Introduction to CFD
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Lecture - 32
Numerical Solution of Linear Wave Equation (Hyperbolic PDE) (continued)

In this lecture, we will continue our discussion on dissipation and dispersion behavior in linear wave equation.

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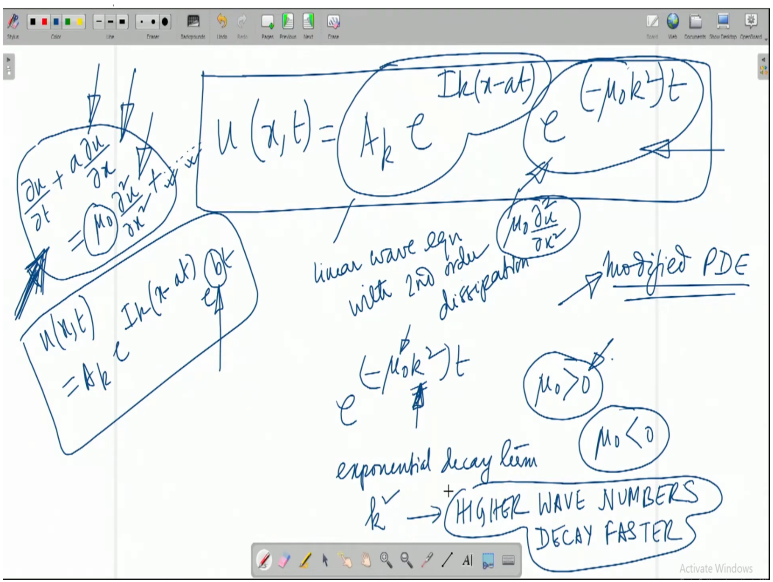

Last time, we had modeled an equation of this kind. So, we slightly modified the original linear wave equation by introducing a term on the right hand side, which looks like this. And we had proposed a trial solution of this form. And then what we did was, we evaluated the value of the which would end up satisfying this equation exactly. And the background behind this activity was that we had learned in the previous lecture about the modified partial differential equation approach.

And we found that when we have an approximate numerical scheme, which is trying to model the linear wave equation behavior. We essentially end up solving the modified partial differential equation. We do not end up solving the exact equation and we pay the maximum importance to the leading truncation error term. So, if the leading truncation error term happens to be of this form.

Then essentially you are modeling a partial differential equation of this kind when you try to solve the linear equation. If you are doing that then how would the solution get affected in the process? So, here what you are trying to figure out is that analytically, how would the solution change? So, now, you understand that there would be an additional term coming in here, because of the presence of the second order term on the right hand side.

Because this term was not figuring when you had the solution for the exact wave equation. The exact wave equation was modeled purely by this part. So, this is essentially the outcome of the presence of the second order term on the right hand side on the solution and mind it, this is an analytical solution. Now, what is the difference between this and say, the finite difference scheme that one uses.

Remember that there are more terms here on the right hand side when you are actually using a finite difference form. So, here we have a much truncated form. So, if you are using an equation of this kind, you are able to generate an exact solution. While, if we were to include more and more terms on the right hand side, it would get increasingly difficult to come up with exact solutions. That is going to be a formidable task.

So, because of that we are just considering one term on the right hand side and trying to find out what would be the impact on the exact solution. And that would give us a big indication towards how the modified partial differential equation would be held in case of a finite difference approximation. So, that is the utility and the significance of this exercise. So, now we understand that if you look at this term, what role is it going to play?

So, let us look at the term very carefully and try to see its behavior. So, we have this term, this exponential term. Let us look at this term very carefully. What does it contain? It contains some kind of a viscosity coefficient. Most often, if we are trying to link with the modified partial differential equation, this is a numerical viscosity coefficient. It is not connected with physical viscosity, but it is analogous.

So, most often this would be a positive number. So, we have to ensure that it is actually positive. If it is, then what do we have further, you have k square that means the square of the wave number which is always going to be positive. And then you have time, time is always going to be approaching towards the future. That means you are looking at future times means you are always going to have positive times there.

