Introduction to CFD
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Lecture - 29
Numerical Solution of Liner Wave Equation (Hyperbolic PDE) (continued)

In the last lecture, we had looked at the geometrical interpretation of the CFL condition; we will continue that discussion in this lecture.

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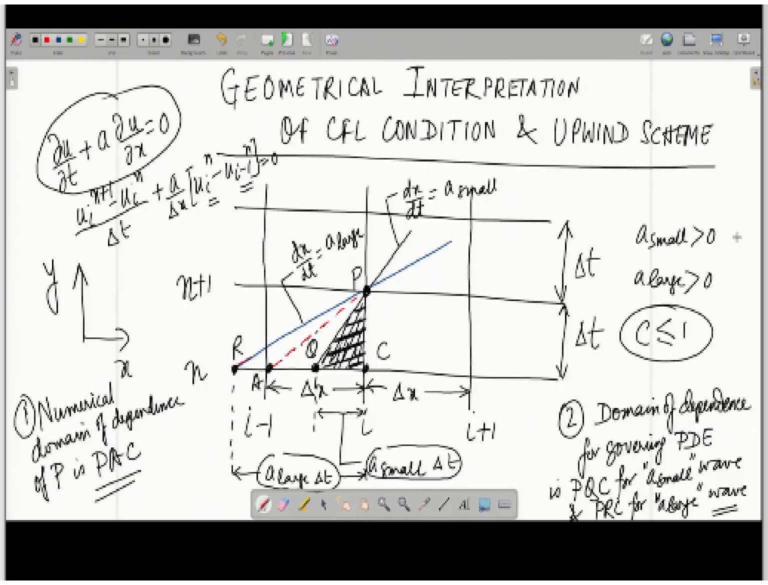

So, this is the diagram we drew to explain the domain of dependence in terms of the numerical scheme and in terms of the governing partial differential equation for the point P. The point P is important for us, because that is the point at which we are trying to build up the solution and the next time step that is n + 1 and add the spatial grid point i. So, how does that point P depend on values which are coming from earlier time steps.

And different spatial locations that is what we have explained through this diagram. And we explain for 2 different wave speeds which can be accounted suitably or not accounted suitably by the numerical scheme in this diagram. So, we explain that wave which can have comparatively smaller velocity a small is bounded by the numerical domain of dependence of point P and therefore, can be correctly resolved using this kind of a grid.

Grid in the sense both space and time grid, because remember that finally, it is the CFL condition and CFL condition is a collective representation in terms of wave speed in terms of time step as well as space step. So, as a combination whether they are satisfying the condition that C is less than equal to 1 as the upper bound is what matters that whether the numerical scheme can properly account for the physics which is involved; which is essentially defined by the characteristic lines.

So, for the a small wave, we saw that the domain of dependence of the governing partial differential equation can be contained within the numerical domain of dependence while when the wave speed is larger. The a large wave cannot be contained by the numerical domain of dependence. So, based on that we argued that the limiting case could be that when dx dt exactly conforms with the dashed red line, that is the limiting case up to which the numerical scheme can handle.

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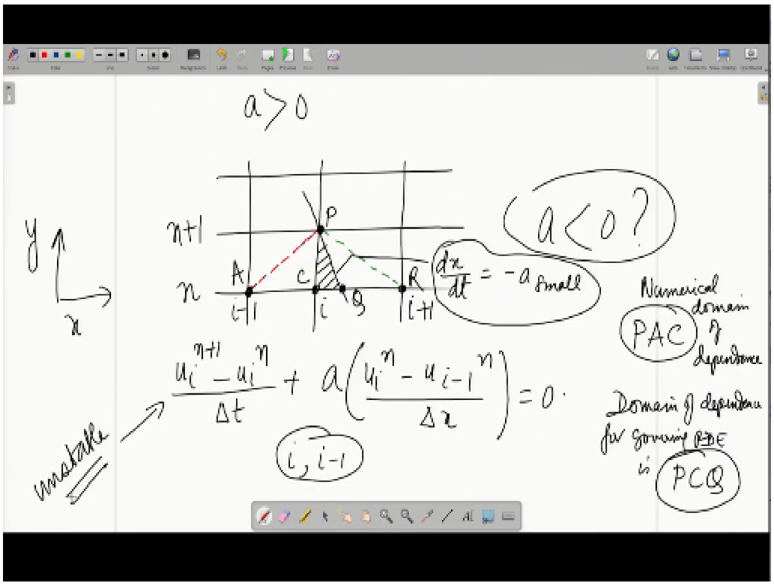

