Introduction to CFD
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Lecture - 33
Numerical Solution of One Dimensional Convection-Diffusion Equation

In the next few lectures, we will discuss about one dimensional convection diffusion equation.

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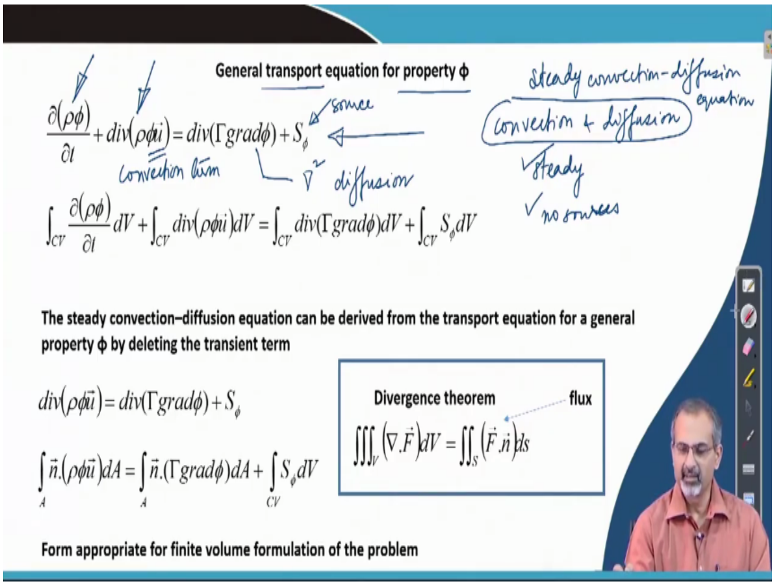

In some earlier lectures, we had talked about a general transport equation for property phi. You may recall that a differential form of such a transport equation, which essentially models the transport of the property phi could contain different terms. The first term contains a time derivative. So, the property phi may change in terms of strength as a function of time and space both.

So, the first derivative at a certain spatial location would indicate the time wise change of the property; the second term is showing up as the divergence of an expression which contains density the property as well as the local flow velocity. So, because the local flow velocity is involved in transporting the property phi; this is often called as the convection term or convection derivatives or convective derivatives.

Because it is a divergence term therefore, there would be spatial derivatives **(()) (02:13)** of the argument, the rho, phi, u. Again remember, u is essentially a vector. So, you could write down this transport equation in terms of components along different orthogonal directions. So, in the case of Cartesian frame of reference, you would have x, y, z components of this governing transport equation.

But when you write it in the form of a vector you have a more compressed way of representing. On the right hand side, you have again a divergence; but this time, divergence applies on a gradient of a scalar so that essentially ends up producing Laplacian. But then you also have a coefficient gamma so that coefficient gamma could be a function of space. If it is a function of space, then it remains within that bracket.

If it happens to be a constant, it could come out of that bracket. Now, this term is called as the diffusion term, because it involves second order derivative of the property and then you can finally have a term which represents a so called source term. So, there could be some region of the flow field from where the property is getting created, the property is getting introduced into the flow field. So, that could be treated as a source point for that property.

You may also have antagonistic points where it acts as sink of the property from where the property can vanish. So, all these physical phenomena get modeled into a general transport equation for property phi. So, some or all of these effects may be responsible for the transport of the property. Most importantly, we are bringing in the effects of convection and diffusion. If the transport is steady then the time derivative would not exist.

If there are no sources for the property file, then the source term x phi also goes away from the governing equation and then it can become a transport equation involving only convection and diffusion on the property. In such a situation, we would call it as a steady convection diffusion equation. We will just take a small detour at this point and come back to this slide once again in a few minutes. It could be a good time now, to discuss about what we had talked about in the previous module on linear wave equation.