So, the only factor which can influence the sign of this expression is mu not as long as mu not is positive. What do we have here? This is going to be an exponential decay term that means the solution will get damped, if you have something like this. What would happen if in case mu not is negative, then this will exponentially blow up the solution. So, that is what we meant by artificial diffusion and artificial anti diffusion.

Where the solution gets damped and where the solution blows up respectively. And this is a hugely significant term whenever you are dealing with partial differential equations of this form. Where the leading error term, if it is a finite difference approximation is a second order term or a even order term and we need to understand how this coefficient is behaving. So, we have seen in the case of first order upwind scheme for example.

That as long as the CFL number was restricted to a value of 1 bounded between 0 and 1. This would remain positive and that was the necessary condition for stability. So, we have understood now that for a partial differential equation of this form how the solution gets modified if you are using a dissipation term on the right hand side.

Now, let us do a few more experiments, but before we do that we also need to observe another important thing that here, if you look at the exponential town here you have k square. K square means the decay will be stronger for larger wave numbers. So, higher wave number or numbers decay faster, because higher wave numbers means higher k. Hard k means higher k square and therefore, a very strong decay.

So, always we have to keep in mind that these kind of terms will damp the high wave number components to the greatest extent and the low wave number components that means, more gradually varying part of the solution to a lesser extent. That means, you are unlikely to find high frequency oscillations in the solution. This is a very, very important feature.

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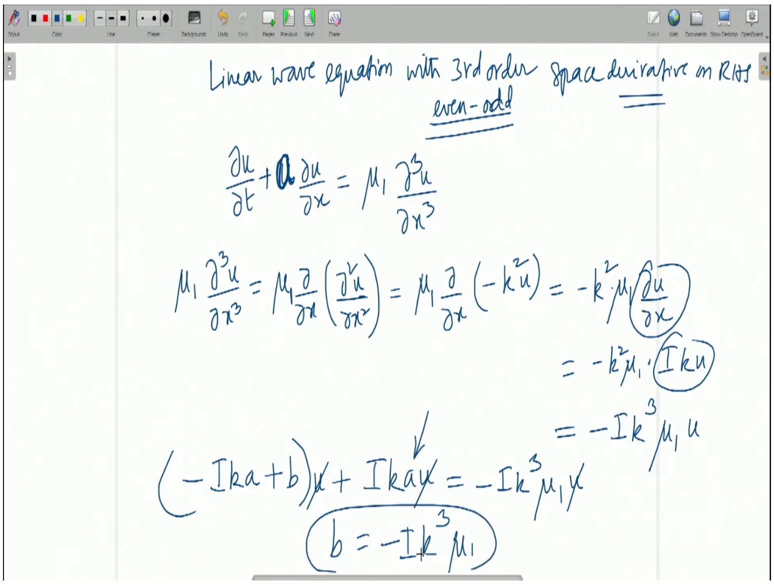

We look at linear wave equation with 3rd order space derivative on the right hand side. When we move from second order to 3rd order, we are essentially going from even to odd that is important. So, whatever we find here, by doing an experiment with 3rd order space derivative on the right hand side of the equation would apply in general for all odd order derivatives that means 5th, 7th and so on.

So, how does the equation now look like? Let us change the coefficient to mu 1. And of course, the derivative is a 3rd derivative in space. Now, as usual, will go ahead with the same tans solution like we did before with the e to the power of bt term additionally, and again we are going to find out what that b is, which satisfies this equation. So, for doing that you need to first figure out how the 3rd derivative calculations work out because the rest of it the left hand side part is a very routine calculation.

So, let us do the right hand side calculations. For doing that we already know the expression for the second order derivative which we, which we had derived earlier. So, we will just substitute that and this coefficient is mu 1. So, del del x operates on - k square u. So, - k square u was found to be the expression for the second derivative.