So, we discussed the conditions a > 0. And we put our arguments together through the geometrical approach. What if a is less than 0? How would we tackle that situation? Let us look at the grid once more. So, we remember that along the x axis, we have the spatial points; along the y axis we have the temporal levels. And now, we are defining a characteristic line with a negative slope.

Let us say - a small and then let us mark a few points here. So, last time we had a dashed red line. Let us use red color here. Additionally, we will create a green color dashed line on the other side because we will end up needing it later. And then let us mark a few important points as usual. So, P, C, Q, R and A. So, let us define these points. Now, if we were to continue with the backward differencing.

Let us see what happens. So, we will write down the backward difference formula once again, which worked as the upwind scheme when a was greater than 0. Now, this uses a stencil comprising of i an i – 1 spatial. That means the numerical domain of dependence will continue to be the same triangular region that we discussed PAC that is the numerical domain of dependence.

That is the region from where it draws information for doing the calculations in order to calculate the values at point P. Does it work for this case? That is the question to ask. So, we see the domain of dependence of the governing PDE in this case. So, domain of, for governing PDE is which region it is PCQ. It is clear from the characteristic line that we have drawn.

That this is the region from where information will propagate to point P as far as the governing equation is concerned. Is it accounted or enveloped by the numerical domain of dependence? The answer is no. The numerical domain of dependence is looking at the other direction altogether. So, it is not looking at the upwind direction as far as a < zero situation is concerned. What happens?

This scheme becomes unstable. So, which scheme would work the schemes at least the ones that we have tried.

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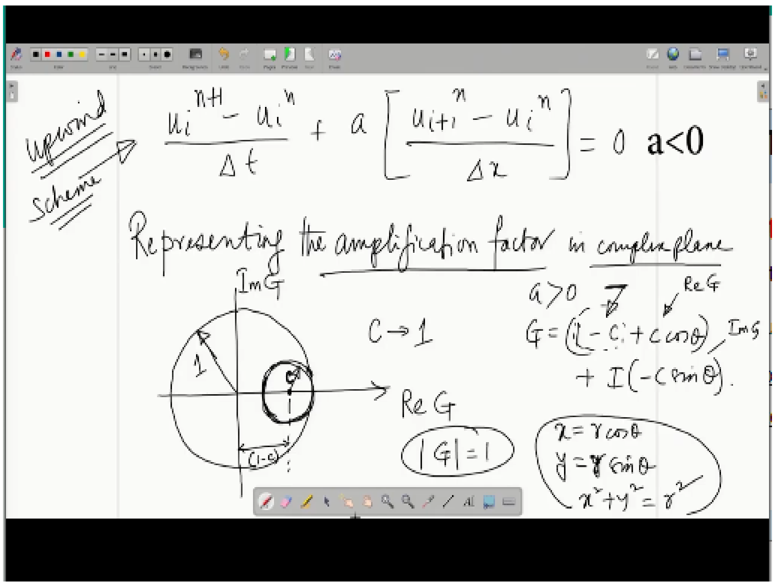

Now, it is the forward differencing that we had tried which will work because that is the scheme which can account for the physics consistently by looking at the proper domain of dependence and because the wave speed is small enough. It can also be bounded by that reason. If the wave speed additionally was large, the wave was moving from right to left and it was large enough in comparison with the spatial and temporal grid that we have provided.

