising of the coefficients A_{jk} and we extracted the eigenvalues. We would end up doing similar procedure here.

We take matrix **A** and calculate the eigenvalues by using the characteristic equation. So, what are we doing? We are actually coming up with this equation, which ends up giving us two roots, which are plus and minus c and you know that if you were to deal with the linear wave equation by using the $b^2 - 4ac$ approach, which we learnt in the beginning, you would end up getting the same outcome.

We see that whether we follow that earlier approach by using a single second order partial differential equation or we follow this approach where we have replaced it by a system of first order partial differential equations, we end up getting the same outcome. The outcome is that you essentially have a dual wave system, one moving towards the positive x axis, the other moving towards negative x axis with c velocity along the two directions.

We call it as +c and -c. The equivalent first order representations will look like this if you split them apart. Now you know how to categorize a system of first order partial differential equations. This could be a good time to do a small homework problem. We have the very famous Cauchy-Riemann equations, which you must have come across while studying complex variables. So, these are the two equations.

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

You can put them into a framework of a system of first order partial differential equations and try to find out their behavior, whether they have hyperbolic, elliptic or parabolic behavior. So, you must follow the matrix approach and extract the eigenvalues of the coefficient matrix to derive the solution.

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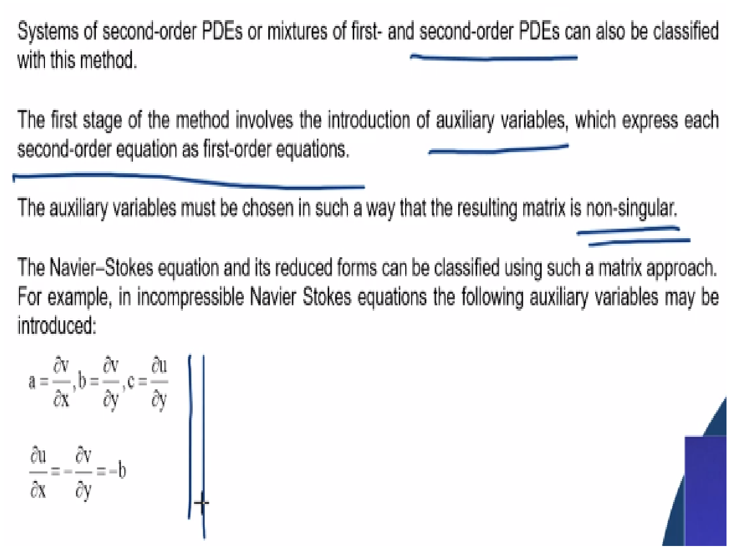

Second order partial differential equations: you had a brief look at the incompressible Navier Stokes equations and you saw that because of the presence of the viscous stresses, there are second order partial derivatives. So, how do we handle such equations because we just learnt to handle first order system of PDEs, but not second order PDEs.

We introduce so called auxiliary variables so that we can convert the second order partial differential equations into first order equations and the auxiliary variables of course have to be chosen carefully because the resulting matrix **A** that we saw a few minutes back needs to be non-singular. This is a very important issue. Now, if you look at the incompressible Navier Stokes equations, this is how we generally introduce the auxiliary variables.

So, you see additional variables a, b, c has now been defined, which are essentially representing first order velocity derivatives along different directions; $a=\frac{\partial v}{\partial x}, b=\frac{\partial v}{\partial y}, c=\frac{\partial u}{\partial y}$. By defining them you could end up forming a system of partial differential equations which are essentially first order to replace the original second order system of equations.

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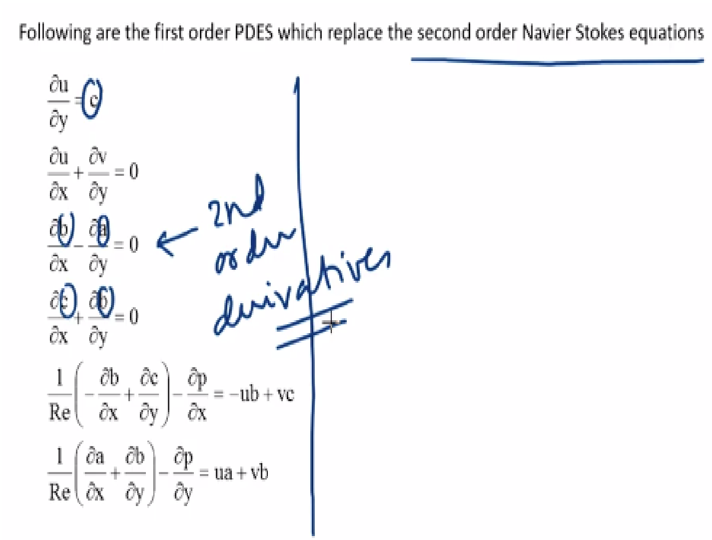

You find that when you are taking derivatives of these variables, they are representing second order derivatives of velocity components because the auxiliary variables are already representing first order derivatives of velocity components. That is how you end up reducing the second order partial derivatives of velocity components to first order ones.

Therefore you finally have a system of first order partial differential equations to solve instead of a second order system of equations. So, with the knowledge that you have gained, you can now go ahead and try classifying the incompressible Navier Stokes equations in this manner. With this, we end the discussion on classifying partial differential equations. Thank you.