So, when you apply the 3rd derivative, then you will be able to show it as - k square mu 1 then del u del x that is I k u. We have already derived del u del x earlier and we showed it to be equal to I k u. So, finally, what are we having? We are having I k cubed mu 1 u. This is the expression for the right hand side term. So, what do we do know? We just substitute all the expressions for the left and the right hand side remember that we had – I k a + b times u for the time derivative.

I k a u for the space derivative are more importantly the term. So, here there is a small change. So, this should have been a not u have to make that change. So, this that term is I k a u and then we have derived this expression I k cubed mu 1 u. So, of course, you cancels out from all the terms. And what are we left with. We are left with b is equal to – I k cubed mu 1. So, that is the solution for this case.

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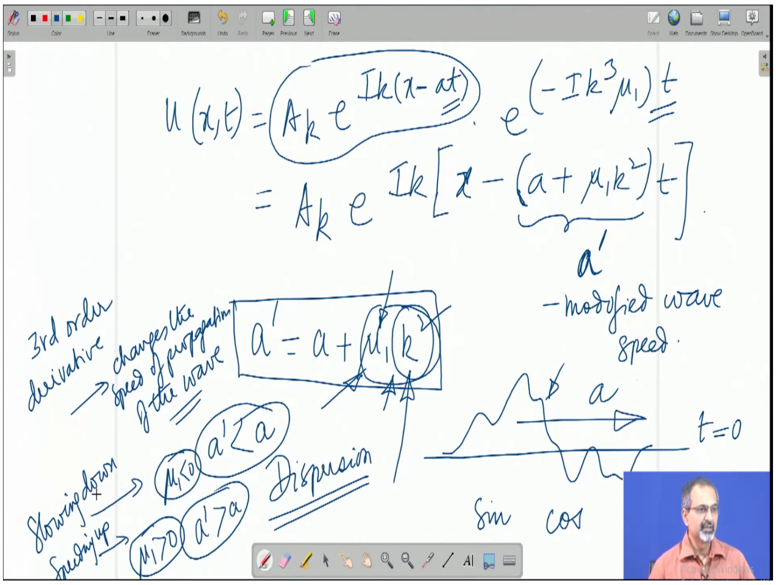

So, what do we have finally, we have a solution which looks like this. So, we have an additional t term, which we can club with this t term. So, let us try to collect all the terms together. We will be left with an expression which looks like this. Now, what we have over here? Remember that in the original wave equation, the solution look like this. And what do we have now? We have an altered expression here.

Most importantly, we have an alteration in the time part. We have only an alteration here that means, earlier this used to be a, while it is now something else. So, let us call it as say, say a dash. So, it is a modified wave speed that we have. We do not have our earlier wave speed anymore. So, the modified wave speed is a + mu 1 k square. Now, what happens if you have a modification here?

One thing that we can first of all figure out is that presence of the 3rd order derivative. What does it do? It changes the speed of propagation of the wave. This is the most important thing that it does. It has changed the way propagation speed from a to a dash because, we can say that mu 1 for all practical purposes must be positive or other non-zero more importantly non-zero.

Only then do you have 3rd order derivative present on the right hand side of the equation, Right. So, you have a non-zero mu 1 over here means that gets multiplied with the square of the wave number that means, this is a change which the presence of the town brings to what, to the wave speed. So, that means the wave would propagate at a different speed now. And that speed will vary for different wave numbers, the different wave number components of the wave.

So, we can say that the wave has a certain distribution, a spatial distribution. Now; that spatial distribution embeds a lot of wave numbers for complicated waveforms say a wave looking like this. If this is the form you are defining at t = 0. Then this is not a very simple wave, which you can say generate with purely 1 sin term or a single cosine. So, it could probably come up with a combination of number of sin and cosine terms.

If that is the case, you would see that a lot of wave numbers get embedded to give rise to this waveform. Now, as you propagate this waveform at the theoretical wave speed or analytical wave speed a does it really move as a packet? The answer is no, because every wave number would start moving at different speeds. So, that is the distortion that the presence of the 3rd order derivative is bringing in.