Like what we saw in the previous lecture, where we defined a large then it would be difficult even with this scheme to give you a proper result. But as long as the wave speed is like what we have shown as a small, but this time with a negative sign. This scheme is actually going to consistently address the physics. So, in this context, this is the upwind scheme. So, we have the convenience of defining these schemes in a more general sense, irrespective of which is the direction of the wave propagation.

So, we do not have to call it forward differencing or backward differencing. We can in general say upwinding. And then the scheme adapts based on what is the local direction of propagation of the wave. This is a more generalized framework. So, we understood that the backward scheme works for $a > 0$; the forward scheme works for $a < 0$. There are a few more things we can do, which is going to help us understand the different aspects of our analysis better.

So, we have been talking about complex expressions for the amplification factor. So, representing the amplification factor in complex plane, because till now, we have just written down the expression for G in order to get modulus of G. We have defined the complex conjugate multiplied it with G to get modulus of G square and so on. But we have never thought about a geometrical representation of the amplification factor.

So, let us try to draw the axis in complex plane. So, this is real G. This is the imaginary part of G. And we can pick up the amplification factor, let us say for $a > 0$ for the forward time backward space scheme. The expression for G will be used here. So, the expression of G is 1 + or other, we will put it slightly differently we put it as 1 - C + C cos theta + i times - C sin theta. And if you watch the expression for G very carefully.

You can figure out that there is a parametric equation of a circle hiding here, in the expression for G. Because you may remember that an expression like x is equal to r cos theta, y is equal to y, or say r sine theta can satisfy the equation of a circle, x square + y square is equal to r square. So, taking a clue from there, which we have done in our +2 coordinate geometry.

We also remember that these 2 portions of the G expression, this is the real part of G; this is the imaginary part of G. These are like the 2 coordinates and in the x coordinate sense, there is an offsetting of the origin by an amount 1 - C. So, if you did not have this offsetting, then the coordinates of the center of the circle would have lied at 0 0. But because you have a nonzero value there and that corresponds to the real axis.

You will have a shifting of the center of the circle along the real axis. So, taking these clues, you could try representing it here in the complex plane. And you will find that in the complex plane, you will have a representation looking like this you essentially have 2 circles, one with radius immunity, another the smaller one with radius C and the center of the smallest circle is offset by an amount 1 – C. That is essentially the representation of G in complex plan.

So, as C grows, what happens is the value of 1 - C will become smaller and smaller. And in the limit as C approaches 1, 1 - C will tend to 0 and this small circle will limit toward the large circle with radius unity. As long as a C is not, C does not exceed the value of 1, you can lie within the large circle or at most margin the large circle. Even if you merge with the large circle, there is no issue because in that case, modular G will be equal to 1 which is still stable.

But if you exceed that then you will end up having instability in the calculation. So, that was what we learned from the CFL condition that the CFL number needs to be restricted to 1. It should not go beyond 1 and this is the way you tell the story through geometrical means by representing it in the complex plane. So, this is a presentation of amplification factor in the complex plane.

So, this is something that we did not look at. Since, we were looking at the geometric representation of the CFL condition, which we did with the spatial and temporal grids earlier. This is another geometric representation which may be of interest.

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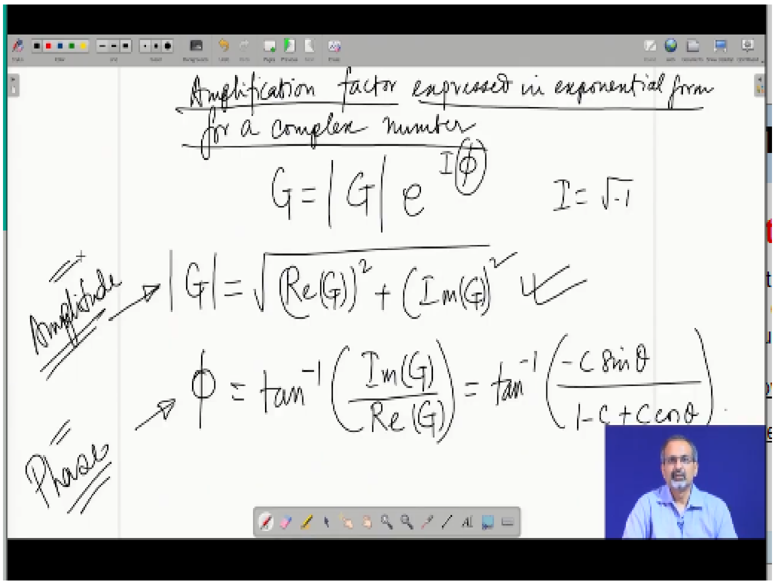