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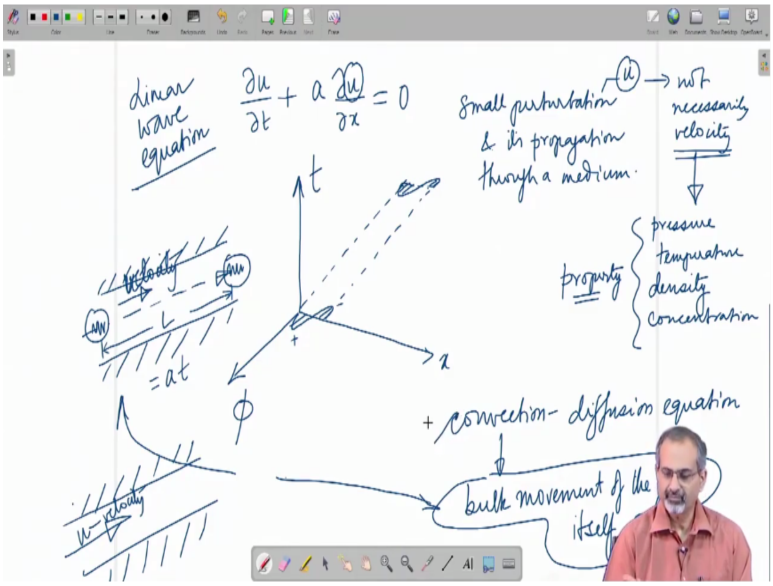

So, when we were discussing about the linear wave equation, this was our governing equation. So, this equation, what does it model? It models a small perturbation and its propagation through a medium. The small perturbation occurs in the dependent variable u. u is not necessarily velocity, it could be other properties as well; it could be pressure, could be temperature, could be density, could be concentration.

So, in general, it is a property of the flow field which is getting transported in the form of a wave; in the form of a small ripple or a perturbation. That does not bring in bulk movement of the fluid medium itself. So, going back to the physics again through a simple plot, we had drawn this plot earlier as well. So, if we call this general property as phi, then at t = 0, let us say we have some distribution of this property phi, a little positive ripple and a little negative ripple just to give it a simple waveform.

Then this distribution would just translate through the medium and get located at a different x as time elapses. As we said that this is just a small perturbation of the local pressure, the local temperature, local density and things like that. And that small perturbation just moves through the medium without inducing bulk movement of the medium itself. That means, this should not be confused with velocity.

When we use the nomenclature u, it does not necessarily mean velocity, it is just a general property. And this needs to be kept in mind when we are talking about convection; when we are discussing convection diffusion equation, because when we discuss convection diffusion equation then it is very important to understand that convection is brought about by bulk movement of the fluid itself.

It is not that a perturbation is just moving through the medium at speed a like what we saw in the case of linear wave equation, but rather there is bulk movement of the fluid itself through certain region and that is carrying along with it the property say phi. So, the fluid movement is responsible for transporting the property. It no longer is the small ripple which is moving at a certain speed a, irrespective of the fact that there is no bulk movement of the fluid medium.

That means, if it is some kind of a passage through which the wave moves, then you have a small disturbance here that disturbance will find its way to another location in an unattenuated manner and this length will be traveled at a predefined speed a. So, this L is nothing but a times a certain t and this packet moves from here to here without causing any bulk movement of the fluid or neither is the bulk movement of the fluid necessary for carrying this small ripples.

So, this has to be very well understood in contrast to the situation that we are talking about in convection diffusion. In convection diffusion, if you have a passage like this, then the first thing that we have to understand is there is bulk movement of the fluid existing in the passage. So, there is a u which now means velocity which is existing in the passage and that in general is a vector it would have multiple components.

And then you are talking about a property phi and its transport through that passage under this the influence of this velocity and not only this velocity, which is causing convection, but also the diffusion which would occur even if this velocity field is not existing. So, diffusion is something which we have discussed at length when we discussed about say elliptic equations. And that kind of a mechanism can exist even in the absence of convection.

So, we are now talking about a scenario where there is diffusion inherently and over and above that there is bulk movement of the fluid which will in induce advection which will induce a movement of the property by virtue of the movement of the fluid through that domain. So, the fluid movement will be responsible for taking the property through the flow field. So, once again we are carefully trying to contrast this with what we studied in the linear wave equation.