So, now, just going back to the concept of modified partial differential equation, if you have the leading error term showing up as a 3rd order derivative. What would that do to the solution? It will create different wave speeds and that is what creates dispersion. So, the concept of dispersion comes from here. Again, remember that if mu 1 is positive, then we have speeding up of a certain wave number component.

If new mu 1 is negative, then we have slowing down of a certain wave number component with respect to the exact wave because then if mu 1 is negative, we are looking at a dash which is less than a. So, this is the case for mu 1 less than 0. Right? And remember that mu 1 is less than 0 is not to be confused with an anti diffusion case, because anti diffusion was occurring with even derivative. Here we have an odd derivative.

If mu 1 is greater than 0, what does it do? a dash becomes greater than a. So, you have a slowing down here; you have a speeding up. So, at this point, you may be recalling that when we had earlier talked about phase errors. We talked about leading phase error and lagging phase error for different ranges of values of the CFL number for the first order upwind scheme.

So, we once more visit that concept here, but now in terms of solution of the exact partial differential equation. So, in the previous instance, when we were talking about the dispersion or dissipation. We were talking about the approximate solution. Here, we are talking about the exact solution. We will show once more another example.

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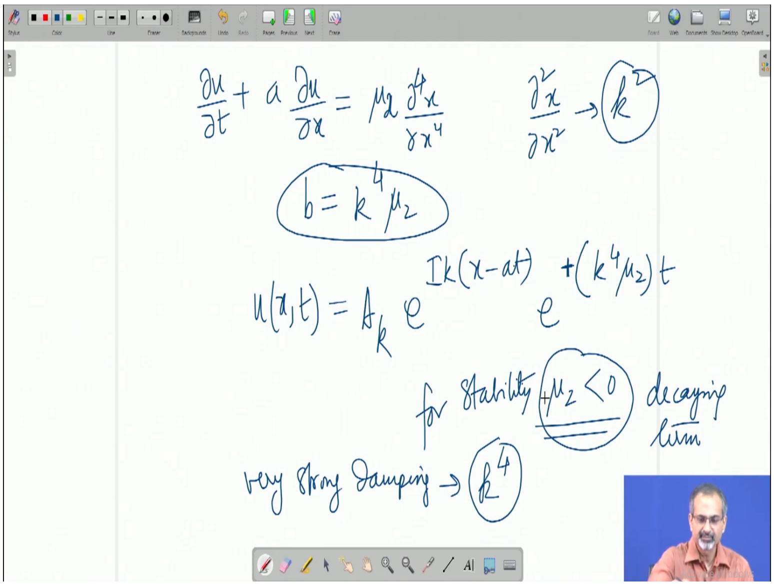

So, here, we go back to an even order derivative. Let us, try with a 4th order derivative now. Just to check that what does the effect of a higher order even derivative have on the solution. So, in this case, if you do a simple calculation yourself, you can show that the b term will come out to be k to the power of 4 times mu 2. And if that is the case, you will have a solution looking like this. This should be in the subscript.

So, this essentially becomes like this. And of course, for the sake of stability mu 2 has to be less than 0. You can see that this has a positive sign. It does not have an inherent negative sign like you saw in the second order derivative case. So, what will happen is you have to explicitly state that mu 2 has to be negative. So that you can effectively have a negative exponent here and it actually behaves like a decaying term.

Otherwise, there will be a blowing up of the solution. That is from the stability perspective. What about the damping part? This would damp very severely. So very strong damping because you have a k to the power of 4 term here. So, in the case where you had a second order derivative on the right hand side. If you saw that in the exponent there was a k square term, Right.

Now, you have a k to the power of 4 term, which means it will damp very strongly compared to a second order derivative term, but it will damp. So, now we see that even order derivatives are all damping the solution; odd order derivative at least for the 3rd order derivative. We saw that it is dispersing the wave. So, these are very important properties, which we now saw through some model equations.

And we will remember these trends when we are looking at modified partial differential equations.