There is another way. We can look at the amplification factor in express it in exponential form when it turns out to be a complex number. So let us see how we do it. We express the amplification factor G as a product of its modulus and an exponential term, which we will define as e to the power of I phi. Of course, I is under root - 1. So, how do we define phi, of course, modulus of G is already known to us.

So, we take the real part of G, square it, add it to the imaginary part of G and if you take under root you will get more G. How about the angle phi that comes from the tan inverse of the imaginary part of G divided by the real part of G. Of course, if you wish you could put brackets here. So, this is how we can express the amplification factor in an exponential form for representing a complex number or a complex expression for G.

And let us say, if you were to do it for the first order upwind scheme, then this will come out to be as far as mod G is concerned, we have already shown the expression earlier for first order upwind scheme, so, we are not repeating here. So, additionally we have the definition phi here. So, essentially this gives us information about amplitude and this gives us information about the phase.

So, we may put it this way that for the exact equation will have a certain definition for the amplitude; will have a certain definition for the phase and then for any given numerical discretization. We can have a definition for its amplitude and its phase. So, if we compare amplitudes between the exact scheme and a particular numerical discretization. We can see whether we are getting amplitude errors.

If we are comparing between the face expression for an exact scheme and the phase expression for a numerical discretization. We will get whether there are any phase wise errors which are committed. So, how would it matter? It would matter enormously as to how we capture the wave propagation. So, as we capture the moment of the wave, if the numerical scheme is artificially dissipating it, then the amplitudes will decay.

If the numerical scheme is creating some phase errors, then the wave will get distorted. There would be certain wiggles which would be formed. We will talk more about it later, but, there would be phase distortions in the wave and therefore, the wave might show existence of certain oscillations. So, these errors would have to be minimized and that is done through more advanced numerical schemes.

But the target should always be to first of all quantify them and then try to minimize them. Now, since we discussed about possible comparison of a numerical scheme's performance with the exact solutions. Then we need to have a basis on which we compare.

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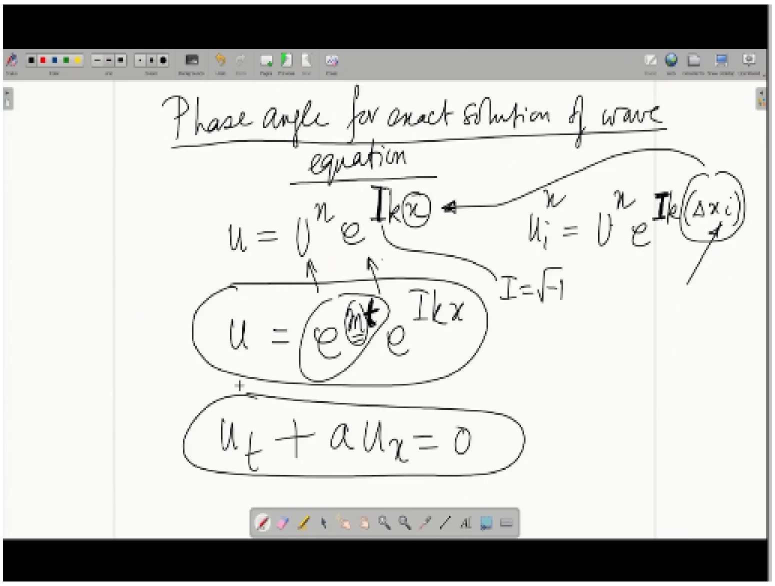

So, let us try to write down the phase angles for exact solution of wave equations. We are not concern much about the amplitude because linear wave equation by definition will not attenuate the amplitude of the wave that you define at t = 0. It will propagate on attenuated through the domain with wave speed a by definition.