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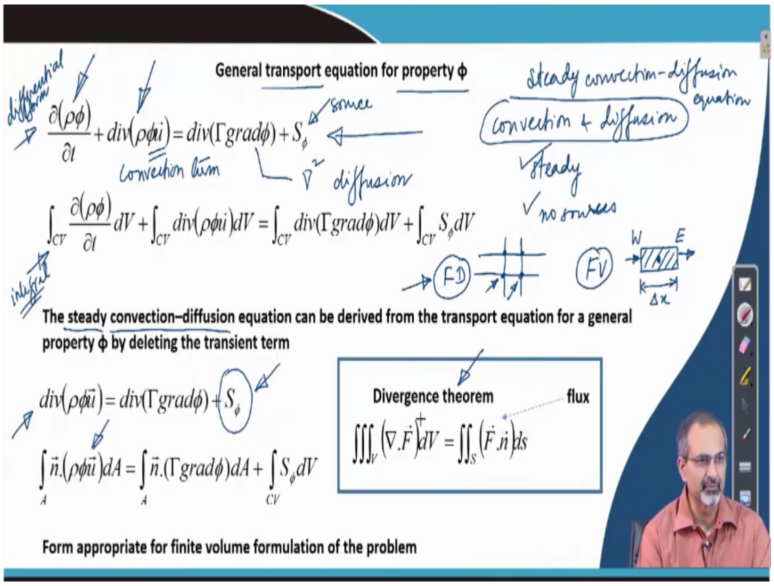

So, coming back to the general transport equation involving the transport of the property phi under the combined influence of convection and diffusion, we can say that such a general transport equation can be represented both in a differential form in the first line and then if you are approximately trying to solve such an equation, you would be invoking say a finite difference discretization in order to solve for that transport equation.

If you want to solve it using a finite volume form, which we had reviewed earlier, then you will look forward to an integral formulation. And while a differential formulation will involve grid points, this is how a finite difference grid would look like. We recall that in finite volume, you are talking about volumes with finite extent. So, there could be a control point embedded inside the finite volume.

But more importantly, we are concerned about the transport of the property through the different phases of the control. Now, when we look at a one dimensional problem, then of course, we could have two phases located at say, the east and the west ends of the control volume. And then we try to work out expressions for the fluxes through those phases. And then we have to work it out in terms of convection in terms of diffusion, if we are handling a one dimensional convection diffusion equation.

Similarly, if we were to work it out from the perspective of finite difference, we would have to take the transport equation in its differential form and then approximate the derivatives for the convection terms and the diffusion terms at these grid points. And of course, we have to ensure that we have a stable scheme; we have an accurate scheme. So, all these properties, we have to make sure that they are incorporated.

Like we discussed before that steady form of the convection diffusion equation would look like this where the transient term has been taken away. And then if you have no source terms, then even this term may be taken away. And then what we end up doing when we try to integrate this equation is that for the given control volume, we try to write these divergence terms in the form of flux terms by using the divergence theorem.

So, this would be a very important theorem to apply, when we try to think about a formulation from the perspective of finite volume method. Because, instead of calculating derivatives, what we are doing is we are trying to calculate fluxes and the fluxes are taking place through the phases of the control volume.

So, this is a very important part of the finite volume approach that we have to keep in mind, if we are trying to solve a convection diffusion equation through the finite volume approach. Of course, for the finite difference approach, if we want to solve it, we have to approximate the derivatives using the finite difference formula.

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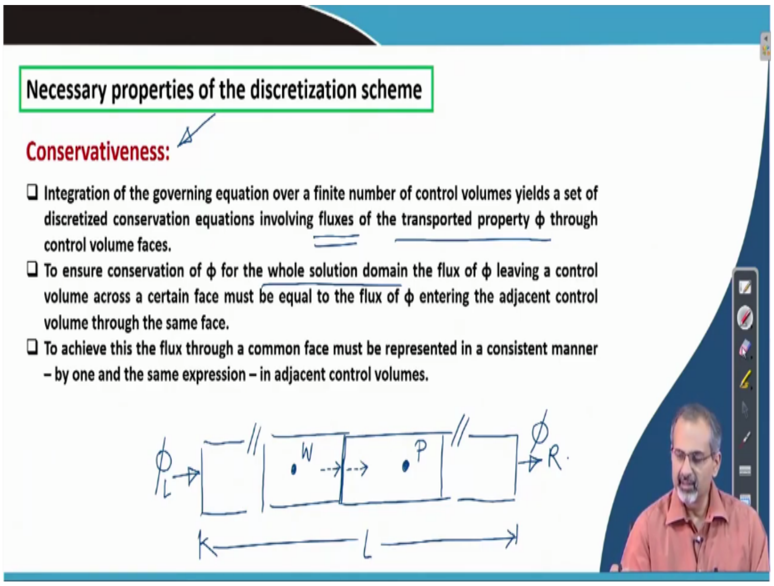

This could be a good time to also recall some of the necessary properties of the discretization scheme that we want to use. If we are using say a finite volume technique, then let us try to see how these properties would matter. This we had discussed earlier when we discussed about the finite volume technique, the first property that is of interest and importance is conservativeness.