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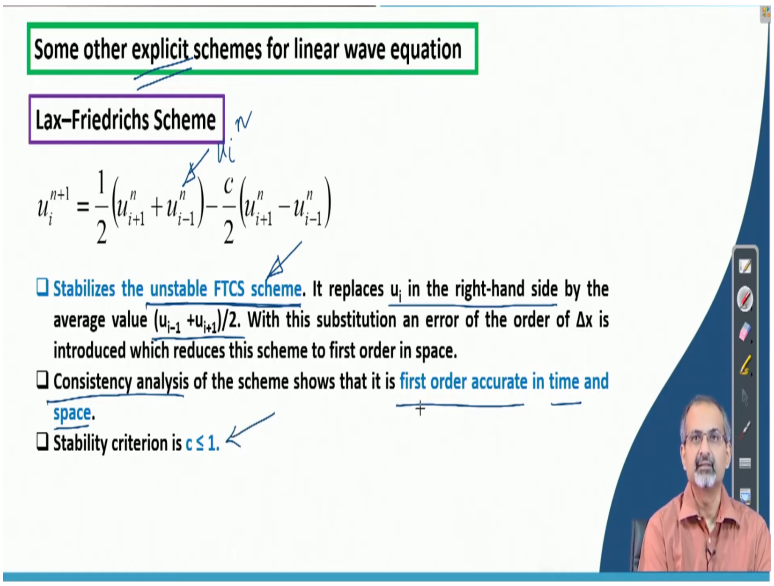

Now, before we complete our discussion on linear wave equation. We would do a quick review of, some of the schemes which are existing beyond the first order upwind scheme. We have mainly dealt with the first order upwind scheme. But, as we can understand that first order upwind scheme has first order accuracy both in space and time. So, there are other schemes, we first look at a few explicit ones.

So, in the explicit category, we have a scheme called as the Lax-Friedrichs scheme. Now; that is a scheme which essentially stabilizes the unstable FTCS scheme by replacing the u i term on the right hand side by an average. So, you can imagine that if you were looking at the FTCS scheme, instead of this term, you would actually have a u i n term. So, instead of that it is replaced by an average and that happens to stabilize the solution, but in the process, it reduces the order of accuracy.

It no longer has second order accuracy, like what FTCS scheme would have though FTCS scheme essentially is not a usable scheme at all. While this is a usable scheme, because it stabilizes the solution. So, order of accuracy is first order both in time and space that can be shown through a consistency analysis. We have discussed about the consistency property already. And stability criteria is that the CFL number should be less than equal to 1.

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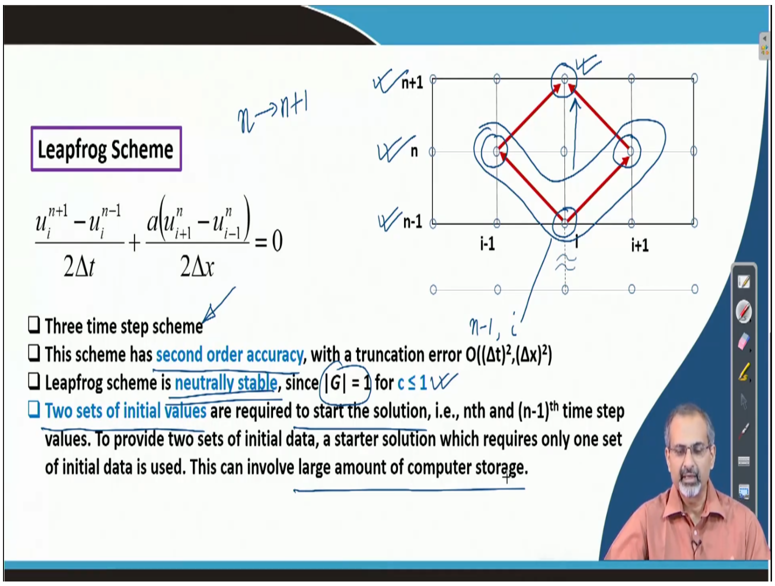

If we are looking at more higher order accuracy in the explicit category and linear wave equation, we can look at other schemes an interesting scheme to look at is, what is called as a Leapfrog scheme. It has its name by virtue of the typical nature of the stencil, where there is a kind of jumping around in the form the frogs do as they move. So, if you look at a point here, it is at a time level n - 1 and at a space location i.