So, the main thing that we need to be concerned about is how we ended up defining the phase angle for the exact solution, because that would be the basis for comparing the phase angle as per the numerical scheme, and we need to see whether there is a gap between the two. And as far as the modulus of G is concerned, we know modulus of G for the exact scheme will definitely be 1.

So, we only need to care about the modulus of the numerical scheme and then that gives you the difference the gap. So, to begin with, we propose a form for the dependent variable U and going by our experience of doing the Von Neumann stability analysis. We begin with a proposal of this kind that they would have to be an amplitude part and a phase part because we are drawing from our experience that when we did it for discrete terms.

We did it this way. So, only thing that we have done is that instead of writing it as delta x into i, we have written it as x here. And now, we need to think that whether we will keep the U n term as it is, but it turns out that we can replace it with an exponential term, because that will give us a more convenient form to work with, by the way, this should be capital I, because that is what represents under root - 1 while i here is the grid index.

So, this should be capital I as well. So, we are proposing an alternative form for U n with an unknown parameter m, but of course, it has to be multiplied with the time instant t. So, that we can show the dependence of the term U n on the time. So, if you propose a form for u like this and you robot substituting that form in the governing partial differential equation, can you work out the value of m which is showing up in the index in terms of the other known parameters. Let us try to do that.

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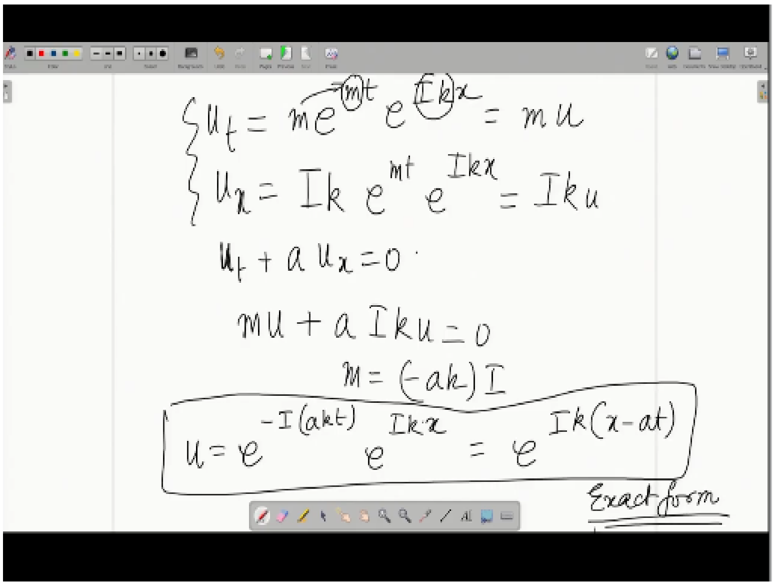

So, if you substitute that form in the partial derivative u t, then it comes out to be this which is nothing but m times u itself. And then you work out u x. You need these because you have to substitute it in the governing partial differential equation. So, if you take a partial derivative with respect to x, you will get I times k which is coming from here. And this m essentially came from there.

So, when you do u x, you get I k times e to the power of m t times e to the power of I k x. So, that is nothing but i times k times u. So, if you substitute these two into the governing partial differential equation, what do you get? You will get m into u + a I k u = zero, which means m is equal to - a k times I. If you substitute it back into the original equation, you will get - i times a k t into e to the power of I k x.

So, if you just club the terms together, you get I k x – a t. So, that that is the form for u. This is the exact form of u and in any numerical discretization, this is not exactly going to be how you behaves, there will be a difference. We need to quantify what that difference works out to be in terms of the amplitude, and in terms of the phase. So, if you were to look at the exact amplification factor.