So, when we are going to integrate the governing partial differential equation over a finite number of control volumes, because, when we take a certain domain, we may not be able to cover that domain with a single control volume. We may actually have to discretize that domain into a number of portions and each of the portions would be defined on a separate control volume.

Again, you may recall that these control volumes may or may not be of equal size. This is one of the flexibilities that we have with finite volume method. So, when you apply the integrated form of the governing equations to these control volumes, they would end up building a set of discretized conservation equations. And these equations will involve fluxes what we discussed already in the context of applying the divergence theorem.

We can show that the derivatives essentially are expressed in the form of flux transports and these fluxes would involve the transport of the property phi through the control volume faces. Now, in order to ensure this conservativeness property an important conservation of phi that we should involve where phi is being transported is that when we consider the whole solution domain the flux of phi leaving a control volume across a certain face must be equal to the flux of phi entering the adjacent control volume through the same face.

So, what do you mean by that? If we have to adjust and control volumes looking like this and this is a common face that they share. There is a certain flux which leaves this face from say this West control volume and is expected to enter the control volume P. The numerically whatever flux that you define, which is leaving from this face from the cell W should be exactly equal to the flux that enters to this face into the cell P.

So, though may it may sound a little trivial, we must take particular care in terms of the numerical calculations to ensure that. And if that happens, what we finally get out of accounting for these fluxes over the entire domain length is that over all the intermediate faces, there is a flux balance this way, whatever goes out of one face from one cell enters through that same face into the adjacent cell and so on.

We can monitor the conservation of the overall fluxes through the ends of the domain. So, whatever flux had entered from this end, and whatever flux had left from this end, so, this is the left end and this is the right end we should be able to exactly compare and balance those fluxes. Because all other intermediate fluxes automatically get balanced. So, the conservation works this way, when you are having a finite volume technique in operation.

And as long as we are representing the fluxes through the common faces in a consistent manner, then we should not have any issues in with conserving the flux overall in the domain. This is incidentally one of the strong points of finite volume technique, where conservation of fluxes is automatically ensured. Remember that this is not necessarily ensured through finite difference calculations. Flux conservation is inherently ensured in a finite volume technique.

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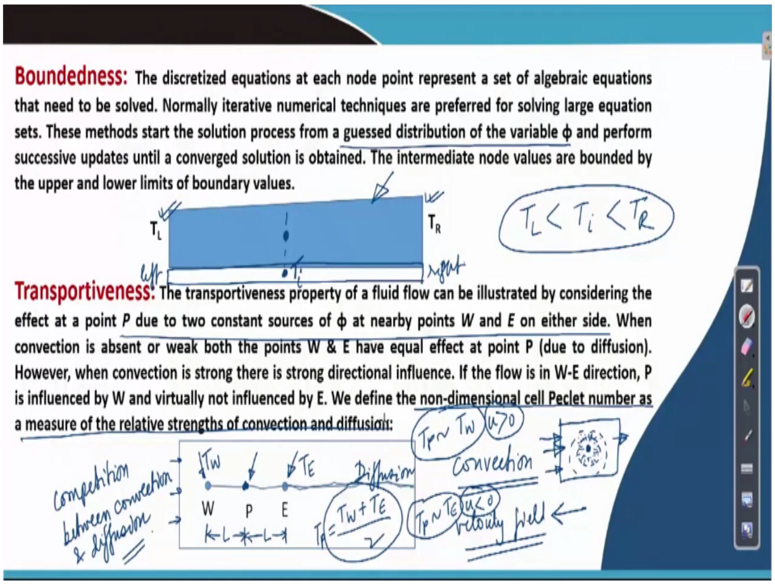

There are other properties of importance as well which the numerical scheme should be taking care of, that is one of them is boundedness. So, when we have the discretized equations at each one of the node points. The node points happen to be the cell centers of the control volumes. So, they represent a set of algebraic equations which need to be solved.

And we may end up using some iterative numerical techniques for solving large number of such equations, because we have a large number of control volumes. So, these techniques may actually start from a guest distribution of the variable phi and perform successive updates until the converged solution is actually obtained. Now, as we do that the intermediate nodal values have to remain bounded by the upper and lower limits of the boundary values.