From there, if you look at a solution which you are developing at the higher grid point and n + 1 at level, then what are the points which are contributing. A stencil looking like this is contributing. So, it is almost like a frog like stencil with jumps to that point. It leads to that point, to give you a solution. So, there are some interesting aspects to it. One is that you are using three time steps.

If you have noted, there is n - 1, there is n, n + 1. So, in the previously discussed schemes, we have always talked about going from nth level to the n + 1th level. So, there were only 2 time steps involved in the solution. While here we are talking about 3 time steps. The scheme has a second order accuracy which we of course, can show through the Taylor series expansions. Another very interesting property is that it is neutrally stable.

That means the amplitude of G is equal to 1 which means, it will never artificially damp the solution. So, this is neutral stability when mod G is equal to 1. And it has a stability condition that several number should be less than equal to 1 because it deals with 3 different time steps. You would need a 2 steps of initial conditions or initial values to start the solution, because if you are at the first time step, and you want to go to the second time step.

You would need a zeroth time step value to be fed into the algorithm in order to generate the second time step value, which is not available with you. So, first what you do is? You use a scheme which can work based on 2 time steps to generate a solution at the second time step just by using the first time step value. Once you are done with that you have the first and the second time step values available with you.

And then you can generate the 3 time step using the Leapfrog scheme. And then the solution propagates that way. If you are not carefully programming, then this could involve large amount of computer storage, because you are involving more number of time step data. Another very popular explicit scheme which has overall second order accuracy both in space and time is the Lax-Wendroff scheme which shows up like this. Now, we would like to very briefly discuss how the scheme is derived?

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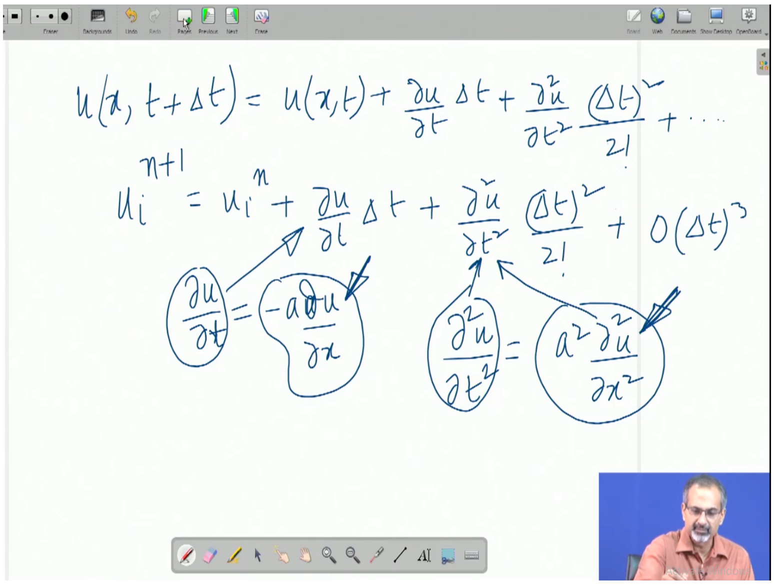

So, we look at a Taylor series approximation at x and t + delta t. This should be time derivative, second order time derivative or you can replace it by an index now, u i n + 1. This is u i n and then the derivatives. Of course, you will have order delta t cube terms here. Now, we remember that the model equation is this. There is a linear wave equation. So, that this expression for del u del t replaces the term as - a del u del x.

And additionally, we know that the second order form of the wave equation looks like this. So, the second order time derivative is now replaced by this form that means, the temporal derivatives will now be replaced by these 2 forms in terms of spatial derivatives. If you do that what are you left with?