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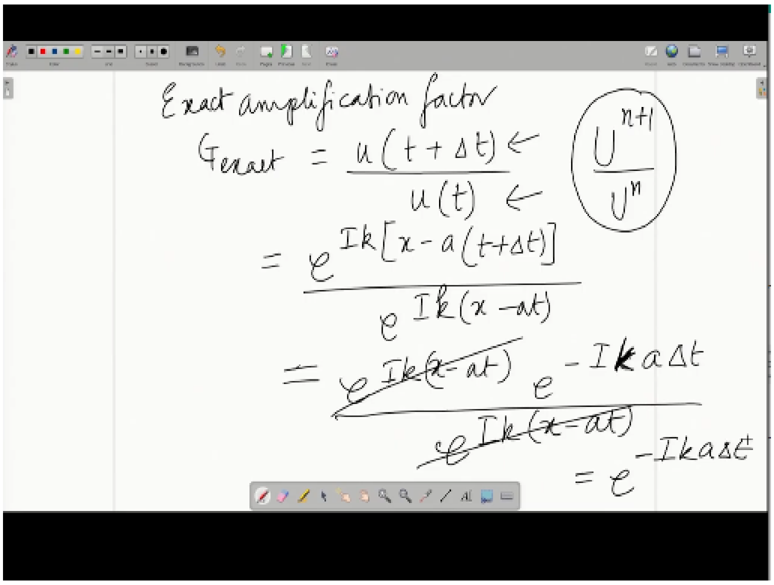

So, that G exact, let us say, it is u t + delta t and divided by u t. Remember that for a discretized case, we do it as U n + 1 by U n here. We are doing it as a ratio of u at 2 different time instants separated by a small interval delta t. This is how it works out and as you can figure out the exponent in the numerator can be worked out like this.

So, that way one of the terms in the numerator will get cancel with 1 term with the term in the denominator. So, leaving behind only 1 term, that is, e to the power of - I k a delta t.

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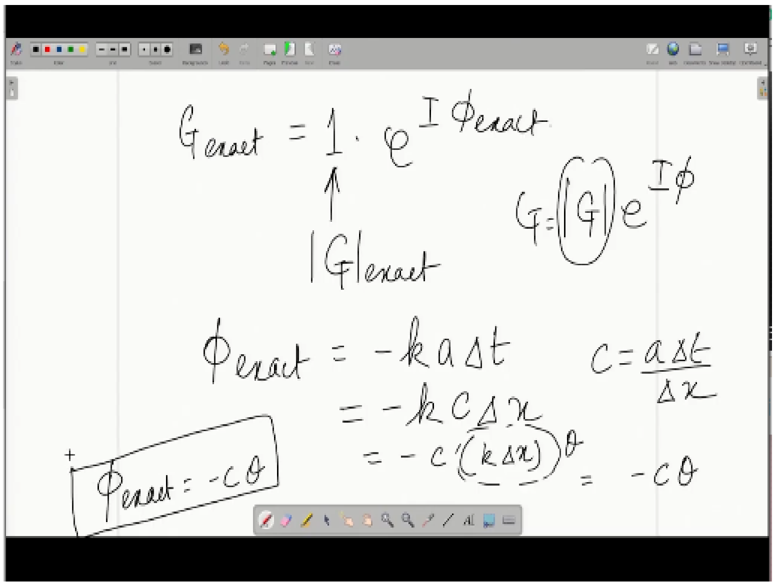

So, we can write this as, say, G exact is, say, 1 into e to the power of I phi exact. Now, one in the sense that this is modulus of G exact because you did not have any number other than 1 here as a coefficient of the exponential term and remember that this what is the basis of this.

We have defined G to be equal to mod G into e to the power of I phi. So, that mod G here comes out to be 1.

What have you got for the phi exact that is nothing but k a delta t and then that can be written as - k C delta x. C being equal to a delta t by delta x. And we remember that k delta x again is phase angle theta in the wave number space. So, that is - C theta. So, phi exact is equal to - C times theta. So, now is the time when we can compare between the exact and the numerical scheme.