So, in the figure that we have shown over here, the control volume exists over this interval physically and we are talking about transport of a property T capital T. So, that may be temperature that may be some other property that may be enthalpy it may be some other property as well just given a nomenclature T and this happens to be the left end of the control volume, this happens to be the right end of the control volume.

And incidentally, these values that we have obtained over here. They define the limits for the value of the node which may be lying at the center of this control volume. So, if I am having a value of T at the node here, the node being called as the ith node. Then obviously, the value of the property T add the ith node remains bounded by the upper and lower values existing at the two ends of that control volume.

This ensures the conservation of a property which we are calling as boundedness or ensuring that property boundedness is satisfied. Since we are dealing with convection, we are very much careful about the presence of a velocity field which we discussed at length a few minutes back and we have to be careful that velocity field or its existence will have a big role to play in transporting that property.

Diffusion would exist irrespective of whether velocity is existing or not. So, diffusion is there inherently. But in addition, if velocity field is there, then there is a kind of a competition between convection and diffusion. So, if there is no velocity existing in a certain flow region that means, the fluid is stagnant. Now, if you talk about distribution or transport of a property, how does it work?

So, let us say you have a certain domain and this is containing a fluid and you have a spot from where heat is emanating, it may be a hot wire, which has been immersed into the fluid. So, you know that heat would dissipate or rather spread around this hot wire heating up the ambient fluid. The fluid closest to the wire will get heated to the largest extent and so on. And if you monitor the temperature distribution in the vicinity of the wire, you will see that gradually fluid at different radial locations will start showing increasing temperature.

So, irrespective of the presence of velocity, you see propagation of heat through the medium. This is typically diffusion, it does not even have a directional bias it is expected to heat up the fluid uniformly in all possible directions. However, if you not begin a movement of this bulk flu bulk movement of this fluid then we will see gradually that the temperature distribution is changing and more heat tends to get transported towards the downstream direction that is being swept by the velocity field.

So, this is precisely the combined action of convection and diffusion that we are talking about. And the numerical scheme should be such that it should have the capability of capturing the transportiveness property which is in particular able to cater to the capturing of the convective effect. So, transportiveness property of fluid flow can be illustrated by considering the effect at a point P.

So, we are talking about a point P here due to two constant sources of phi at nearby points W and E. So, these may be now two hot wires, which are exactly equidistant from this point P. So, let us say you have a certain length L and the same here. That means if the fluid is not moving, then the heat from points W and E should equally affect the point P. So, that is what happens when you have only diffusion in play.

And now, if there is a bulk movement of the fluid in a very weak manner to begin with, then the point P still continues to be affected by both points W and E. But gradually more effect comes from the point W then point E because there is a gradual movement of the fluid. So, effect of W will now start spreading downstream along the movement of the flow velocity.

And then as the velocity grows, you make the flow move more strongly with higher velocity then there would be a point where W will influence a point P almost entirely as compared to the point E. And now, if you had maintained different temperatures at these points W and E you would be able to make out the differences more because if T W and T E are different and if the flow velocity is large, you will find that the point P is recording almost entirely T W and almost no part of T E.

As you slow down the velocity and make it stagnant, it starts getting influenced in this manner an average effect coming from T W and T E. So, in a situation where it is purely diffusion, you can expect that the point P may show up an average temperature T W + T E by

2. While if it is a strongly convective flow for a strongly convective effect that is dominating the point P, then you might see that the temperature at the point P may be almost approximately equal to T W.

When you have a u > 0 that means a u which is moving from left to right and it will be T P approximately equal to T E if u < 0. That means, u is moving from right to left then T P may be almost equal to T E. So, this is the kind of situation which you expect to see when there is a transportation which is either dominated by convection or dominated by diffusion.

So, when you talk about a mixed convection diffusion scenario, you would of course, always like to compare their strengths. And then we almost inevitably define so called non dimensional cell Peclet number, which would measure the relative strengths of convection and diffusion. So, the Peclet number is a very important number which compares between the two in terms of strength.

And then that gives an indication that what we can expect out of that kind of a transport, that whether diffusion will play the major role or convection will play the major role or whether it is a mix and match of both. So, we will discuss more on convection and diffusion equation in the next lecture. Thank you.