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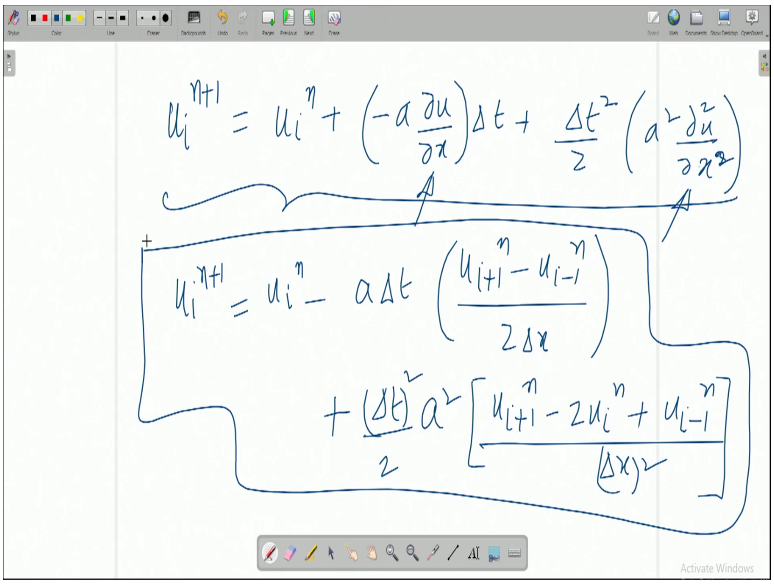

You are left with this and of course, 3rd order truncation error, which we do not mention anymore. So, we use this form to be now discretize for the first and the second order spatial derivatives in the form of central differencing to give rise to the Lax-Wendroff scheme. So, we finally have. So, this is essentially the scheme and that is what you see over here. So, this of course, has second order accuracy in time and space and has a stability criterion of CFL number less than or equal to 1.

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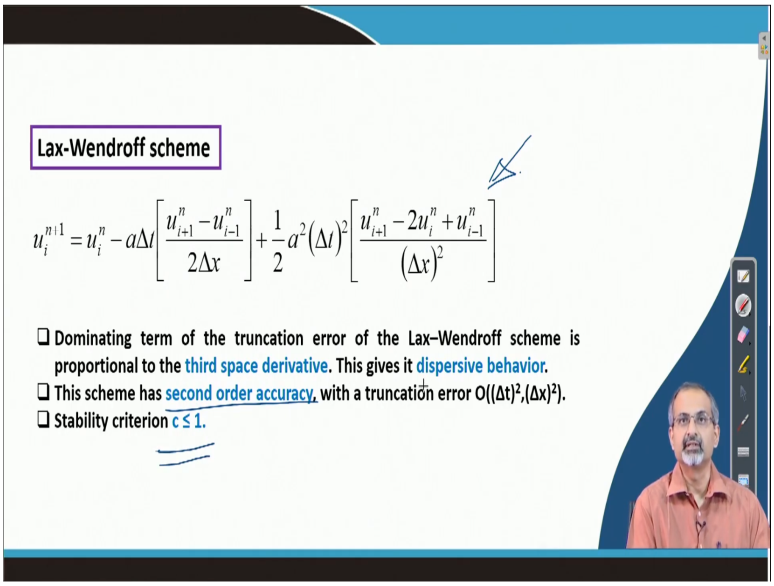