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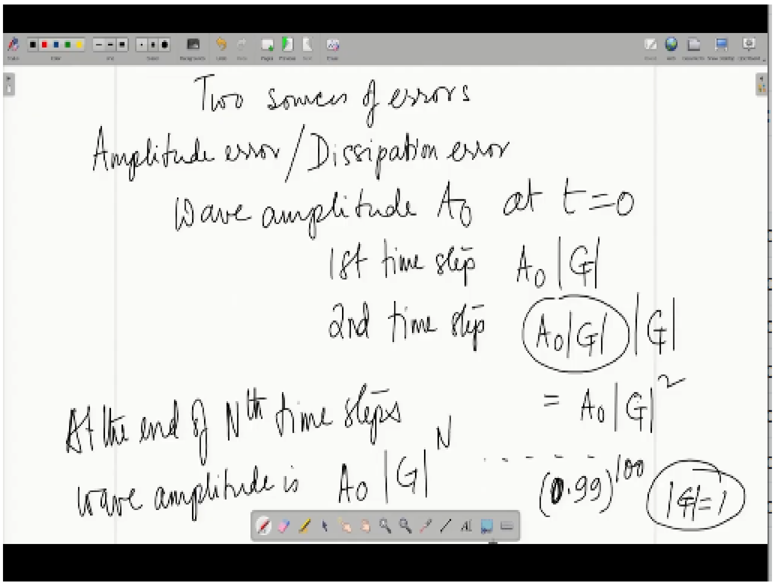

So, if we say that there are 2 errors, 2 sources of errors then we would say that there is an amplitude error which is often called as dissipation error because, if your mod G is too far below 1, then on repeated application of mod G over time steps. The wave amplitude can come down with time. How it applies is like this that if you have a wave amplitude, say, A not at t = 0.

When you are setting the initial waveform, then at the first time step, what will happen to the amplitude is that A not will be multiplied by mod G. That will be the new amplitude. What will happen to the amplitude at the second time step you are now starting with the reduced amplitude already, which comes from the first time step. To that the mod G will again get applied in the seconds time, time step and therefore, it will become A not into mod G square and so on.

So, at the end of, say, Nth time steps, what will be the wave amplitude is nothing but A not mod G to the power of N. So, you can imagine that if mod G happens to be a number close enough to 1, but by repeated application of that mod G is step by step, step by step. Let us say, at the end of 100 time steps, this would not remain very close to 1 anymore.

That means, when you are computing for a long enough time, in order to ensure that the wave goes through the numerical domain entirely. It starts somewhere and it ends somewhere in the sense that it has coming into the domain at some point and it has traversed through the domain at wave speed a, and is moving out of the domain. So, it takes a number of time steps to do that. So, as it does, it is also getting attenuated and that is not a good message.

So, you need to remain as close as possible to 1. So that you are not getting sufficiently at innovated. Typically, if you are able to maintain mod G = 1, that is the best scenario. That is the way you ensure that amplitude wise you match the exact solution. If you do, then you then bother that how good are you doing in terms of phases. So, this is the amplitude issue.

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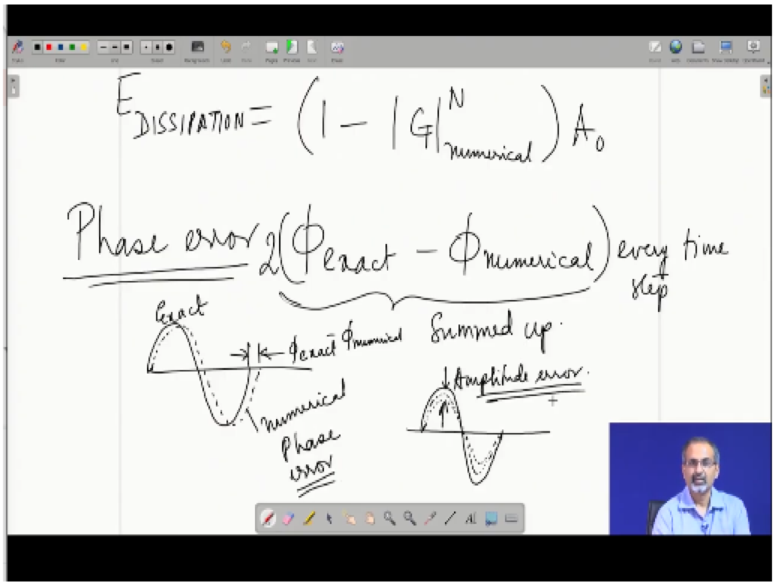

So, if you were to count quantify the error between exact and exact and the numerical you can say that this is the diffusion error. So, E for error and the subscript DIF for diffusion. So, one can define it like this say 1- mod of G and that it is numerical scheme. So, G numerical raise to the power of n into A not. So, that would be the diffusion. So, this should be dissipation. So, let us write it completely.

So, this is the dissipation error. So, as we said that there could be also phase error. So, what is the phase error? So, phase error is essentially the phi exact – phi numerical that is what you are committing at every time step. Incidentally, this is incremental phase shifting that is happening every time step that means, in 1 time step, it is this; in the subsequent time step, it is 2 times this. it gets summed up.

So, dissipation error is like particular factor getting multiplied over and over again as time steps elapse where as phase error gets summed up. So, if there is a phase shifting, let us say, you have a wave like this, which is exact and initially the numerical result also looks like this. But in the next time step, the numerical wave goes ahead of the exact wave. So, there is a certain amount of phase error that you have committed.

So, this would be the phi exact - phi numerical, let us say. And then in the subsequent time step, you will notice that this has become more, say, 2 times of what happened in 1 time step and so on. So, the phase error will keep accruing like that while the dissipation error is of a different kind. So, if this is the exact 1 and the numerical 1 also matches it initially in the first time step, if there is a reduction in amplitude of this kind.

In the second time step, the reduction may be more and so on. This way the wave gets dissipated. So, this is like an amplitude error while this is phase error. This is how things show up. So, just trying to complete on the phase error part.

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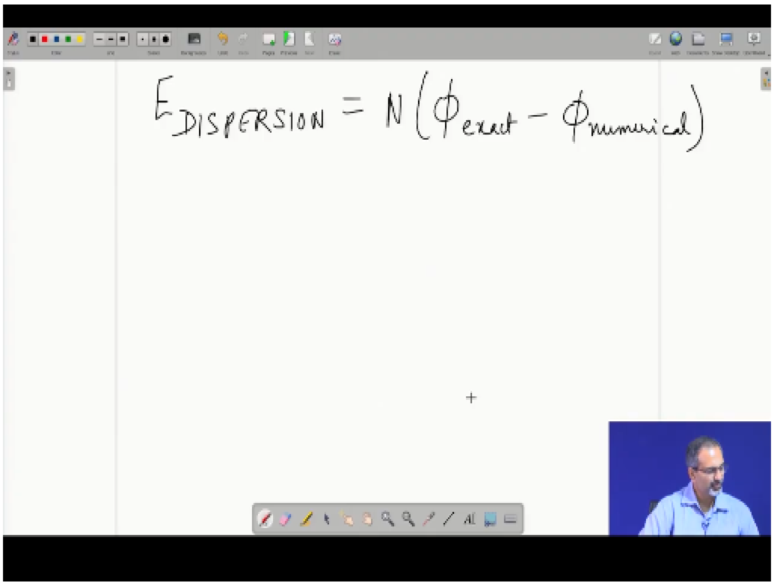

We would say that because of the phase error, there is something called as dispersion of the wave. And that dispersion over N time steps can be quantified as the difference between the phi for the exact and the phi for the numerical in 1 time step multiplied by the number of time steps. So, that gives you an idea about the kind of errors which could be committed, in course of the numerical computations. We will discuss more on this in the next lecture. Thank you.