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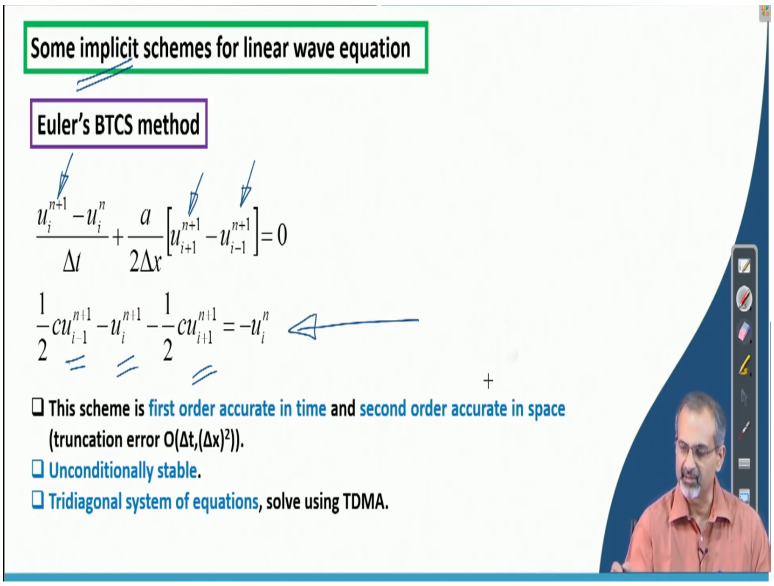

We can also think about implicit schemes where we do not have any issues with stability. So, one possibility is the Euler's backward time central space method. So, as you can understand that you have at more than one grid points, spatial grid points you are using n + 1 at time level in your discretization. So, that would generate an implicit formulation.

So, what you have essentially is a tridiagonal system of equations formed through that formulation and this formulation would give you first order accuracy in time, second order accuracy in space and it is of course, unconditionally stable. So, this is Euler's backward time central space method. Since, you have a tridiagonal system, of course, you can go ahead using the TDMA.

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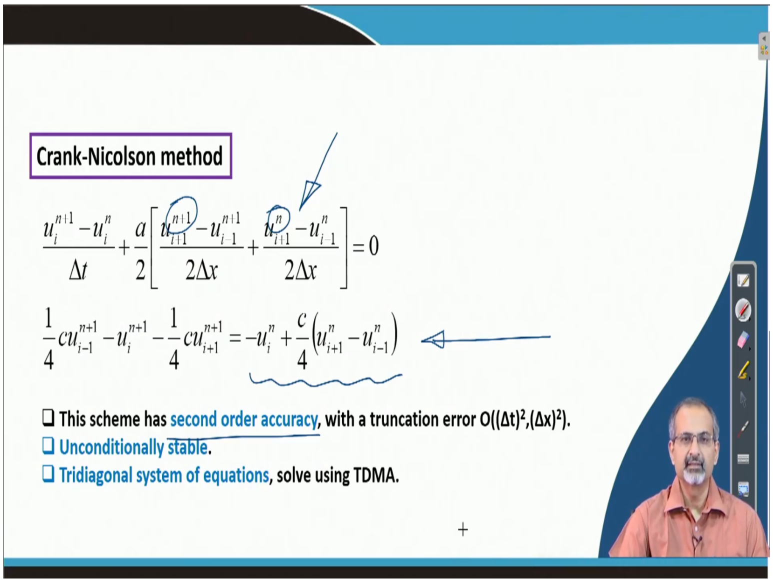

The last method that we discuss before closing the lecture is the Crank-Nicholson method being applied here. Crank-Nicholson, of course, we have learned earlier when we discussed about parabolic partial differential equations. And we already saw that that gives an implicit formulation and if you recall, there we had used the nth and the n + 1th time level for discretizing the spatial derivative in an averaged form.

So, the same formulation comes over here. So, you have contributions from both the n + 1th as well as nth time levels in discretizing the time derivative and thereby, you get the tridiagonal form again in this case. You have more terms on the right hand side contributing from the nth level which are the known terms. It has second order accuracy in both space and time which is superior than the Euler backward time central spacing that we discussed earlier and again it is unconditionally stable.

So, these are a few methods, which you can think of for discretizing linear wave equation both from the explicit side as well as implicit side and they would have different formal orders of accuracy. But it would be more important to see that in terms of dissipation and dispersion behavior how these schemes perform. So, these could be explored further by the students as homework exercises, taking up each one of these schemes.

And trying to explore the dissipation and dispersion behavior, the form of the modified partial differential equation. And that would give more insight into how these schemes function as we use them for solving linear wave equation. Thank you